Anal jfz Imérnököknek 2. zh. 2019. dec. 6. Aesop Mo

(1.)

$$f(x) = \log_2 \sqrt{x^2-3x+4} + e^{(x-3)^2} = \frac{1}{2} \log_2 (x^2-3x+4) + e^{(x-3)^2}$$

$$f'(x) = \frac{1}{2} \frac{2x-3}{x^2-3x+4} \frac{1}{\ln 2} + e^{(x-3)^2} \cdot 2(x-3)$$

$$f(3) = \frac{1}{2} \log_2 (4) + e^0 = \frac{1}{2} \cdot 2 + 1 = \underline{\underline{2}}$$

$$f'(3) = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{\ln 2} + 0 = \frac{3}{8} \frac{1}{\ln 2}$$

Érintő egyenes!

$$y = f(x_0) + f'(x_0)(x-x_0)$$

$$\underline{\underline{y = 2 + \frac{3}{8}\frac{1}{\ln 2}(x-3)}}$$

(2.)

$$f(x) = \begin{cases} x \cos\sqrt{x} & , x > 0 \\ Ax+B & , x \le 0 \end{cases}$$

$f$ $x>0$ ill $x<0$ tart-on diffható kell $x=0$-ban is diffható legyen.

Diffhatósághoz szükséges a folytonosság.

$\underline{\underline{f \text{ folyt } 0\text{-ban}}} \iff \lim_{x\to 0-} f(x) = f(0) = \lim_{x \to 0+} f(x)$.

$$\lim_{x\to 0-} Ax+B = 0 = \lim_{x\to 0+} x\cos\sqrt{x}$$

$$B = 0 = 0 \qquad \underline{\underline{B=0}}$$

$f$ diffható 0-ban $\iff f'_+(0) = f'_-(0)$

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = (Ax+0)'\Big|_{x=0}$$

$$\lim_{x\to 0+} \frac{x\cos\sqrt{x} - 0}{x} = A$$

$$\lim_{x\to 0+} \cos\sqrt{x} = A$$

$$\underline{\underline{1 = A}}$$

$f$ diffható 0-ban és $\forall$hol $\iff$ $\underline{\underline{A=1, B=0}}$.

3.

$f(x)=xe^{\frac{1}{x}}$ $\quad D_f=\mathbb{R}\setminus\{0\}$ (nem ps, ptlan, per)

$f(x)=0 \iff x=0$ lenne, de $0\notin D_f$

$\Rightarrow f(x)\neq 0 \quad \forall x\in D_f$

| $x$ | | 0 | |
|---|---|---|---|
| $f$ | $-$ | $\nexists$ | $+$ |

$f'(x)=e^{\frac{1}{x}}+xe^{\frac{1}{x}}\cdot\frac{-1}{x^2}=e^{\frac{1}{x}}\left(1-\frac{1}{x}\right)$

$f'(x)=0 \iff x=1$

| $x$ | | 0 | | 1 | |
|---|---|---|---|---|---|
| $f'$ | $+$ | $\nexists$ | $-$ | 0 | $+$ |
| $f$ | $\nearrow$ | $\nexists$ | $\searrow$ | $(1,e)$ min | $\nearrow$ |

$f''(x)=e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\left(1-\frac{1}{x}\right)+e^{\frac{1}{x}}\frac{1}{x^2}$

$f''(x)=e^{\frac{1}{x}}\frac{1}{x^3}\neq 0$

| | | 0 | |
|---|---|---|---|
| $f''$ | $\ominus$ | $\nexists$ | $\oplus$ |
| $f$ | $\cap$ | $\nexists$ | $\cup$ |

$\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty} xe^{\frac{1}{x}}=\underline{\underline{\infty}}$ $\quad \infty\cdot 1$

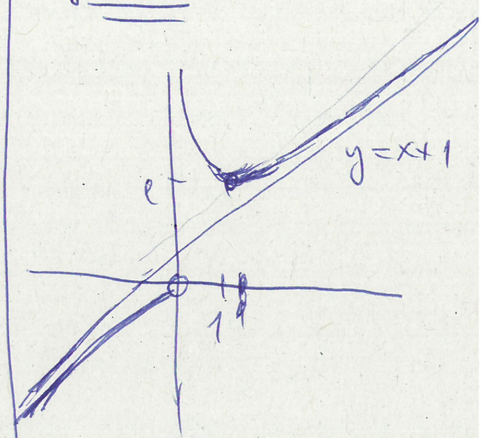
$\lim\limits_{x\to 0+} f(x)=\lim\limits_{x\to 0+} xe^{\frac{1}{x}}=(0+)\cdot\infty$ problémás

