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Equilibrium Fluctuations for a System of Harmonic Oscillators with Conservative Noise

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We investigate the harmonic chain forced by a multiplicative noise, the evolution is given by an infinite system of stochastic differential equations. Total energy and deformation are preserved, the conservation of momentum is destroyed by the noise. Gaussian product measures are the extremal stationary states of this model. Equilibrium fluctuations of the conserved fields at a diffusive scaling are described by a couple of generalized Ornstein-Uhlenbeck processes.

KEY WORDS: Harmonic chain, two-component models, equilibrium fluctuations, non-gradient methods, generalized Ornstein-Uhlenbeck processes.
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1. INTRODUCTION AND MAIN RESULTS

During the last 15 years a great progress has been made in the theory of hydrodynamic limits of one component systems, whereas only a few results are available on two component models, see [4,5,6,9,12], and these latter all concern the hydrodynamic law of large numbers. Here we present an equilibrium fluctuation result for a two component system. Stochastic perturbations of mechanical systems are certainly interesting also from a physical point of view, see [5,8,9] for some examples. The model we discuss here is also of this kind.

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Perhaps the simplest mechanical model is the harmonic chain defined by its formal Hamiltonian

$$H = \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(p_k^2 + (q_{k+1}-q_k)^2\right), \tag{1.1}$$

where p_k and q_k denote the momentum and amplitude of oscillator $k \in \mathbb{Z}$. Newton's equations of motion read as $\dot{p}_k = q_{k+1} + q_{k-1} - 2q_k$ and $\dot{q}_k = p_k$, total energy and momentum are certainly preserved by the dynamics. It is natural to introduce new coordinates $r_k := q_{k+1} - q_k$ called deformation; total deformation, $\sum r_k$ turns out to be a third conserved quantity. In terms of the p_k and r_k variables, Gaussian product measures like (1.3) are stationary states of the Hamiltonian flow.

In this article we consider this linearly ordered system of harmonic oscillators with damping and conservative noise. This model has been considered in a non-equilibrium setting in [(1)] and [(2)]. The oscillators are labelled by $k \in \mathbb{Z}$, the configuration space $\Omega = (\mathbb{R}\times R)^{\mathbb{Z}}$ is equipped with the usual product topology, a typical configuration is of the form $\omega = (p_k, r_k)_{k\in\mathbb{Z}}$ where p_k denotes the velocity of oscillator indexed by $k \in \mathbb{Z}$, and r_k stands for the difference between the amplitudes of oscillators $k+1$ and k. The dynamics is given by the following set of stochastic differential equations:

$$\begin{aligned} dp_k &= (r_k - r_{k-1})\,dt - \gamma\, p_k\,dt + \sqrt{\gamma}\, p_{k+1}\,dW_k - \sqrt{\gamma}\, p_{k-1}\,dW_{k-1}\\ dq_k &= p_k\,dt, \quad \text{i.e.} \quad dr_k = (p_{k+1} - p_k)\,dt, \end{aligned} \tag{1.2}$$

where $\{W_k : k \in \mathbb{Z}\}$ are independent, standard Wiener processes, and $\gamma > 0$ is the coefficient of damping. Since the r.h.s. of (1.2) is uniformly Lipschitz continuous with respect to any of the norms $||\omega||_\alpha$, $||\omega||_\alpha^2 := \alpha\sum_{k\in\mathbb{Z}} e^{-\alpha|k|}(p_k^2 + r_k^2)$, $\alpha > 0$, a standard iteration procedure yields existence of unique strong solutions to (1.2) in the associated weighted ℓ^2 spaces. In fact, the infinite dynamics is approximated by solutions of its finite subsystems. This approach also shows that for any $\beta > 0$ and $\rho \in \mathbb{R}$ the Gaussian product measures $\mu_{\beta,\rho}$ on Ω with marginals

$$\mu_{\beta,\rho}(dp_k, dr_k) = \frac{\beta}{2\pi}\exp\left(-\frac{\beta}{2}\left(p_k^2 + (r_k-\rho)^2\right)\right)dp_k\,dr_k \tag{1.3}$$

are invariant measures of the process. One can also prove that their convex combinations are the only stationary states, but we do not need this fact here.

