Probability 1
CEU Budapest, fall semester 2016
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Final exam, 06.12.2016
Working time: 150 minutes
Every question is worth 10 points. Maximum total score: 40 points.

1. We toss a fair coin infinitely many times and set $X_i = 1$ if the ith toss is heads, and $X_i = 0$, if not. Now let
$$Y_i = \begin{cases} X_i X_{i+1}, & \text{if } i \text{ is odd} \\ X_{i-1} + X_i & \text{if } i \text{ is even} \end{cases}.$$
Let $S_n = Y_1 + \dots + Y_n$. Find and prove the weak limit of $\frac{S_n - \frac{5}{8}n}{\sqrt{n}}$.

2. Let $X_n \sim Bin(n, \frac{1}{n})$. Show that X_n is weakly convergent and describe the limit.

3. A frog performs a discrete time "lazy" and "sticky" symmetric random walk on the set $\{-10, -9, \dots, 9, 10\}$, stating from 0, with time-dependent jump probabilities: At time 0 the frog is in 0. If it reaches -10 or 10, then it stays there forever. If it has not reached -10 or 10, then in the ith time step it jumps one step down with probability $\frac{p_i}{2}$, it jumps one step up with probability $\frac{p_i}{2}$, and stays where it was with the remaining probability $q_i = 1 - p_i$, independently of what happened before.

   Is it possible to choose the sequence p_i so that the frog performs infinitely many jumps?

   And what if p_i can depend on the entire past of the frog's trajectory?

4. Define the notion of conditional expectation with respect to a σ-algebra for integrable random variables, and show that it always exists.

5. *Coupon collector problem.* Bob keeps drawing cards from a pile of n different cards, *with replacement*, meaning that every card drawn is chosen uniformly and independently of the others. Let Y_k^n be the number of draws he needs in order to see at least k different cards, and let $U_n = Y_n^n$ be the number of draws until all cards are seen.

   (a) What is the distribution of $(Y_{k+1}^n - Y_k^n)$, that is, the number of draws he needs to find yet another new card if he has already seen k?

   (b) Calculate the expectation and variance of U_n.

   (c) Find the limit distribution of $\frac{U_n}{n \log n}$.