Probability 1
CEU Budapest, fall semester 2016
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Midterm exam, 25.10.2016
Working time: 120 minutes $\approx \infty$
Every question is worth 10 points. Maximum total score: 30.

1. Is there a sequence X_n of random variables on the same probability space such that

   a.) $X_n \to 0$ almost surely, and $\mathbb{E}X_n^2 \to \frac{1}{2}$?

   b.) $X_n \to 0$ almost surely, and $\mathbb{E}\sin(X_n) \to \frac{1}{2}$?

   If no, why not? If yes, give an example!

2. Let $X_1, X_2, \dots$ be independent, $X_n \sim B(p_n)$ with $p_n \in [0,1]$. Let $Y = \sum_{n=1}^{\infty} X_n$.

   a.) Show that if $\sum_{n=1}^{\infty} p_n < \infty$, then $Y < \infty$ almost surely.

   b.) Show that if $\sum_{n=1}^{\infty} p_n = \infty$, then $Y = \infty$ almost surely.

3. Let $X_1, X_2, \dots$ be random variables on the same probability space, $X_n \sim Exp(\lambda_n)$ with $\lambda_n > 0$. Show that if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, then $\sum_{n=1}^{\infty} X_n < \infty$ almost surely.

4. A kind of molecule is trying to decompose into atoms the following way: At each time $t \in \{\delta, 2\delta, 3\delta, \dots\}$ it tries to decompose, and it always succeeds with probabilty δ, which is very small – if it has not succeeded before. If it fails, it tries again next time, indepenedently of the past attempts. (We measure time in hours).

   Let T_δ denote the random time when it successfully decomposes.

   Find the weak limit of T_δ as $\delta \to 0$. (Find Means: describe in your favourite way, or write down its name.)