Probability 1
CEU Budapest, fall semester 2016
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Homework sheet 7 – due on 28.11.2013 – and exercises for practice

7.1 **(homework)** Let $\mathcal{F}_n$ be a filtration and X any random varibale with $\mathbb{E}|X| < \infty$. Let $X_n = \mathbb{E}(X|\mathcal{F}_n)$.

a.) Show that X_n is a martingale w.r.t. $\mathcal{F}_n$.

b.) Show that X_n converges almost surely to some limit X_∞.

c.) Give a specific example when $X_\infty \neq X$.

d.) Give a specific example when $X_\infty = X$.

7.2 **(homework)** Let X_n be a martingale w.r.t. the filtration $\mathcal{F}_n$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let the random variable $\tau : \Omega \to \mathbb{N}$ be a *stopping time*, meaning

$$\{\tau = k\} := \{\omega \in \Omega \,|\, \tau(\omega) = k\} \in \mathcal{F}_k \quad \text{for every } k.$$

Using the notation $a \wedge b := \min\{a, b\}$, we introduce the process

$$Y_n := X_{\tau\wedge n} = \begin{cases} X_n & \text{if } n < \tau, \\ X_\tau & \text{if } n \geq \tau. \end{cases}$$

Show that Y_n is also a martingale w.r.t. $\mathcal{F}_n$.

7.3 Durrett [1], Exercise 5.3.12 *(Hint: Use $Z_{n+1} = \sum_{k=1}^{Z_n} \xi_k^{n+1}$ (with the notation of Durrett) or $Z_{n+1} = \sum_{k=1}^{Z_n} X_{n,k}$ (with the notation of the lecture) to show that $r_* := \mathbb{P}(\lim Z_n/\mu^n > 0)$ is a fixed point of the generating function (which is φ in the book and was g on the lecture).)*

7.4 Durrett [1], Exercise 5.3.13

7.5 **(homework)** Harry is organizing a *pyramid scheme* in his family.
(See http://en.wikipedia.org/wiki/Pyramid_scheme) The participants are not too persistent: every participant keeps trying to recruit new participants until the first failure (i.e. until he is first rejected). The probability of such a failure is p at every recruit attempt, independently of the history of the scheme.

The first participant is Harry, he forms the 0-th generation alone. The first generation consists of those recruited (directly) by Harry. The second generation consists of those recruited (directly) by members of the first generation, and so on.

Let Z_k denote the size of the k-th generation ($k = 0, 1, 2, \dots$), and let N denote the total number of participants in the scheme (meaning $N = \sum_{k=0}^{\infty} Z_k$).

0-th question: What is the distribution of Z_1 (which is the same as the distribution of the number of participants recruited by any fixed member of the scheme)? This distribution has a name.

Answer the questions below

I. for $p = \frac{2}{3}$,
II. for $p = \frac{1}{2}$,
III. for $p = \frac{1}{3}$:

a.) What is the probability that the scheme dies out (that is, one of the generations will already be empty)?

b.) What is the expectation of N?

c.) In case “not dying out” has positive probability, what is the growth rate of Z_n on this event?

References

[1] Durrett, R. *Probability: Theory and Examples*. **4th** edition, Cambridge University Press (2010)