Probability 1
CEU Budapest, fall semester 2016
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Homework sheet 2 – solutions

2.1 *Continuity of the measure*

(a) Prove the following:

Theorem 1 *(Continuity of the measure)*

i. If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $A_1, A_2, \dots$ is an increasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \subset A_{i+1}$ for all i), then $\mu(\cup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).

ii. If $(\Omega, \mathcal{F}, \mu)$ is a measure space, $A_1, A_2, \dots$ is a decreasing sequence of measurable sets (i.e. $A_i \in \mathcal{F}$ and $A_i \supset A_{i+1}$ for all i) and $\mu(A_1) < \infty$, then $\mu(\cap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$ (and both sides of the equation make sense).

(b) Show that in the second statement the condition $\mu(A_1) < \infty$ is needed, by constructing a counterexample for the statement when this condition does not hold.

2.2 *Usefulness of the linearity of the expectation.* A building has 10 floors, not including the ground floor. On the ground floor, 10 people get into the elevator, and every one of them chooses a destination at random, uniformly out of the 10 floors, independently of the others. Let X denote the number of floors *on which the elevator stops* – i.e. the number of floors that were chosen by at least one person. Calculate the expectation and the variance of X. *(hint: First notice that the distribution of X is hard to calculate. Find a way to calculate the expectation and the variance without that.)*

2.3 Calculate the characteristic function of

(a) The Bernoulli distribution $B(p)$ (see Homework sheet 1)

(b) The "pessimistic geometric distribution with parameter p" – that is, the distribution μ on $\{0, 1, 2 \dots\}$ with weights $\mu(\{k\}) = (1-p)p^k$ $(k = 0, 1, 2 \dots)$.

(c) The "optimistic geometric distribution with parameter p" – that is, the distribution ν on $\{1, 2, 3, \dots\}$ with weights $\nu(\{k\}) = (1-p)p^{k-1}$ $(k = 1, 2 \dots)$.

(d) **(homework)** The Poisson distribution with parameter λ – that is, the distribution η on $\{0, 1, 2 \dots\}$ with weights $\eta(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}$ $(k = 0, 1, 2 \dots)$.

Solution:

$$\psi_{Poi(\lambda)}(t) = \sum_{k=0}^{\infty} e^{itk} e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} e^{itk}\eta(\{k\}) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^{it})^k}{k!} = e^{-\lambda}e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}.$$

(e) **(homework)** The exponential distribution with parameter λ – that is, the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if not} \end{cases}.$$

Solution:

$$\phi_{Exp(\lambda)}(t) = \int_{\mathbb{R}} e^{itx} f_\lambda(x)\,\mathrm{dLeb}(x) = \int_0^{\infty} e^{itx}\lambda e^{-\lambda x}\,\mathrm{d}x = \lambda\left[\frac{e^{(it-\lambda)x}}{it-\lambda}\right]_0^{\infty} = \frac{\lambda}{\lambda - it}.$$

2.4 **(homework)** Calculate the characteristic function of the normal distribution $\mathcal{N}(m, \sigma^2)$. (Remember the definition from the old times: $\mathcal{N}(m, \sigma^2)$ is the distribution on $\mathbb{R}$ with density (w.r.t. Lebesgue measure)

$$f_{m,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

You can save yourself some paperwork if you only do the calculation for $\mathcal{N}(0, 1)$ and reduce the general case to this using the relation between different normal distributions. You can and should use the fact that

$$\int_{-\infty}^{\infty} f_{m,\sigma^2}(x)\,\mathrm{d}x = 1$$

for every m and σ.

Solution: First we reduce the general case to the case of the standard normal distribution using the fact (known from old times, easy to check from the formulas) that if $X \sim \mathcal{N}(0, 1)$ and $Y = m + \sigma X$, then $Y \sim \mathcal{N}(m, \sigma^2)$. As a result, the characteristic function for the normal distribution with expectation m and variance σ^2 is

$$\psi_{\mathcal{N}(m,\sigma^2)}(t) = \mathbb{E}(e^{itY}) = \mathbb{E}(e^{itm+it\sigma X}) = e^{itm}\mathbb{E}(e^{i(t\sigma)X}) = e^{itm}\psi_{\mathcal{N}(0,1)}(\sigma t), \tag{1}$$

where $\psi_{\mathcal{N}(0,1)}(t) := \mathbb{E}(e^{itX})$ is the characteristic function of the standard normal distribution.