The formal generator of the system reads as $\mathcal{L} = \mathcal{A} + \mathcal{S}$, where

$$\begin{aligned} \mathcal{A} &= \sum_{k\in\mathbb{Z}}\{(p_{k+1}-p_k)\partial r_k + (r_k - r_{k-1})\partial p_k\},\\ \mathcal{S} &= \frac{\gamma}{2}\sum_{k\in\mathbb{Z}}(p_{k+1}\partial p_k - p_k\partial p_{k+1})^2. \end{aligned}$$

Here $\mathcal{A}$ is the Liouville operator of a chain of interacting harmonic oscillators, ∂p_k and ∂r_k denote differentiation with respect to p_k and r_k, finally $\mathcal{S}$ is the noise operator. The symmetric $\mathcal{S}$ is acting only on velocities and it couples the neighboring velocities in such a way that kinetic energy of the system is conserved. Actually the model admits two conserved quantities: total deformation (the sum of r_k 's) and total energy (the sum of the H_k 's, where $H_k = \frac{1}{2}p_k^2 + \frac{1}{4}r_k^2 + \frac{1}{4}r_{k-1}^2$).

The model is obviously asymmetric, nevertheless it exhibits a diffusive hydrodynamic behaviour. In fact its hyperbolic (Euler) limit is trivial while a couple of nonlinear parabolic equations

$$\partial_t u = \frac{1}{\gamma}\,\Delta u\,, \tag{1.4}$$
$$\partial_t e = \frac{1+\gamma^2}{2\gamma}\,\Delta e + \frac{1-\gamma^2}{4\gamma}\,\Delta(u^2)\,,$$

are obtained in the hydrodynamic limit under diffusive scaling, see [1] for a partial derivation. At a level of hyperbolic scaling there are no fluctuations either as Euler time is not enough to develop effective randomness.

Our aim is to study the equilibrium fluctuation of the two conserved quantities under diffusive scaling. The fluctuation fields are defined as follows:

$$u_t^\varepsilon(\psi) = \sqrt{\varepsilon}\sum_{k\in\mathbb{Z}} \psi(\varepsilon k)(r_k(t/\varepsilon^2) - \rho)\,, \tag{1.5}$$

$$e_t^\varepsilon(\varphi) = \sqrt{\varepsilon}\sum_{k\in\mathbb{Z}} \varphi(\varepsilon k)\left(H_k(t/\varepsilon^2) - \frac{1}{\beta} - \frac{\rho^2}{2}\right)\,, \tag{1.6}$$

where φ and ψ are smooth functions of compact support. It is straightforward to see that in an equilibrium state $\mu = \mu_{\beta,\rho}$, $\xi_t^\varepsilon := (u_t^\varepsilon, e_t^\varepsilon)$ converges in law at any fixed t to a Gaussian field $\xi_t = (u_t, e_t)$ with mean $(0, 0)$ and covariances

$$\begin{aligned} \mathbb{E}_\mu[u_t(\psi_1)u_t(\psi_2)] &= \frac{1}{\beta}\int \psi_1(x)\psi_2(x)\,dx\,, \\ \mathbb{E}_\mu[e_t(\varphi_1)e_t(\varphi_2)] &= \frac{1+\beta\rho^2}{\beta^2}\int \varphi_1(x)\varphi_2(x)\,dx\,, \\ \mathbb{E}_\mu[u_t(\psi)e_t(\varphi)] &= \frac{\rho}{\beta}\int \psi(x)\varphi(x)\,dx\,, \end{aligned} \tag{1.7}$$

where $\mathbb{E}_\mu$ denotes expectation in a stationary regime specified by an arbitrary, but fixed stationary state $\mu = \mu_{\beta,\rho}$.