$=\lim\limits_{y\to\infty}\frac{e^y}{y}\overset{l'H}{=}\lim\limits_{y\to\infty}\frac{e^y}{1}=\underline{\underline{\infty}}$ $\quad \frac{1}{x}=y \quad \boxed{\frac{\infty}{\infty}}$

$\lim\limits_{x\to 0-} f(x)=\lim\limits_{x\to 0-} xe^{\frac{1}{x}}=0\cdot e^{-\infty}=0\cdot 0=\underline{\underline{0}}$

$\lim\limits_{x\to -\infty} f(x)=\lim\limits_{x\to -\infty} xe^{\frac{1}{x}}=-\infty\cdot e^0=\underline{\underline{-\infty}}$

$f(1)=e$




Ferde aszimptota $y=ax+b$

$\lim\limits_{x\to\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty} e^{\frac{1}{x}}=\underline{\underline{1}}=a$

$\lim\limits_{x\to\infty}(f(x)-ax)=\lim\limits_{x\to\infty} x\left(e^{\frac{1}{x}}-1\right)=\lim\limits_{x\to\infty}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}=\lim\limits_{y\to 0}\frac{e^y-1}{y}=\underline{\underline{1}}=b$ $\quad$ volt EA-on, v. l'H

$y=x+1$ ferde asz. $+\infty$-ben.

$-\infty$-ben is (hasonlóan)

4.

$\ln 8 - \ln 7 = \ln\frac{8}{7}$

$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

$f(x) = \ln x \qquad x_0 = 1 \qquad f(1) = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \qquad \frac{f''(1)}{2!} = -\frac{1}{2}$

$f'''(x) = \frac{2}{x^3} \qquad \frac{f'''(1)}{3!} = \frac{1}{3}$

$f^{IV}(x) = \frac{-6}{x^4} \qquad \frac{f^{IV}(1)}{4!} = \frac{-1}{4}.$

$\vdots$

$T_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}(x-1)^k$ (tanultuk is)

avagy

$T_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \mp \dots$

$f(x) - T_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-1)^{n+1}$ ahol $t$ $x_0$ és $x$ közötti szám

Így $\left|f\left(\frac{8}{7}\right) - T_n\left(\frac{8}{7}\right)\right| = \left|\frac{f^{(n+1)}(t)}{(n+1)!}\left(\frac{1}{7}\right)^{n+1}\right| \le \frac{|f^{(n+1)}(1)|}{(n+1)!}\cdot\left(\frac{1}{7}\right)^{n+1} = \frac{1}{n+1}\left(\frac{1}{7}\right)^{n+1}$

$\underset{x_0}{\overset{\shortparallel}{1}} \le t \le \underset{x}{\overset{\shortparallel}{\frac{8}{7}}}$ $\qquad \underbrace{|f^{(n+1)}(t)|}_{\text{mon fogyó fv-ek}}$ max hogy $t = 1$-nél

kell $\frac{\left(\frac{1}{7}\right)^{n+1}}{n+1} \le 10^{-3}$

$n = 2$ jó, hiszen

$\frac{\left(\frac{1}{7}\right)^3}{3} = \frac{1}{343\cdot 3} < 10^{-3}$

~~T...~~ $T_2(x) = (x-1) - \frac{1}{2}(x-1)^2$

$\ln\frac{8}{7} \approx T_2\left(\frac{8}{7}\right) = \underline{\underline{\frac{1}{7} - \frac{1}{2}\cdot\frac{1}{49}}}$

5.

a) $$\int_0^{\pi} x\sin^2 x\,dx = \int_0^{\pi} x\,\frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\left(\left[x\left(x-\frac{\sin 2x}{2}\right)\right]_0^{\pi} - \int_0^{\pi} x-\frac{\sin 2x}{2}\,dx\right)$$

$$= \frac{1}{2}\left(\pi\left(\pi - \underbrace{\frac{\sin 2\pi}{2}}_{0}\right) - 0 - \left[\frac{x^2}{2}+\frac{\cos 2x}{4}\right]_0^{\pi}\right)$$

$$= \frac{1}{2}\left(\pi^2 - \left(\frac{\pi^2}{2} + \underbrace{\frac{\cos 2\pi}{4}}_{\frac14} - \left(\frac{0^2}{2} + \underbrace{\frac{\cos 0}{4}}_{\frac14}\right)\right)\right) = \frac{1}{2}\,\frac{\pi^2}{2} = \frac{\pi^2}{4}$$

