Probability 1
CEU Budapest, fall semester 2016
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Homework sheet 5 – solutions

5.1 Durrett [1], Exercise 5.1.1

5.2 Durrett [1], Exercise 5.1.3

5.3 **(homework)** Durrett [1], Exercise 5.1.4

Solution:

a.) *Existence:* $Y := \lim_{M\to\infty} Y_M$ exists, because Y_M is increasing in M. The monotone convergence theorem implies that for $A \in \mathcal{F}$

$$\int_A Y \, d\mathbb{P} \overset{Y_M \nearrow Y}{=} \lim_{M\to\infty} Y_M \, d\mathbb{P} \overset{Y_n = \mathbb{E}(X_M|\mathcal{F})}{=} \lim_{M\to\infty} \int_A X_M \, d\mathbb{P} \overset{X_M \nearrow X}{=} \int_A X \, d\mathbb{P},$$

so Y will do.

b.) *Uniqueness:* If Y and Z are both $\mathcal{F}$-measurable and $\int_A Y \, d\mathbb{P} = \int_A Z \, d\mathbb{P}$ for every $A \in \mathcal{F}$, then $Y = Z$ a.s. because for any $\varepsilon > 0$ and $M < \infty$

$$A_{\varepsilon,M} := \{\omega \in \Omega \,|\, Y(\omega) + \varepsilon \le Z(\omega) \le M\}$$

is $\mathcal{F}$-measurable, so

$$0 = \int_A (Z - Y) \, d\mathbb{P} \ge \varepsilon \mathbb{P}(A_{\varepsilon,M}) \quad \Rightarrow \quad \mathbb{P}(A_{\varepsilon,M}) = 0 \, \forall \varepsilon, M.$$

But $\{Z > Y\} = \bigcup_n A_{\frac{1}{n},n}$, so $\mathbb{P}(Z > Y) = 0$. Similarly $\mathbb{P}(Y > Z) = 0$.

5.4 Durrett [1], Exercise 5.1.6

5.5 **(homework)** Let X and Y be independent standard Gaussian random variables. Let $U = X+Y$ and $V = 2X-Y$. Calculate $\mathbb{E}(V|U)$. *(Hint: Example 5.1.2 says that if W is independent of U, then $\mathbb{E}(W|U) = \mathbb{E}W$. If you choose $\lambda \in \mathbb{R}$ cleverly, then $W := V - \lambda U$ will be independent of U. (Since U and W are jointly Gaussian, to show independence it's enough to check that $Cov(U,W) = 0$.) Then write $V = \lambda U + W$.)*

Solution: X and Y are independent standard Gaussians, so $\mathbb{E}X = \mathbb{E}Y = 0$, $Cov(X,X) = Cov(Y,Y) = 1$ and $Cov(X,Y) = 0$. This implies

$$\begin{aligned}
\mathbb{E}U = \mathbb{E}V &= 0 \\
Cov(U,U) &= Cov(X+Y, X+Y) = Cov(X,X) + 2Cov(X,Y) + Cov(Y,Y) = 2 \\
Cov(V,V) &= Cov(2X-Y, 2X-Y) = 4Cov(X,X) - 4Cov(X,Y) + Cov(Y,Y) = 5 \\
Cov(U,V) &= Cov(X+Y, 2X-Y) = 2Cov(X,X) + Cov(X,Y) - Cov(Y,Y) = 1.
\end{aligned}$$

So if we follow the hint and define $W := V - \lambda U$, then $\mathbb{E}W = 0$ and

$$Cov(W,U) = Cov(V - \lambda U, U) = Cov(V,U) - \lambda Cov(U,U) = 1 - 2\lambda.$$

We choose $\lambda = \frac{1}{2}$, so $Cov(W,U) = 0$, and (by the property in the hint) W is independent of U. So

$$\mathbb{E}(V|U) = \mathbb{E}\left(\frac{1}{2}U + W \middle| U\right) = \frac{1}{2}\mathbb{E}(U|U) + \mathbb{E}(W|U) = \frac{1}{2}U + \mathbb{E}W = \frac{1}{2}U.$$

5.6 Durrett [1], Exercise 5.1.8

5.7 Durrett [1], Exercise 5.1.9

5.8 **(homework)** Durrett [1], Exercise 5.1.10

Solution: We will apply the statement of Exercise 5.1.9 with $\mathcal{F} := \sigma(N)$.

Now if we fix $N = n$,then the (conditional) ditribution of X is easier: it's the sum of n i.i.d. random variables. So the conditional expectation and conditional variance are

$$\mathbb{E}(X \mid N = n) = n\mu \quad , \quad \mathrm{var}(X \mid N = n) = n\sigma^2.$$

With the measure theoretical notion of conditional expectation these are written as

$$\mathbb{E}(X \mid N) = \mu N \quad , \quad \mathrm{var}(X \mid N) = \sigma^2 N.$$

We plug these into the statement of Exercise 5.1.9 to get

$$\mathrm{var}(X) = \mathbb{E}(\mathrm{var}(X|N)) + \mathrm{var}(\mathbb{E}(X|N)) = \mathbb{E}(\sigma^2 N) + \mathrm{var}(\mu N) = \sigma^2 \mathbb{E}N + \mu^2 \mathrm{var}(N).$$

5.9 **(homework)** Durrett [1], Exercise 5.1.11

Solution: To understand what's going on, see Theorem 5.1.8, the remark after it, and its proof.

Now, since X is $\mathcal{G}$-measurable,

$$\begin{aligned}\mathbb{E}((Y-X)^2|\mathcal{G}) &= \mathbb{E}(Y^2 - 2XY + X^2|\mathcal{G}) = \mathbb{E}(Y^2|\mathcal{G}) - 2X\mathbb{E}(Y|\mathcal{G}) + \mathbb{E}(X^2|\mathcal{G}) = \\ &= \mathbb{E}(Y^2|\mathcal{G}) - 2X^2 + X^2 = \mathbb{E}(Y^2|\mathcal{G}) - X^2.\end{aligned}$$

So, by assumption

$$\mathbb{E}((Y-X)^2) = \mathbb{E}(\mathbb{E}((Y-X)^2|\mathcal{G})) = \mathbb{E}Y^2 - \mathbb{E}X^2 = 0.$$

Since $(Y-X)^2 \geq 0$, this implies that $Y - X = 0$ almost surely.

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)