In this paper we prove that $\xi_t^\varepsilon = (u_t^\varepsilon, e_t^\varepsilon)$, as a vector of two distribution valued processes converges in law to the solution $\xi_t = (u_t, e_t)$ of the following pair of stochastic partial differential equations of generalized Ornstein-Uhlenbeck type:

$$\partial_t u = \frac{1}{\gamma}\,\Delta u + \sqrt{\frac{2}{\gamma\beta}}\,\nabla j_1\,, \tag{1.8}$$

$$\partial_t e = \frac{1+\gamma^2}{2\gamma}\,\Delta e + \frac{1-\gamma^2}{2\gamma}\,\Delta(\rho u) + \frac{\sqrt{2}\rho}{\sqrt{\gamma\beta}}\,\nabla j_1 + \frac{\sqrt{1+\gamma^2}}{\beta\sqrt{\gamma}}\,\nabla j_2\,,$$

where j_1 and j_2 are independent, standard white noise processes in space and time. According to general principles of hydrodynamics the drift of these equations can be obtained by the linearization of the hydrodynamic equations, (1.4). Of course, (1.8) should be interpreted in a weak sense, and the law of $\xi_t = (u_t, e_t)$ is specified as the unique solution to the martingale problem for (1.8).

To formulate our result more precisely, we have to introduce some notation. Let $\mathcal{H}_m$ be the Sobolev space associated to the scalar product

$$(f, g)_m = \int_{\mathbb{R}} f(q)(q^2 - \Delta)^m g(q)\,dq\,, \tag{1.9}$$

and let $\mathcal{H}_{-m}$ be its dual space with respect to $L^2(\mathbb{R})$. We consider the fluctuation fields $\xi_t^\varepsilon = (u_t^\varepsilon, e_t^\varepsilon)$ as random elements of $C\,(\mathbb{R}_+; \mathcal{H}_{-m}\times\mathcal{H}_{-m})$ with some $k > 0$ large enough, and P^ε denotes the probability distribution of ξ_t^ε in a stationary regime. The parameters $\beta > 0$ and $\rho \in \mathbb{R}$ of the stationary state $\mu_{\beta,\rho}$ are arbitrary; sometimes we put $\beta = 1$ and $\rho = 0$ for convenience. In Section 3 we prove tightness of P^ε for $m > 3$, thus first of all, we have to see that any limit point of P^ε solves the martingale problem related to (1.8), that is for all compactly supported and infinitely differentiable $\psi, \varphi \in C_c^\infty(\mathbb{R})$ we have

$$u_t(\psi) = u_0(\psi) + \frac{1}{\gamma}\int_0^t u_s(\psi'')\,ds + M_t^u(\psi) \tag{1.10}$$

$$e_t(\varphi) = e_0(\varphi) + \frac{1+\gamma^2}{2\gamma}\int_0^t e_s(\varphi'')\,ds + \frac{\rho-\gamma^2\rho}{2\gamma}\int_0^t u_s(\varphi'')\,ds + M_t^e(\varphi)\,,$$

where $M_t^u(\psi)$ and $M_t^e(\varphi)$ are $\mathcal{F}_t$ adapted Gaussian martingales with quadratic and cross variations

$$d\langle M_t^u, M_t^u\rangle = 2(\gamma\beta)^{-1}\|\psi'\|^2\,dt,$$

$$d\langle M_t^e, M_t^e\rangle = \frac{2\rho^2\beta+\gamma^2+1}{\beta^2\gamma}\,\|\varphi'\|^2\,dt, \tag{1.11}$$

$$d\langle M_t^u, M_t^e\rangle = 2\rho(\gamma\beta)^{-1}\langle\psi', \varphi'\rangle\,dt,$$

respectively, where $\|f\|$ and $\langle f, g\rangle$ denote the usual norm and scalar product in $L^2(\mathbb{R})$. In view of the Holley-Stroock theory of generalized Ornstein-Uhlenbeck processes (cf. Chapt. 11 in [7]), this martingale problem is uniquely solved under (1.7). We are now in a position to state our main result:

Theorem 1. *If $m > 3$ and $\epsilon \to 0$, then the distribution P^ϵ of the fluctuation fields ξ^ε converges in $C(\mathbb{R}_+; \mathcal{H}_{-m} \times \mathcal{H}_{-m})$ to the unique solution P of the martingale problem (1.10) specified by (1.7).*