Now we go on to calculate

$$\begin{aligned}\psi_{\mathcal{N}(0,1)}(t) &= \mathbb{E}(e^{itX}) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,\mathrm{d}x = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2-2itx}{2}}\,\mathrm{d}x = \\ &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-it)^2-(it)^2}{2}}\,\mathrm{d}x = e^{-\frac{t^2}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-it)^2}{2}}\,\mathrm{d}x.\end{aligned}$$

We use the substitution $y := x - it$ to get

$$\psi_{\mathcal{N}(0,1)}(t) = e^{-\frac{t^2}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,\mathrm{d}x = e^{-\frac{t^2}{2}}.$$

In the last step we used that the standard normal density function (just like every probability density function) integrates to 1. Writing this back to (1), we get the final result

$$\psi_{\mathcal{N}(m,\sigma^2)}(t) = e^{itm}e^{-\frac{(\sigma t)^2}{2}}.$$

Remark: The substitution $y = x - it$ is not completely trivial to make rigorous. In fact, with this substitution, while x runs over the real line, y will run over a line in the complex plane, namely the line γ of complex numbers with imaginary part $-it$, so leaving the boundaries as $-\infty$ and ∞ after the substitution is cheating. To make the argument precise, one has to show that the integral on γ is equal to the integral on the real line. This is a typical application of a standard, but strong tool of complex analysis, called the *residue theorem*. I will not go into that here, and I don't expect the students to do so either.

2.5 *Dominated convergence and continuous differentiability of the characteristic function.*
The Lebesgue dominated convergence theorem is the following

Theorem 2 (dominated convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \ldots$ measurable real valued functions on Ω which converge to the limit function pointwise, μ-almost* *everywehere. (That is,* $\lim_{n\to\infty} f_n(x) = f(x)$ *for every* $x \in \Omega$*, except possibly for a set of* x*-es with* μ*-measure zero.) Assume furthermore that the* f_n *admit a common* integrable dominating function*: there exists a* $g : \Omega \to \mathbb{R}$ *such that* $|f_n(x)| \le g(x)$ *for every* $x \in \Omega$ *and* $n \in \mathbb{N}$*, and* $\int_\Omega g \,\mathrm{d}\mu < \infty$*. Then (all the* f_n *and also* f *are integrable and)*

$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu.$$

Use this theorem to prove the following

Theorem 3 (differentiability of the characteristic function) *Let* X *be a real valued random variable,* $\psi(t) = \mathbb{E}(e^{itX})$ *its characteristic function and* $n \in \mathbb{N}$*. If the* n*-th moment of* X *exists and is finite (i.e.* $\mathbb{E}(|X|^n) < \infty$*), then* ψ *is* n *times continuously differentiable and*

$$\psi^{(k)}(0) = i^k \mathbb{E}(X^k), \quad k = 0, 1, 2, \ldots, n.$$

2.6 **(homework)** *(Weak convergence and densities)* Let $f_1, f_2, \ldots$ and f be probability densities on $\mathbb{R}$. Let $F_1, F_2, \ldots$ be the respective distribution functions, meaning that $F_n(x) = \int_{-\infty}^x f_n(y)\,\mathrm{d}y$ and $F(x) = \int_{-\infty}^x f(y)\,\mathrm{d}y$. Show that if $f_n(x) \to f(x)$ for every x as $n \to \infty$, then also $F_n(x) \to F(x)$ for every x as $n \to \infty$. *(Hint: use the Fatou lemma.)*

Solution: $F_n(x) = \int_{-\infty}^x f_n(x)\,\mathrm{d}x$ and $f_n(x) \to f(x)$ for every x, so the Fatou lemma says that

$$F(x) = \int_{-\infty}^x f(x)\,\mathrm{d}x = \int_{-\infty}^x \liminf_{n\to\infty} f_n(x)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_{-\infty}^x f_n(x)\,\mathrm{d}x = \liminf_{n\to\infty} F_n(x).$$

Similarly,

$$\begin{aligned} 1 - F(x) &= \int_x^\infty f(x)\,\mathrm{d}x = \int_x^\infty \liminf_{n\to\infty} f_n(x)\,\mathrm{d}x \\ &\le \liminf_{n\to\infty} \int_x^\infty f_n(x)\,\mathrm{d}x = \liminf_{n\to\infty} (1 - F_n(x)) = 1 - \limsup_{n\to\infty} F_n(x), \end{aligned}$$

which implies $\limsup_{n\to\infty} F_n(x) \le F(x)$, so $F_n(x) \to F(x)$ for every x, and we are done.