Probability 1
CEU Budapest, fall semester 2016
Imre Péter Tóth
Final exam, 06.12.2016 – solutions
Working time: 150 minutes
Every question is worth 10 points. Maximum total score: 40 points.

1. **(after correcting an error)** We toss a fair coin infinitely many times and set $X_i = 1$ if the ith toss is heads, and $X_i = 0$, if not. Now let

$$Y_i = \begin{cases} X_i X_{i+1}, & \text{if } i \text{ is odd} \\ X_{i-1} + X_i & \text{if } i \text{ is even} \end{cases}.$$

Let $S_n = Y_1 + \cdots + Y_n$. Find and prove the weak limit of $\frac{S_n - \frac{5}{8}n}{\sqrt{n}}$.

Solution: The Y_i are neither independent, nor identically distributed, so the central limit theorem can not be applied directly. However, if we add them up pairwise, defining

$$Z_1 := Y_1 + Y_2,\ Z_2 := Y_3 + Y_4,\ Z_3 := Y_5 + Y_6, \dots,$$

then $Z_1, Z_2, Z_3, \dots$ are mutually independent and identically distributed, so the central limit theorem applies. To understand the distribution of Z_i, maybe the easiest is to calculate al possible values explicitly:

X_1	X_2	Y_1	Y_2	Z_1
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	2	3

Since $X_i \sim B(\frac{1}{2})$ and they are independent, we have

k	0	1	3
$\mathbb{P}(Z_1 = k)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

This gives $\mathbb{E}Z_1 = \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 3 = \frac{5}{4}$, $\mathbb{E}Z_1^2 = \frac{1}{4}\cdot 0^2 + \frac{1}{2}\cdot 1^2 + \frac{1}{4}\cdot 3^2 = \frac{11}{4}$, $Var Z_1 = \frac{11}{4} - \left(\frac{5}{4}\right)^2 = \frac{19}{16}$.

If we define

$$\tilde{S}_k := Z_1 + Z_2 + \cdots + Z_k,$$

then we can describe $S_n = X_1 + \cdots + X_n$ at least when n is even: $S_{2k} = \tilde{S}_k$. Now the central limit theorem says that

$$\frac{\tilde{S}_k - \frac{5}{4}k}{\sqrt{k}} \Rightarrow \mathcal{N}\left(0, \frac{19}{16}\right),$$

so for $n = 2k$ we get

$$\frac{S_n - \frac{5}{8}n}{\sqrt{n}} = \frac{S_{2k} - \frac{5}{8}\cdot 2k}{\sqrt{2k}} = \frac{\tilde{S}_k - \frac{5}{4}k}{\sqrt{2}\sqrt{k}} \Rightarrow \frac{1}{\sqrt{2}}\mathcal{N}\left(0, \frac{19}{16}\right) = \mathcal{N}\left(0, \frac{19}{32}\right).$$

Now if $m = n + 1 = 2k + 1$ is odd, then $S_m - S_n = X_m$ always has the same distribution, so it cannot spoil the weak convergence:

$$\begin{aligned} \frac{S_m - \frac{5}{8}m}{\sqrt{m}} &= \frac{S_{n+1} - \frac{5}{8}(n+1)}{\sqrt{n+1}} = \frac{\sqrt{n}}{\sqrt{n+1}}\left(\frac{S_n - \frac{5}{8}n}{\sqrt{n}} + \frac{X_m - \frac{5}{8}}{\sqrt{n}}\right) \Rightarrow \\ &\Rightarrow\ 1\cdot\left(\mathcal{N}\left(0, \frac{19}{32}\right) + 0\right) = \mathcal{N}\left(0, \frac{19}{32}\right). \end{aligned}$$