What makes this system interesting is the fact that one of the conserved quantities, namely energy, is not a linear functional of the coordinates of the system, and the investigation of its fluctuation field is not trivial. While the derivation of the stochastic PDE for u is straightforward, to obtain that for e, one has to overcome two difficulties. First we have to get rid of the singularity coming from the asymmetric Hamiltonian part of the generator by means of some non-gradient analysis. The second crucial step is the verification of the Boltzmann-Gibbs principle, we have to replace the microscopic currents with linear functionals of the conserved quantities, see Section 2 for further details. In Section 3 tightness of the distribution of the fluctuation fields is proven, finally in Section 4, by means of the a priori bounds of Sections 2 and 3, we prove convergence to the solution of the martingale problem (1.10) related to (1.8).

2. TIME EVOLUTION OF THE FLUCTUATION FIELDS

To make computations more transparent, we introduce:

$$\nabla_1 a_k := a_{k+1} - a_k,\ \nabla_1^* a_k := a_{k-1} - a_k,\ \Delta_1 a_k := a_{k+1} + a_{k-1} - 2a_k,$$
$$\nabla_\varepsilon f(x) := \varepsilon^{-1}(f(x+\varepsilon) - f(x)),\ \nabla_\varepsilon^* f(x) := \varepsilon^{-1}(f(x-\varepsilon) - f(x)),$$
$$\Delta_\varepsilon f(x) := \varepsilon^{-2}(f(x+\varepsilon) + f(x-\varepsilon) - 2f(x)).$$

From the evolution law 1.2

$$dr_k = \frac{1}{\gamma}\Delta_1 r_k dt - \frac{1}{\gamma}\nabla_1 dp_k + \frac{1}{\sqrt{\gamma}}\nabla_1\left(p_{k+1}dW_k - p_{k-1}dW_{k-1}\right), \tag{2.1}$$

thus the deformation fluctuation field satisfies

$$u_t^\varepsilon(\psi) = u_0^\varepsilon(\psi) + \frac{1}{\gamma}\int_0^t u_s^\varepsilon(\Delta_\varepsilon\psi)\,ds + \frac{\varepsilon}{\gamma}\left[\pi_t^\varepsilon(\nabla_\varepsilon\psi) - \pi_0^\varepsilon(\nabla_\varepsilon\psi)\right] + M_t^{u,\varepsilon}(\nabla_\varepsilon\psi),$$

where

$$\pi_s^\varepsilon(\psi) := \sqrt{\varepsilon}\sum_{k\in\mathbb{Z}}\psi(\varepsilon k)p_k(s/\varepsilon^2),$$

$$M_t^{u,\varepsilon}(\psi) := \sqrt{\frac{\varepsilon}{\gamma}}\int_0^t\sum_{k\in\mathbb{Z}}(\psi(\varepsilon k)p_k - \psi(\varepsilon k - \varepsilon)p_{k+1}))\,d\bar{W}_k \tag{2.2}$$

is a martingale, and $\bar{W}_k(s) := \varepsilon W_k(s/\varepsilon^2)$. Note that here and below the time argument of all microscopic observables is speeded up by a factor ε^{-2}. It is easy to see that $\varepsilon[\pi_t^\varepsilon(\nabla_\varepsilon \psi) - \pi_0^\varepsilon(\nabla_\varepsilon \psi)]$ vanishes in mean square as $\varepsilon \to 0$. The replacements $\nabla_\varepsilon \psi \sim \psi'$ and $\Delta_\varepsilon \psi \sim \psi''$ are also immediate, thus we have proven

Proposition 2. *For any $\psi \in C_c^2(\mathbb{R})$ we have*

$$\lim_{\varepsilon \to 0} \mathbb{E}_\mu \left[u_t^\varepsilon(\psi) - u_0^\varepsilon(\psi) - \frac{1}{\gamma} \int_0^t u_s^\varepsilon(\psi'')\, ds - M_t^{u,\varepsilon}(\psi') \right]^2 = 0\,.$$

