Probability 1
CEU Budapest, fall semester 2016
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Homework sheet 3 – solutions

3.1 *Exchangeability of integral and limit.* Consider the sequences of functions $f_n : [0,1] \to \mathbb{R}$ and $g_n : [0,1] \to \mathbb{R}$ concerning their pointwise limits and the limits of their integrals. Do there exist integrable functions $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$, such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for Lebesgue almost every $x \in [0,1]$? What is $\lim_{n\to\infty} \left(\int_0^1 f_n(x)dx \right)$ and $\lim_{n\to\infty} \left(\int_0^1 g_n(x)dx \right)$? Are the conditions of the dominated and monotone convergence theorems and the Fatou lemma satisfied? If yes, what do these theorems ensure about these specific examples?

(a)

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < 1/n, \\ 2n - n^2 x & \text{if } 1/n \le x \le 2/n, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Write n as $n = 2^k + l$, where $k = 0, 1, 2\dots$ and $l = 0, 1, \dots, 2^k - 1$ (this can be done in a unique way for every n). Now let

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{l}{2^k} \le x < \frac{l+1}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

3.2 *Exchangeability of integrals.* Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 \le x - y \le 1, \\ -1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 < y - x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dx \right) dy$ and $\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dy \right) dx$. What's the situation with the Fubini theorem?

3.3 **(homework)** For real numbers $a_1, a_2, a_3, \dots$ define the infinite product $\prod_{k=1}^{\infty} a_k$ as

$$\prod_{k=1}^{\infty} a_k := \lim_{n\to\infty} \prod_{k=1}^{n} a_k,$$

whenever this limit exists.

Let $p_1, p_2, p_3, \dots$ satisfy $0 \le p_k < 1$ for all k. Show that $\prod_{k=1}^{\infty}(1-p_k) > 0$ if and only if $\sum_{k=1}^{\infty} p_k < \infty$.

(Hint: estimate the logarithm of $(1-p)$ with p.)

Solution: For $0 \le p_k \lneqq 1$ we have that $\prod_{k=1}^{\infty}(1-p_k) > 0$ if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln(1-p_k) > -\infty. \tag{1}$$

Now if $p_k \nrightarrow 0$, then this is clearly false. If $p_k \to 0$, then we get from the linear approximation of $x \mapsto \ln(1+x)$ near $x_0 = 0$ that – except possibly for finitely many k-s –

$$-p_k \geq \ln(1-p_k) \geq -2p_k.$$

This implies that

$$C - \sum_{k=1}^{n} p_k \geq \sum_{k=1}^{n} \ln(1-p_k) \geq C - 2\sum_{k=1}^{n} p_k,$$

which means that (1) holds if and only if $\lim_{n\to\infty} \sum_{k=1}^{n} p_k < \infty$.

3.4 Let $X_1, X_2, \ldots$ be independent random variables such that

$$\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}.$$

Show that $\mathbb{E}X_n = 0$ for every n, but

$$\lim_{n\to\infty} \frac{X_1 + \ldots X_n}{n} = -1$$

almost surely.

3.5 Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables. Prove that the following two statements are equivalent:

(i) $\mathbb{E}|X_i| < \infty$.

(ii) $\mathbb{P}(|X_n| > n$ for infinitely many n-s$) = 0$.

3.6 **(homework)** Prove that for *any* sequence $X_1, X_2, \ldots$ of random variables (real valued, defined on the same probability space) there exists a sequence $c_1, c_2, \ldots$ of numbers such that

$$\frac{X_n}{c_n} \to 0 \text{ almost surely.}$$

Solution: let a_n be so big that $\mathbb{P}(|X_n| > a_n) \leq \frac{1}{n^2}$, and let $c_n = na_n$. Then by the first Borel-Cantelli lemma, almost surely $|X_n| \leq a_n$ for all but finitely many values of n, so $\left|\frac{X_n}{n}\right| \leq \frac{1}{n}$ for all but finitely many values of n. This implies

$$\mathbb{P}\left(\frac{X_n}{c_n} \to 0\right) = 1.$$

3.7 Let the random variables $X_1, X_2, \ldots, X_n, \ldots$ and X be defined on the same probability space. Prove that the following two statements are equivalent:

(i) $X_n \to X$ in probability as $n \to \infty$.

(ii) From every subsequence $\{n_k\}_{k=1}^{\infty}$ a sub-subsequence $\{n_{k_j}\}_{j=1}^{\infty}$ can be chosen such that $X_{n_{k_j}} \to X$ almost surely as $j \to \infty$.