avagy $\frac{1}{2}\int_0^{\pi} x(1-\cos 2x)\,dx$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{2}\left(\left[x\,\frac{-\sin 2x}{2}\right]_0^{\pi} - \int_0^{\pi} \frac{-\sin 2x}{2}\,dx\right) =$$

$$= \frac{\pi^2}{4} + \frac{1}{2}\left(0 - 0 - \underbrace{\left[\frac{\cos 2x}{4}\right]_0^{\pi}}_{0}\right) = \frac{\pi^2}{4}$$

b) 1.Mo $$\int_0^{1/2} \sqrt{\frac{\arcsin x}{1-x^2}}\,dx = \int_0^{1/2} \sqrt{\arcsin x}\;\underbrace{\frac{1}{\sqrt{1-x^2}}}_{(\arcsin x)'}\,dx = \left[\frac{(\arcsin x)^{3/2}}{3/2}\right]_0^{1/2} = \frac{2}{3}\left(\left(\arcsin\frac{1}{2}\right)^{\frac32} - \left(\arcsin 0\right)^{\frac32}\right)$$

$$= \frac{2}{3}\left(\left(\frac{\pi}{6}\right)^{\frac32} - 0^{\frac32}\right) = \frac{2}{3}\left(\frac{\pi}{6}\right)^{3/2}$$

2.Mo Uez helyettesítéssel!

$\arcsin x = t$ alapján

$$\frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx}$$

$$\int_{t=\arcsin 0}^{t=\arcsin\frac12} \sqrt{t}\,\frac{dt}{dx}\,dx = \int_0^{\pi/6} \sqrt{t}\,dt = \left[\frac{t^{3/2}}{3/2}\right]_0^{\pi/6} = \frac{2}{3}\left(\frac{\pi}{6}\right)^{3/2}$$

c) 1.Mo $$\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx = \int \frac{(t-1)^2}{\sqrt{t}}\,\underbrace{\frac{1}{t-1}}_{\frac{dx}{dt}}\,dt =$$

$1+e^x = t$

$e^x = t-1$

$x = \ln(t-1)$

$\frac{dx}{dt} = \frac{1}{t-1}$

$$= \int \frac{t-1}{\sqrt{t}}\,dt = \int \sqrt{t} - \frac{1}{\sqrt{t}}\,dt$$

$$= \frac{t^{3/2}}{3/2} - 2\sqrt{t} + C$$

$$= \frac{2}{3}\left(1+e^x\right)^{3/2} - 2\sqrt{1+e^x} + C$$

2.Mo $$\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx = \int \frac{t^2}{\sqrt{1+t}}\,\frac{1}{t}\,dt = \int \frac{t}{\sqrt{1+t}}\,dt = \int \frac{t+1}{\sqrt{1+t}} - \int \frac{1}{\sqrt{1+t}}\,dt$$

$t = e^x$

$\ln t = x$

$\frac{1}{t} = \frac{dx}{dt}$

$$= \int (t+1)^{1/2} - (t+1)^{-1/2}\,dt = \frac{(t+1)^{3/2}}{3/2} - 2(t+1)^{1/2} + C$$

$$= \frac{2}{3}\left(e^x+1\right)^{3/2} - 2\sqrt{e^x+1} + C$$

d) $$\int \frac{1}{x(x^2-1)} = \int \frac{1}{x(x+1)(x-1)} = \int \frac{a}{x} + \frac{b}{x+1} + \frac{c}{x-1} =$$

$$1 = a(x^2-1) + b(x^2-x) + c(x^2+x)$$

| | |
|---|---|
| $x^2$ | $0 = a+b+c$ |
| $x$ | $0 = -b+c \Rightarrow c=b$ |
| $1$ | $1 = -a$ |

$0 = -1+2b$ $\quad$ $b=c=\frac{1}{2}$

$a = -1$

$$= -1\ln|x| + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \ln\frac{\sqrt{x^2-1}}{|x|} + C$$

Anal fz/mérnököknek 2. zh. Mo Bcsop 2019. dec. 6.

(1.) $f(x) = \operatorname{arctg}(x^2-4) + 3^{(x-2)^2} \sin\frac{\pi}{x}$ , $x_0 = 2$

$$f'(x) = \frac{1}{1+(x^2-4)^2}\, 2x + 3^{(x-2)^2} \cdot \ln 3 \cdot 2(x-2) \sin\frac{\pi}{x} + 3^{(x-2)^2} \cos\frac{\pi}{x} \cdot \left(\frac{-\pi}{x^2}\right)$$

$$f(2) = \operatorname{arctg} 0 + 3^0 \cdot \sin\frac{\pi}{2} = 0 + 1 = \underline{\underline{1}}$$

$$f'(2) = 4 + \underbrace{3^0 \cdot \ln 3 \cdot 2 \cdot 0 \cdot 1}_{0} + 3^0 \underbrace{\cos\frac{\pi}{2}}_{0} \cdot \frac{-\pi}{4} = \underline{\underline{4}}$$

érintőegyenes $y = f(2) + f'(2)(x-2)$

$$\underline{\underline{y = 1 + 4(x-2)}}$$

(2.) $f(x) = \begin{cases} (\sin\sqrt{x})^2 & , x \geq 0 \\ Cx + D & , x < 0 \end{cases}$ diffható az $x>0$ és az $x<0$ tart-on. kell $x=0$-ban is diffható legyen.