Our next aim is to prove a similar result for the fluctuation field of energy. Set $h_k = H_k - 1/\beta - \rho^2/2$, from (1.2) we get

$$\begin{aligned} dh_k &= \frac{1}{2}[r_{k-1}(p_k - p_{k-1}) + r_k(p_{k+1} - p_k)]\, dt + p_k(r_k - r_{k-1})\, dt \\ &\quad + \frac{\gamma}{2}\left(p_{k+1}^2 + p_{k-1}^2 - 2p_k^2\right) dt + \sqrt{\gamma}\,(p_k p_{k+1} dW_k - p_{k-1} p_k dW_{k-1})\,, \end{aligned}$$

that is

$$dh_k = \mathcal{A} h_k + \mathcal{S} h_k + \sqrt{\gamma}\, \nabla_1 (p_{k-1} p_k\, dW_{k-1})\,, \tag{2.3}$$

where

$$\mathcal{A} h_k = \nabla_1 J_{k-1}\,, \quad \mathcal{S} h_k = \frac{\gamma}{2} \Delta_1 p_k^2\,, \quad J_k := \frac{1}{2} r_k\, (p_{k+1} + p_k)\,. \tag{2.4}$$

Using (2.3) and (2.4) we rewrite the energy fluctuation field as

$$\begin{aligned} e_t^\varepsilon(\varphi) &= e_0^\varepsilon(\varphi) - \frac{1}{\sqrt{\varepsilon}} \int_0^t \sum_{k \in \mathbb{Z}} \nabla_\varepsilon \varphi(\varepsilon k) J_k(s/\varepsilon^2)\, ds \\ &\quad + \frac{\gamma \sqrt{\varepsilon}}{2} \int_0^t \sum_{k \in \mathbb{Z}} \Delta_\varepsilon \varphi(\varepsilon k) \left(p_k^2(s/\varepsilon^2) - \frac{1}{\beta} \right) ds - K_t^{e,\varepsilon}(\nabla_\varepsilon \varphi)\,, \end{aligned} \tag{2.5}$$

where

$$K_t^{e,\varepsilon}(\varphi) := \sqrt{\gamma \varepsilon} \int_0^t \sum_{k \in \mathbb{Z}} \varphi(\varepsilon k)\, p_k\, p_{k+1}\, d\bar{W}_k \tag{2.6}$$

is a martingale. Since the Hamiltonian flux is not a gradient, the first integrand of (2.5) containing J_k is rapidly oscillating. In the following proof we use some elementary tricks to find cancellation of singularities. To simplify computations we assume that $\rho = 0$ and $\beta = 1$; the general case reduces to this one by a linear transformation, see the end of Section 4.

Proposition 3. *If $\beta = 1$, $\rho = 0$ and $\varphi \in C_c^2(\mathbb{R})$, then*

$$\lim_{\varepsilon\to 0} \mathbb{E}_\mu\big[e_t^\varepsilon(\varphi) - e_0^\varepsilon(\varphi) - A_t^\varepsilon(\Delta_\varepsilon\varphi) + K_t^{e,\varepsilon}(\nabla_\varepsilon\varphi) + N_t^{e,\varepsilon}(\nabla_\varepsilon\varphi)\big]^2 = 0,$$

where

$$A_t^\varepsilon(\varphi) = \frac{\sqrt{\varepsilon}}{2\gamma}\int_0^t \sum_{k\in\mathbb{Z}} \varphi(\varepsilon k)\big[(1+\gamma^2)\big(p_k^2 - 1\big) + r_{k-1}r_k\big]\, ds,$$

while $K_t^{e,\varepsilon}$ and $N_t^{e,\varepsilon}$ are martingales defined by 2.6 and 2.11.