2. Let $X_n \sim Bin(n, \frac{1}{n})$. Show that X_n is weakly convergent and describe the limit.

Solution: This is a special case of Homework 4.1. The solution copied from there: Let $p_n = \frac{1}{n}$ and set $q_n = 1 - p_n$, so X_n has characteristic function

$$\psi_{X_n}(t) = \left(q_n + p_n e^{it}\right)^n = \left[\left(1 + \frac{e^{it}-1}{1/p_n}\right)^{1/p_n}\right]^{np_n}.$$

The base of the power converges to $\exp(e^{it}-1)$ as $p_n \to 0$ by standard elementary calculus, while the exponent is to $np_n = 1 =: \lambda$, so

$$\psi_{X_n}(t) \to e^{\lambda(e^{it}-1)},$$

which is exactly the characteristic function of the $Poi(\lambda)$ distribution with $\lambda = 1$, so the continuity theorem ensures that X_n converges to $Poi(\lambda)$ weakly.

Alternative solution: For every fixed $k \in \{0, 1, 2, \dots\}$

$$\begin{aligned}
\mathbb{P}(X_n = k) &= \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{n-k} = \\
&= \frac{n(n-1)\cdots(n+k+1)}{k!}\frac{1}{n^k}\left(1-\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^{-k} = \\
&= \frac{1}{k!}\frac{n(n-1)\cdots(n+k+1)}{n^k}\left(1-\frac{1}{n}\right)^{-k}\left(1-\frac{1}{n}\right)^n \to \\
&\to \frac{1}{k!}\cdot 1\cdot 1\cdot e^{-1} = e^{-1}\frac{1^k}{k!} = \mathbb{P}(Poi(1) = k).
\end{aligned}$$

By Exercise 3.2.11 from Durrett [1], this means that $X_n \sim Poi(1)$.

3. A frog performs a discrete time "lazy" and "sticky" symmetric random walk on the set $\{-10, -9, \dots, 9, 10\}$, stating from 0, with time-dependent jump probabilities: At time 0 the frog is in 0. If it reaches -10 or 10, then it stays there forever. If it has not reached -10 or 10, then in the ith time step it jumps one step down with probability $\frac{p_i}{2}$, it jumps one step up with probability $\frac{p_i}{2}$, and stays where it was with the remaining probability $q_i = 1 - p_i$, independently of what happened before.

   Is it possible to choose the sequence p_i so that the frog performs infinitely many jumps?

   And what if p_i can depend on the entire past of the frog's trajectory?

   Solution: *No, It is not possible.* No matter what strategy the frog follows, its position is a bounded martingale, so it has to be convergent almost surely by the matringale convergence theorem. Since the values are discrete, this means that is has to be almost surely eventually constant, meaning finitely many jumps.

4. Define the notion of conditional expectation with respect to a σ-algebra for integrable random variables, and show that it always exists.

   Solution:

   Definition: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X : \Omega \to \mathbb{R}$ be a random variable with $\mathbb{E}|X| < \infty$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. The random variable $Y : \Omega \to \mathbb{R}$ is said to be the conditional expectation of X with respect to $\mathcal{G}$ if

   a.) Y is $\mathcal{G}$-measurable,

   b.) $\int_A Y \,\mathrm{d}\mathbb{P} = \int_A X \,\mathrm{d}\mathbb{P}$ for all $A \in \mathcal{G}$.

Notation: $Y = \mathbb{E}(X|\mathcal{G})$.

Proof of existence: First for $X \geq 0$. Consider the measurable space $(\Omega, \mathcal{G})$, and on this space consider the set functions $\mu, \nu : \mathcal{G} \to \mathbb{R}^+$ defined for $A \in \mathcal{G}$ as

$$\begin{aligned} \mu(A) &:= \mathbb{P}(A) = \int_A 1 \, \mathrm{d}\mathbb{P} \\ \nu(A) &:= \int_A X \, \mathrm{d}\mathbb{P}. \end{aligned}$$

It is easy to check thet these are finite measures and ν is absolutely continuous with respect to μ. So the Radon-Nikodym theorem says that there exists a Radon-Nikodym derivative $f : \Omega \to \mathbb{R}$, which, by definiton, has the properties that

a.) f is $\mathcal{G}$-measurable,

b.) $\nu(A) = \int_A f \, \mathrm{d}\mu$ for all $A \in \mathcal{G}$.