3.8 **(homework)** Let $X_1, X_2, \ldots$ be independent such that X_n has *Bernoulli*(p_n) distribution. Determine what property the sequence p_n has to satisfy so that

(a) $X_n \to X$ in probability as $n \to \infty$

(b) $X_n \to X$ almost surely as $n \to \infty$.

Solution: First, let's see what the possible limits can be. If X_n converges either strongly or in probability, then it also has to converge weakly, so p_n also has to converge to some $p \in [0,1]$, and $X \sim B(p)$. Now if $p = \lim_{n\to\infty} p_n \notin \{0,1\}$, then the sequence of *independent* X_n has no chance to converge to X in probability, since $\mathbb{P}(|X_n - X_{n+1}| = 1) \not\to 0$ (that is, the sequence X_n makes big jumps often).

So we need either $p_n \to 0$ and $X_n \equiv 0$, or $p_n \to 1$ and $X_n \equiv 1$.

I. Let's see $X_n \to 0$ first.

a.) $X_n \to 0$ in probability iff $\forall \varepsilon > 0$ we have $\mathbb{P}(|X_n| < \varepsilon) \to 0$. but $X_n \in \{0,1\}$, so for $0 < \varepsilon < 1$, $\{|X_n| > \varepsilon\} = \{X_n = 1\}$, so

$$X_n \to 0 \text{ in probability } \Leftrightarrow \mathbb{P}(X_n = 1) \to 0 \Leftrightarrow p_n \to 0.$$

b.) Since $X_n \in \{0,1\}$, $X_n \to 0$ almost surely iff $X_n = 0$ for all but finitley many n-s, almost surely. By independence and the Borel-Cantelli lemmas, this happens iff

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n \neq 0) = \sum_{n=0}^{\infty} p_n < \infty.$$

II. Similarly for $X_n \to 0$

a.) $X_n \to 1$ in probability iff $\mathbb{P}(X_n \neq 1) \to 0$ iff $1 - p_n \to 0$ iff $p_n \to 1$.

b.) $X_n \to 1$ almost surely iff $\sum_n \mathbb{P}(X_n \neq 1) < \infty$ iff $\sum_n (1 - p_n) < \infty$.

3.9 Let $X_1, X_2, \dots$ be independent random variables. Show that $\mathbb{P}(\sup_n X_n < \infty) = 1$ if and only if there is some $A \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \mathbb{P}(X_n > A) < \infty$.

3.10 Let $X_1, X_2, \dots$ be independent exponentially distributed random variables such that X_n has parameter λ_n. Let $S_n := \sum_{i=1}^{n} X_i$. Show that if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$, then $S_n \to \infty$ almost surely, but if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, then $S_n \to S$ almost surely, where S is some random variable which is almost surely finite.

3.11 Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution *Bernoulli*(p) for some $p \in (0;1)$ but $p \neq \frac{1}{2}$. Let $Y := \sum_{n=1}^{\infty} 2^{-n} X_n$. (The sum is absolutely convergent.) Show that the distribution of Y is continuous, but singular w.r.t. Lebesgue measure.

3.12 **(homework)** Let the random variables $X_1, X_2, \dots, X_n, \dots$ and X be defined on the same probability space and suppose that $X_n \to X$ in probability as $n \to \infty$.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $Y_n = f(X_n)$ and $Y = f(X)$, show that $Y_n \to Y$ in probability as $n \to \infty$.

Solution: We nned to see that for any $\delta > 0$ and $\varepsilon > 0$ if n is big enough, then $\mathbb{P}(|Y_n - Y| > \delta) < \varepsilon$. since $Y_n = f(X_n)$ and $Y = f(X)$ where f is continuous, if $|Y_n - Y|$ can only be big when $wX_n - X|$ is also big.

More precisely, if f is **uniformly** continuous, so that for any $\delta > 0$ there is a $\delta' > 0$ such that $|f(x) - f(x_0)| \le \delta$ whenever $|x - x_0| \le \delta'$, then

$$\mathbb{P}(|Y_n - Y| > \delta) \le \mathbb{P}(|X_x - X| > \delta') < \varepsilon \text{ if } n \text{ is big enough.}$$

since $X_n \to X$ in probability.