Diffhatósághoz kell, h. folyt is legyen.

$f$ folyt 0-ban $\iff$ $\lim\limits_{x\to 0-} f(x) = f(0) = \lim\limits_{x\to 0+} f(x)$

$$\lim_{x\to 0-} (Cx + D) = 0 = \lim_{x\to 0+} (\sin\sqrt{x})^2$$

$$\underline{\underline{D = 0 = 0}}$$

$\underline{\underline{D=0}}$ kell folyt-hoz.

$f$ diffható 0-ban $\iff$ $f'_+(0) = f'_-(0)$

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{(\sin\sqrt{x})^2 - 0}{x} = (Cx+D)'\Big|_{x=0}$$

$$\lim_{x\to 0+} \left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right)^2 = C$$

$$1^2 = C$$

$f$ diffható $x=0$-ban is $\forall$hol $\iff$ $1 = C$

$$\underline{\underline{C = 1 \text{ és } D = 0}}$$

(3.) $f(x) = x^2 \ln x \qquad D_f = (0,+\infty)$ ln x miatt.

$f(x) = 0 \iff x = 0$ lenne, de $0 \notin D_f$

v.

$\ln x = 0 \iff \underline{\underline{x = 1}}$

| x | $(0,1)$ | 1 | $(1,+\infty)$ |
|---|---|---|---|
| f | $-$ | 0 | $+$ |

$$f'(x) = 2x \ln x + \frac{x^2}{x} = x(2\ln x + 1) = 0 \underset{x \neq 0}{\iff} \ln x = -\frac{1}{2} \iff x = e^{-1/2} = \frac{1}{\sqrt{e}}$$

$$f''(x) = 2\ln x + 1 + x \cdot \frac{2}{x} = 2\ln x + 3 = 0$$

$$\Updownarrow$$

$$\ln x = -\frac{3}{2}$$

$$x = e^{-\frac{3}{2}}$$

| x | $(0,1)$ | $\frac{1}{\sqrt{e}}$ | $\left(\frac{1}{\sqrt{e}}, \infty\right)$ |
|---|---|---|---|
| f' | $-$ | 0 | $+$ |
| f | ↘ | lok min | ↗ |

$$f(e^{-1/2}) = \frac{1}{e} \cdot \frac{-1}{2} = \underline{\underline{\frac{-1}{2e}}}$$

$$f(e^{-3/2}) = e^{-3} \cdot \frac{-3}{2} = \underline{\underline{\frac{-3}{2e^3}}}$$

| x | $(0,$ | $e^{-\frac{3}{2}}$ | $(\ ,\infty)$ |
|---|---|---|---|
| f'' | $-$ | 0 | $+$ |
| f | $\cap$ | infl pont | $\cup$ |

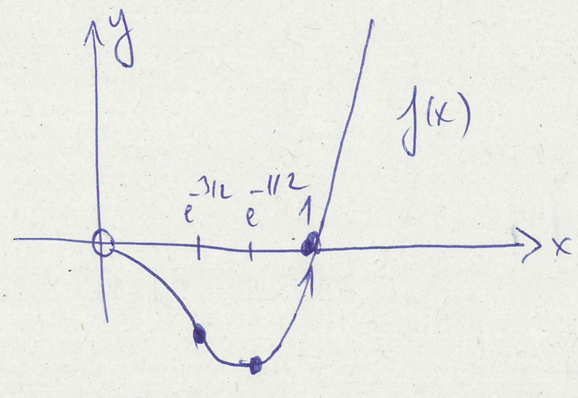
$$\lim_{x \to \infty} f(x) = \infty$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x^2 \ln x \quad \underbrace{0 \cdot -\infty}_{\text{problémás}}$$

$$= \lim_{x \to 0+} \frac{\ln x}{\left(\frac{1}{x}\right)^2} \overset{L'H}{=} \lim_{x \to 0+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x \to 0+} \frac{-x^2}{2} = \underline{\underline{0}}$$

$$\boxed{\frac{-\infty}{\infty}}$$

$\lim_{x \to \infty} \frac{f(x)}{x} = +\infty$, így ferde aszimptota nincsen.

