**Probability 1**
**CEU Budapest, fall semester 2016**
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**Homework sheet 1 – due on 03.10.2016**

1. Define a $\sigma$-algebra as follows:

   **Definition 1** *For a nonempty set $\Omega$, a family $\mathcal{F}$ of subsets of $\omega$ (i.e. $\mathcal{F} \subset 2^\Omega$, where $2^\Omega := \{A : A \subset \Omega\}$ is the power set of $\Omega$) is called a $\sigma$-algebra over $\Omega$ if*

   - $\emptyset \in \mathcal{F}$
   - *if $A \in \mathcal{F}$, then $A^C := \Omega \setminus A \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under complement taking)*
   - *if $A_1, A_2, \dots \in \mathcal{F}$, then $(\cup_{i=1}^\infty A_i) \in \mathcal{F}$ (that is, $\mathcal{F}$ is closed under countable union).*

   Show from this definition that a $\sigma$-algebra is closed under countable intersection, and under finite union and intersection.

2. (a) We toss a biased coin, on which the probability of heads is some $0 \le p \le 1$. Define the random variable $\xi$ as the indicator function of tossing heads, that is

   $$\xi := \begin{cases} 0, & \text{if tails} \\ 1, & \text{if heads} \end{cases} .$$

      i. Describe the distribution of $\xi$ (called the Bernoulli distribution with parameter $p$) in the "classical" way, listing possible values and their probabilities,
      ii. and also by describing the distirbution as a measure on $\mathbb{R}$, giving the weight $\mathbb{P}(\xi \in B)$ of every Borel subset $B$ of $\mathbb{R}$.
      iii. Calculate the expectation of $\xi$.

   (b) We toss the previous biased coin $n$ times, and denote by $X$ the *number of heads* tossed.

      i. Describe the distribution of $X$ (called the Binomial distribution with parameters $(n, p)$) by listing possible values and their probabilities.
      ii. Calculate the expectation of $X$ by integration (actually summation in this case) using its distribution,
      iii. and also by noticing that $X = \xi_1 + \xi_2 + \dots + \xi_n$, where $\xi_i$ is the indicator of the $i$-th toss being heads, and using linearity of the expectation.

3. *The Fatou lemma* is the following

   **Theorem 1** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1$, $f_2$, ... a sequence of measureabale functions $f_n : \Omega \to \mathbb{R}$, which are nonneagtive, e.g. $f_n(x) \ge 0$ for every $n = 1, 2, \dots$ and every $x \in \Omega$. Then*

   $$\int_\Omega \liminf_{n\to\infty} f_n(x)\,\mathrm{d}\mu(x) \le \liminf_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}\mu(x)$$

   *(and both sides make sense).*

   Show that the inequality in the opposite direction is in general false, by choosing $\Omega = \mathbb{R}$, $\mu$ as the Lebesgue measure on $\mathbb{R}$, and constructing a sequence of nonnegative $f_n : \mathbb{R} \to \mathbb{R}$ for which $f_n(x) \xrightarrow{n\to\infty} 0$ for every $x \in \mathbb{R}$, but $\int_\mathbb{R} f_n(x)\,\mathrm{d}x \ge 1$ for all $n$.

4. The *ternary* number $0.a_1a_2a_3\dots$ is the analogue of the usual decimal fraction, but writing numbers in base 3. That is, for any sequence $a_1, a_2, a_3, \dots$ with $a_n \in \{0,1,2\}$, by definition

$$0.a_1a_2a_3\cdots := \sum_{n=1}^{\infty} \frac{a_n}{3^n}.$$

Now let us construct the ternary fraction form of a random real number $X$ via a sequence of fair coin tosses, such that we rule out the digit 1. That is,

$$a_n := \begin{cases} 0, & \text{if the } n\text{-th toss is tails,} \\ 2, & \text{if the } n\text{-th toss is heads} \end{cases},$$

and setting $X = 0.a_1a_2a_3\dots$ (ternary). In this way, $X$ is a "uniformly" chosen random point of the famous *middle-third Cantor set* $C$ defined as

$$C := \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n},\ a_n \in \{0,2\}\ (n = 1, 2, \dots) \right\}.$$

Show that

(a) The distribution of $X$ gives zero weight to every point – that is, $\mathbb{P}(X = x) = 0$ for every $x \in \mathbb{R}$. (As a consequence, the cumulative distribution function of $X$ is continuous.)

(b) The distribution of $X$ is not absolutely continuous w.r.t the Lebesgue measure on $\mathbb{R}$.