Proof: As $\mathcal{S}J_k = -\gamma J_k$, we have

$$J_k = \frac{1}{\gamma}(\mathcal{A}J_k - \mathcal{L}J_k) \quad \text{and} \quad \mathcal{A}J_k = \frac{1}{2}\nabla_1 p_k^2 + \frac{1}{2}\nabla_1(r_{k-1}r_k). \tag{2.7}$$

Moreover,

$$\begin{aligned} dJ_k = \mathcal{L}J_k dt &+ \frac{\sqrt{\gamma}}{2} r_k\left(p_{k+2}\, dW_{k+1} - p_k\, dW_k\right) \\ &+ \frac{\sqrt{\gamma}}{2} r_k\left(p_{k+1}\, dW_k - p_{k-1}\, dW_{k-1}\right), \end{aligned} \tag{2.8}$$

while from 2.7

$$\begin{aligned} -\frac{1}{\sqrt{\varepsilon}}\int_0^t \sum_{k\in\mathbb{Z}} \nabla_\varepsilon\varphi(\varepsilon k) J_k(s/\varepsilon^2)\, ds &= \frac{1}{\gamma\sqrt{\varepsilon}}\int_0^t \sum_{k\in\mathbb{Z}} \nabla_\varepsilon\varphi(\varepsilon k)\mathcal{L}J_k(s/\varepsilon^2)\, ds \\ &+ \frac{\sqrt{\varepsilon}}{2\gamma}\int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k)(p_k^2 - 1 + r_{k-1}\, r_k)\, ds. \end{aligned} \tag{2.9}$$

Since dJ_k integrates out, in view of (2.8) we have

$$\lim_{\varepsilon\to 0}\mathbb{E}_\mu\left[\frac{1}{\gamma\sqrt{\varepsilon}}\int_0^t \sum_{k\in\mathbb{Z}} \nabla_\varepsilon\varphi(\varepsilon k)\mathcal{L}J_k(s/\varepsilon^2)ds + N_t^{e,\varepsilon}(\nabla_\varepsilon\varphi)\right]^2 = 0, \tag{2.10}$$

where N^ε is the associated martingale,

$$N_t^{e,\varepsilon}(\varphi) := \frac{\sqrt{\varepsilon}}{2\sqrt{\gamma}}\int_0^t \sum_{k\in\mathbb{Z}} G_k(s/\varepsilon^2)\, d\bar{W}_k, \tag{2.11}$$

$$G_k := \varphi(\varepsilon k - \varepsilon)r_{k-1}p_{k+1} - \varphi(\varepsilon k)r_k p_k + \varphi(\varepsilon k)r_k p_{k+1} - \varphi(\varepsilon k + \varepsilon)r_{k+1}p_k.$$

Comparing (2.5), (2.9) and (2.10) we obtain the desired statement. □

In view of the Boltzmann-Gibbs principle, in an asymptotic sense we have to represent $A_t^\varepsilon(\varphi)$ as a linear functional of the conserved quantities. The first step in this direction is the treatment of kinetic energy.

Lemma 4. *Let $\beta = 1$ and $\rho = 0$, then for any $\varphi \in C_c^2(\mathbb{R})$*

$$\lim_{\varepsilon\to 0} \mathbb{E}_\mu \left[\sqrt{\varepsilon} \int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k) \left(p_k^2 - 1 - h_k - \frac{1}{2} r_k r_{k-1} \right) ds \right]^2 = 0$$

Proof: Since $\mathcal{L}(p_k r_k) = p_k p_{k+1} - \gamma p_k r_k - (p_k^2 - r_k^2 + r_{k-1} r_k)$,

$$p_k^2 = \frac{p_k^2 + r_k^2}{2} + \frac{p_k p_{k+1} - \gamma p_k r_k - \mathcal{L}(p_k r_k)}{2} - \frac{r_{k-1} r_k}{2}.$$

It is easy to see that the martingale part of the contribution of $d(p_k r_k)$ vanishes in mean square, and that of $d(p_k r_k)$ integrates out in the limit which yields

$$\lim_{\varepsilon\to 0} \mathbb{E}_\mu \left[\sqrt{\varepsilon} \int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k) \left(\frac{p_k^2 + r_k^2}{2} - 1 - h_k - \frac{1}{2} \mathcal{L}(p_k r_k) \right) ds \right]^2 = 0.$$

The remainder we still have to treat comes from

$$a_\varepsilon := \sqrt{\varepsilon} \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k)(p_k p_{k+1} - \gamma p_k r_k).$$