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(4.)

$x_0 = 0$

$x = 0,1$

$f(x) = \operatorname{sh} x \qquad f(0) = 0$

$f'(x) = \operatorname{ch} x \qquad f'(0) = 1$

$f''(x) = \operatorname{sh} x \qquad \frac{f''(0)}{2} = 0$

$\vdots \qquad \frac{f'''(0)}{3!} = \frac{1}{3!}$

$\frac{f^{IV}(0)}{4!} = 0$

$\frac{f^{V}(0)}{5!} = \frac{1}{5!}$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$\operatorname{sh} 0,1 \approx T_n(0,1) \qquad n = ?$

$$|f(x) - T_n(x)| = \left| \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1} \right|$$

$0 \le t \le 0,1 = x$

$$|f(0,1) - T_n(0,1)| = \left| \frac{f^{(n+1)}(t)}{(n+1)!} 0,1^{n+1} \right|$$

$$\left|f^{(n+1)}(t)\right| \le \max_{0\le t\le 0{,}1} \overset{\overset{|sh\,t|\le ch\,t\ \forall t}{\downarrow}}{ch\,t} = ch\,0{,}1 = \frac{e^{0{,}1}+e^{-0{,}1}}{2} < e = 4$$

Így $\quad \left|\frac{f^{(n+1)}(t)}{(n+1)!}\,0{,}1^{n+1}\right| \le \frac{4}{(n+1)!}\,0{,}1^{n+1} < 10^{-5}$ ok, ha $\underline{\underline{n=4}}$

(n=3 is jó
pontosabb
becsléssel)

$T_4(x) = x + \frac{x^3}{3!}$ $\qquad \underline{\underline{sh(0{,}1) \approx 0{,}1 + \frac{0{,}1^3}{6}}}$

5.

a) 1. Mo $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx = \underline{\underline{-\cos(\ln x) + C}}$

2. Mo $\displaystyle\int \sin(\ln x)\cdot\underbrace{\frac{1}{x}}_{(\ln x)'}\,dx = \int \sin t\,\frac{dt}{dx}\,dx = \int \sin t\,dt = -\cos t + C$

$= \underline{\underline{-\cos(\ln x) + C}}$

$t = \ln x$

$\frac{dt}{dx} = \frac{1}{x}$

b) $\displaystyle\int_0^1 \underset{f'}{2x}\,\underset{g}{\ln(x+1)}\,dx = \left[x^2\ln(x+1)\right]_0^1 - \int_0^1 x^2\frac{1}{x+1}\,dx = \ln 2 - 0 - \int_0^1 \frac{x^2-1+1}{x+1}\,dx$

$\displaystyle = \ln 2 - \int_0^1 x-1+\frac{1}{x+1}\,dx = \ln 2 - \left[\frac{x^2}{2} - x + \ln(x+1)\right]_0^1$

$\displaystyle = \ln 2 - \left[\tfrac{1}{2} - 1 + \ln 2 - (0-0+0)\right] = \underline{\underline{\frac{1}{2}}}$

c) $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} \frac{\sin^2 t}{\cos t}\,\underbrace{\cos t}_{\frac{dx}{dt}}\,dt = \int_0^{\pi/6} \frac{1-\cos 2t}{2}\,dt = \frac{\pi}{12} - \left[\frac{\sin 2t}{4}\right]_0^{\pi/6}$

$\displaystyle = \frac{\pi}{12} - \frac{\sin(\pi/3)}{4} + \frac{\sin 0}{4} = \underline{\underline{\frac{\pi}{12} - \frac{\sqrt{3}}{8}}}$

$x = \sin t$

$\frac{dx}{dt} = \cos t$

d) $\displaystyle\int \frac{x+4}{x^3+2x} = \int \frac{x+4}{x(x^2+2)} = \int \frac{a}{x} + \frac{bx+c}{x^2+2}\,dx$

$\displaystyle = \int \frac{2}{x} + \int \frac{-2x+1}{x^2+2} = 2\ln|x| - \ln(x^2+2) + \int \frac{1/2}{\left(\frac{x}{\sqrt{2}}\right)^2+1}$

$\displaystyle = \underline{\underline{2\ln|x| - \ln(x^2+2) + \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right) + C}}$

$x+4 = a(x^2+2) + x(bx+c)$

| | | |
|---|---|---|
| $x^2$ | $0 = a+b$ | $b=-2$ |
| $x$ | $1 = c$ | $c=1$ |
| $1$ | $4 = 2a$ | $a=2$ |