Unfortunately, a continuous $f : \mathbb{R} \to \mathbb{R}$ is in general *not* uniformly continuous. To treat this problem, for any $\varepsilon > 0$ choose K so big that $\mathbb{P}(|X| > K) < \frac{\varepsilon}{2}$. Now the interval $I := [-K-1, K+1]$ is compact, so f is uniformly continuous on I: for any $\delta > 0$ there is a $\delta' > 0$ such that $|f(x) - f(x_0)| \le \delta$ whenever $|x - x_0| \le \delta'$ and $x_0, x \in I$. We can safely assume that $\delta' < 1$.

Now if $|Y_n - Y| = |f(X_n) - f(X)| > \delta$, then either $|X_n - X| > \delta'$, or $|X| > K$. This means that

$$\mathbb{P}(|Y_n - Y| > \delta) \le \mathbb{P}(|X_n - X| > \delta') + \mathbb{P}(|X| > K) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if n is big enough. □

(b) Show that if the X_n are almost surely uniformly bounded [that is: there exists a constant $M < \infty$ such that $\mathbb{P}(\forall n \in \mathbb{N}\, |X_n| \le M) = 1$], then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

Solution: Since $X_n \to X$ in probability, $X_n \Rightarrow X$ weakly as well. The expectations are $\mathbb{E}X_n = \mathbb{E}f(X_n)$ and $\mathbb{E}X = \mathbb{E}f(X)$ where $f : \mathbb{R} \to \mathbb{R}$ is the identity function: $f(x) = x$. If this f was *bounded*, then $X_n \Rightarrow X$ would imply $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$. Of course, f is *not* bounded, but if all the X_n are a.s. bounded by M, then so is X, and f can "replaced" by some *bounded* continuous f_M, for which $f = f_M$ on $[-M, M]$. E.g.

$$f_M(x) := \begin{cases} -M & \text{if } x < -M \\ x & \text{if } -M \le x \le M \\ M & \text{if } x > M \end{cases}$$

will do. So $\mathbb{E}X_n = \mathbb{E}f_M(X_n)$, $\mathbb{E}X = \mathbb{E}f_M(X)$, and $X_n \Rightarrow X$ implies $\mathbb{E}f_M(X_n) \to \mathbb{E}f_M(X)$.

(c) Show, through an example, that for the previous statement, tha condition of boundedness is needed.

Solution: Let $\mathbb{P}(X_n = n) = \frac{1}{n}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$. then $X_n \to 0$ in probability, but $\mathbb{E}X_n = 1$ for every n.

3.13 Let the random variables $X_1, X_2, \dots, Y_1, Y_2, \dots, X$ and Y be defined on the same probability space and assume that $X_n \to X$ and $Y_n \to Y$ in probability. Show that

(a) $X_n Y_n \to XY$ in probability.

(b) If almost surely $Y_n \ne 0$ and $Y \ne 0$, then $X_n/Y_n \to X/Y$ in probability.

3.14 **(homework)** Prove that

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 \frac{x_1^2 + x_2^2 + \dots x_n^2}{x_1 + x_2 + \dots x_n} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \dots \mathrm{d}x_n = \frac{2}{3}.$$

Solution: Let $X_1, X_2, \dots$ be i.i.d. $\sim Uni([0,1])$. Then the joint density of $X_1, \dots, X_n$ is

$$f_n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } 0 \le x_1, \dots, x_n \le 1 \\ 0 & \text{if not} \end{cases}.$$

So

$$\begin{aligned}
e_n &:= \int_0^1 \dots \int_0^1 \frac{x_1^2 + x_2^2 + \dots + x_n^2}{x_1 + x_2 + \dots + x_n} \, \mathrm{d}x_1 \dots \mathrm{d}x_n = \\
&= \int_{\mathbb{R}^n} \frac{x_1^2 + x_2^2 + \dots + x_n^2}{x_1 + x_2 + \dots + x_n} f_n(x_1, \dots, x_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_n = \\
&= \mathbb{E} \frac{X_1^2 + X_2^2 + \dots + X_n^2}{X_1 + X_2 + \dots + X_n} = \mathbb{E} \frac{\frac{X_1^2 + X_2^2 + \dots + X_n^2}{n}}{\frac{X_1 + X_2 + \dots + X_n}{n}}.
\end{aligned}$$