Comparing these with the definition of the conditional expectation, we see that $Y := f$ will do.

In the general case when X may be negative, we write $X = X^+ - X^-$ and apply the previous construction for X^+ and X^- separately, to get some Y^+ and Y^+. Then $Y := Y^+ - Y^-$ will do.

5. *Coupon collector problem.* Bob keeps drawing cards from a pile of n different cards, *with replacement*, meaning that every card drawn is chosen uniformly and independently of the others. Let Y_k^n be the number of draws he needs in order to see at least k different cards, and let $U_n = Y_n^n$ be the number of draws until all cards are seen.

   (a) What is the distribution of $(Y_{k+1}^n - Y_k^n)$, that is, the number of draws he needs to find yet another new card if he has already seen k?

   (b) Calculate the expectation and variance of U_n.

   (c) Find the limit distribution of $\frac{U_n}{n \log n}$.

   Solution:

   (a) $\tau_k^n := Y_{k+1}^n - Y_k^n \sim Geom\left(\frac{n-k}{n}\right)$ for $k = 0, 1, \ldots, n-1$, because $n-k$ out of n cards are unseen, so the success probability is $p_k := \frac{n-k}{n}$.

   (b) $U_n = \sum_{k=0}^{n-1} \tau_k^n$ and the τ_k^n are independent, so we just need to add the expectations and variances. These moments of the geometric distribution can be calculated in many ways (e.g. using the definition, or the characteristic function, or the generating function), and the result is

$$\mathbb{E}(Geom(p)) = \frac{1}{p} \quad , \quad Var(Geom(p)) = \frac{1-p}{p^2}.$$

   So

$$\begin{aligned} \mathbb{E}U_n &= \sum_{k=0}^{n-1} \mathbb{E}\tau_k^n = \sum_{k=0}^{n-1} \frac{n}{n-k} \overset{j=n-k}{=} n \sum_{j=1}^{n} \frac{1}{j} = n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\ VarU_n &= \sum_{k=0}^{n-1} Var\tau_k^n = \sum_{k=0}^{n-1} \frac{nk}{(n-k)^2} \overset{j=n-k}{=} n \sum_{j=1}^{n} \frac{n-j}{j^2} = \\ &= n\left(\frac{n-1}{1^2} + \frac{n-2}{2^2} + \frac{n-3}{3^2} + \cdots + \frac{1}{(n-1)^2} + \frac{0}{n^2}\right) \end{aligned}$$

(c) If we approximate the sum in $\mathbb{E}U_n$ with an integral, we get

$$\lim_{n\to\infty} \frac{\mathbb{E}U_n}{n\log n} = 1. \tag{1}$$

On the other hand

$$VarU_n = n\sum_{j=1}^{n} \frac{n-j}{j^2} \le n^2 \sum_{j=1}^{\infty} \frac{1}{j^2} = Cn^2,$$

where $C := \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$. (Actually $\lim_{n\to\infty} \frac{VarU_n}{n^2} = C = \frac{\pi^2}{6}$.) This means that

$$\sqrt{VarU_n} \ll \mathbb{E}U_n,$$

so

$$\frac{U_n}{\mathbb{E}U_n} \Rightarrow 1$$

because $\mathbb{E}\frac{U_n}{\mathbb{E}U_n} = 1$ and $Var\frac{U_n}{\mathbb{E}U_n} \to 0$. Together with (1) this gives

$$\frac{U_n}{n\log n} \Rightarrow 1.$$

References

[1] Durrett, R. *Probability: Theory and Examples.* **4th** edition, Cambridge University Press (2010)