At this point we use an H^{-1} bound, namely Theorem 2.2 of [(10)]. In view of $\mathcal{S} p_k p_{k+1} = -3\gamma p_k p_{k+1}$ and $\mathcal{S} p_k r_k = -\gamma p_k r_k$ we get

$$\begin{aligned} \mathbb{E}_\mu \left[\int_0^t a_\varepsilon(\omega(s/\varepsilon^2))\, ds \right]^2 &\le 4t\varepsilon^2 \mathbb{E}_\mu \left(a_\varepsilon (-\mathcal{S})^{-1} a_\varepsilon \right) \\ &\le 4t\varepsilon^3 \mathbb{E}_\mu \left[\sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k) \frac{p_k p_{k+1}}{3\gamma} \right]^2 + 4t\varepsilon^3 \mathbb{E}_\mu \left[\sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k) \frac{p_k r_k}{\gamma} \right]^2, \end{aligned}$$

and the right hand side goes to 0 as $\varepsilon \to 0$. □

The following lemma shows that the contribution of the $r_{k-1} r_k$ terms of A_t^ϵ vanishes in the limit, too.

Lemma 5. *Since $\rho = 0$, for $\varphi \in C_c^2(\mathbb{R})$*

$$\lim_{\varepsilon\to 0} \mathbb{E}_\mu \left[\sqrt{\varepsilon} \int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k) r_k r_{k-1}\, ds \right]^2 = 0 \tag{2.12}$$

Proof: We have three auxiliary functions and a clever decomposition

$$r_{k-1} r_k = -\mathcal{L} F_{n,k}^{(1)} + F_{n,k}^{(2)} + F_{n,k}^{(3)} - \gamma F_{n,k}^{(1)}$$

for $n > 2$, where

$$F^{(1)}_{k,n} := \frac{1}{n}\sum_{l=1}^{n}(n+1-l)p_{k+l}r_{k-1}\,,$$

$$F^{(2)}_{k,n} := \frac{1}{n}\sum_{l=1}^{n}(n+1-l)(p_k - p_{k-1})p_{k+l}\,, \tag{2.13}$$

$$F^{(3)}_{k,n} := \frac{1}{n}\sum_{l=1}^{n} r_{k+l}r_{k-1}\,.$$

Let $n \to \infty$ as $\varepsilon \to 0$ such that $\varepsilon n \to 0$. Computing the stochastic differential of $F^{(1)}$ we see that the related martingale vanishes, consequently

$$\mathbb{E}_\mu\left[\sqrt{\varepsilon}\int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k)\mathcal{L}F^{(1)}_{k,n}(s/\varepsilon^2)\,ds\right]^2 = (t\gamma+1)\|\varphi''\|^2\; O(\varepsilon^2 n^2)\,.$$

The contribution of $F^{(3)}$ can directly be estimated by Schwarz:

$$\mathbb{E}_\mu\left[\sqrt{\varepsilon}\int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k)F^{(3)}_{n,k}(s/\varepsilon^2)\,ds\right]^2 = t^2\|\varphi''\|^2\; O(1/n)\,.$$

The remaining two terms are treated by means of the H^{-1} bound, Theorem 2.2 in [10]. Taking into account $\mathcal{S}(p_i p_{i+1}) = -3\gamma p_i p_{i+1}$, $\mathcal{S}(p_k r_k) = -\gamma p_k r_k$ and $\mathcal{S}(p_i p_j) = -2\gamma p_i p_j$ for $i, j \in \mathbb{Z}$, $|i-j| \geq 2$ we obtain that

$$\mathbb{E}_\mu\left[\sqrt{\varepsilon}\int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k)F^{(2)}_{n,k}(s/\varepsilon^2)\,ds\right]^2 = \frac{t}{\gamma}\;\|\varphi''\|^2\; O(\varepsilon^2 n)$$

and

$$\mathbb{E}_\mu\left[\sqrt{\varepsilon}\int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k)\gamma F^{(1)}_{n,k}(s/\varepsilon^2)\,ds\right]^2 = \frac{t}{\gamma}\;\|\varphi''\|^2\; O(\varepsilon^2 n)\,,$$

which complete the proof of (2.12). □

We are now in a position to state the main result of this section on the energy fluctuation field.