Now the strong law of large numbers says that

$$\frac{X_1 + X_2 + \dots + X_n}{n} \to \mathbb{E}X = \frac{1}{2} \quad \text{and} \quad \frac{X_1^2 + X_2^2 + \dots + X_n^2}{n} \to \mathbb{E}X^2 = \frac{1}{3}$$

almost surely. this implies that

$$\frac{X_1^2 + X_2^2 + \dots + X_n^2}{X_1 + X_2 + \dots + X_n} \to \frac{1/3}{1/2} = \frac{2}{3}$$

almost surley, and thus also in probability. To get the convergence of the expectations, one way is to check that $\frac{X_1^2 + X_2^2 + \dots + X_n^2}{X_1 + X_2 + \dots + X_n}$ is bounded. Indeed, $0 \le X_i^2 \le X_i \le 1$, so $0 \le \frac{X_1^2 + X_2^2 + \dots + X_n^2}{X_1 + X_2 + \dots + X_n} \le 1$ and Homework 12 gives that $e_n \to \mathbb{E}\frac{2}{3} = \frac{2}{3}$. $\square$

3.15 Let $f : [0; 1] \to \mathbb{R}$ be a continuous function. Prove that

(a)
$$\lim_{n \to \infty} \int_0^1 \int_0^1 \dots \int_0^1 f\left(\frac{x_1 + x_2 + \dots x_n}{n} \right) \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n = f\left(\frac{1}{2} \right).$$

(b)
$$\lim_{n \to \infty} \int_0^1 \int_0^1 \dots \int_0^1 f\left((x_1 x_2 \dots x_n)^{1/n} \right) \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n = f\left(\frac{1}{e} \right).$$

3.16 Let the random variables $X_1, X_2, \dots, X_n, \dots$ be defined on the same probability space and let $Y_n := \sup_{m \ge n} |X_m|$. Prove that the following two statements are equivalent:

(i) $X_n \to 0$ almost surely as $n \to \infty$.

(ii) $Y_n \to 0$ in probability as $n \to \infty$.

3.17 *Weak convergence and densities, again.*

(a) Prove the following

Theorem 1. *Let $\mu_1, \mu_2, \dots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continuous w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \dots$ and f, respectively. Suppose that $f_n(x) \overset{n \to \infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

(Hint: denote the cumulative distribution functions by $F_1, F_2, \dots$ and F, respectively. Use the Fatou lemma to show that $F(x) \le \liminf_{n \to \infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.

(b) Show examples of the following facts:

i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.

ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.

iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.

3.18 **(homework)** Let $X_1, X_2, \dots$ be independent and uniformly distributed on $[0,1]$. Let $M_n = \max\{X_1, \dots, X_n\}$ and let $Y_n = n(1-M_n)$. Find the weak limit of Y_n. *(Hint: Calculate the distribution functions.)*

Solution: Let F_X be the common distribution function of the X_i:

$$F_X(x) = \mathbb{P}(X_i \le x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}.$$

Now $M_n = \max\{X_1, \dots, X_n\}$, so $M_n \le x$ iff $X_i \le x$ for all i. So the distribution function of M_n is

$$\begin{aligned} F_{M_n}(x) &:= \mathbb{P}(M_n \le x) = \mathbb{P}(X_1 \le x, \dots, X_n \le x) = \mathbb{P}(X_1 \le x) \cdots \mathbb{P}(X_n \le x) = \\ &= (F_X(x))^n = \begin{cases} 0 & \text{if } x \le 0 \\ x^n & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}. \end{aligned}$$

We have used the independence of the X_i. Now the distribution function of Y_n is

$$\begin{aligned} F_n(y) &:= \mathbb{P}(Y_n \le y) = \mathbb{P}(n(1-M_n) \le y) = \mathbb{P}\left(M_n \ge 1 - \frac{y}{n}\right) = 1 - F_{M_n}\left(1 - \frac{y}{n}\right) = \\ &= \begin{cases} 0 & \text{if } y \le 0 \\ 1 - \left(1 - \frac{y}{n}\right)^n & \text{if } 0 < y < n \\ 1 & \text{if } y \ge n \end{cases}. \end{aligned}$$

Given any $y > 0$, as n grows, we will eventually have $y < n$, so the second case matters, and $F_n(y) \to \lim_{n\to\infty} 1 - \left(1 - \frac{y}{n}\right)^n = 1 - e^{-y}$. All in all, we got that

$$\lim_{n\to\infty} F_n(y) = F(y) := \begin{cases} 0 & \text{if } y \le 0 \\ 1 - e^{-y} & \text{if } y > 0 \end{cases}$$

for every $y \in \mathbb{R}$, so $F_n \Rightarrow F$. This F is exactly the distribution function of the exponential distribution with parameter 1, so we have shown that $Y_n \Rightarrow Exp(1)$.