Proposition 6. *Let $\beta = 1$ and $\rho = 0$, then for every $\varphi \in C_c^2(\mathbb{R})$*

$$\lim_{\varepsilon\to 0}\mathbb{E}_\mu\left[e^\varepsilon_t(\varphi) - e^\varepsilon_0(\varphi) - \frac{1+\gamma^2}{2\gamma}\int_0^t e^\varepsilon_s(\varphi'')\,ds + K^{e,\varepsilon}_t(\varphi') + N^{e,\varepsilon}_t(\varphi')\right]^2 = 0,$$

where $K^{e,\varepsilon}_t$ and $N^{e,\varepsilon}_t$ are the martingales defined by (2.6) and (2.11).

Proof: This statement is a direct consequence of Proposition 3, Lemma 4 and Lemma 5. □

Let us postpone the easy calculation of the quadratic and cross variations of the underlying martingales $M^{u,\varepsilon}$, $K^{e,\varepsilon}$ and $N^{e,\varepsilon}$ to Section 4.

3. TIGHTNESS

Here we prove the tightness of the family P^ε, $0 < \varepsilon \leq 1$ of the measures of the conservative fields. Since energy is a nonlinear function of the configuration, some lengthy computations are required. Actually we are extending the proof of [(8)], calculations are based on a representation of Sobolev spaces $\mathcal{H}_m$ in terms of Hermite polynomials. For any $k \geq 0$ and $f, g \in C_c^\infty(\mathbb{R})$ consider the scalar product $(f, g)_m$, see (1.9), and denote by $\mathcal{H}_m$ the corresponding closure. For any positive m, $\mathcal{H}_{-m}$ is its dual with respect to $L^2(\mathbb{R}) \equiv \mathcal{H}_0$, i.e. $\|f\| \equiv \|f\|_0$. It is convenient to represent $(\cdot, \cdot)_m$ in terms of Hermite polynomials, which are the eigenfunctions of $q^2 - \Delta$. Let h_n denote the n^{th} normalized Hermite polynomial, each is an infinitely differentiable real function with Gaussian tail, and h_n, $n \in \mathbb{Z}_+$ form a complete orthogonal base of $L^2(\mathbb{R})$. Since $q^2 h_n - \Delta h_n = (2n+1)h_n$, for every $m \geq 0$ and $f \in L^2$ we have

$$\|f\|_m^2 = \int_{\mathbb{R}} f(q)(q^2 - \Delta)^m f(q)\, dq = (2n+1)^m \sum_{n \in \mathbb{N}} \langle f, h_n \rangle^2 ,$$

and this is valid also for negative m, thus the $\mathcal{H}_{-m}$-norm of a distribution $\zeta = \zeta(\phi)$ can be written as

$$\|\zeta\|_{-m}^2 = \sum_{n \in \mathbb{N}} (2n+1)^{-m} \zeta(h_n)^2. \tag{3.1}$$

In an equilibrium state p_k and r_k can not grow faster than $\log(1 + |k|)$, thus the fluctuation fields u_t^ε and e_t^ε can be considered as distributions, i.e. elements of the Schwartz space $S'(\mathbb{R})$; S' is the dual of the space $S(\mathbb{R})$ of smooth and rapidly decreasing functions. It is plain that, as far as $\varepsilon > 0$, the probability distribution P^ε of the equilibrium process $\xi_t^\varepsilon = (u_t^\varepsilon, e_t^\varepsilon)$ is concentrated on the space $C(\mathbb{R}_+, S'(\mathbb{R}) \times S'(\mathbb{R}))$. The basic result of this section is

Proposition 7. *For any $m > 3$ and every $T > 0$, the family P^ε, $0 < \varepsilon < 1$ of probability measures has support in $C([0, T], \mathcal{H}_{-m} \times \mathcal{H}_{-m})$, and it is relatively compact in this space.*

In view of the Holley-Stroock theory of Generalized Ornstein-Uhlenbeck processes (cf. Chapt. 11 in [(7)]), Proposition 7 is a consequence of the following result. To avoid too big expressions, let $\sup_{(\delta,T)}$ denote the least upper bound over the set $\{s, t : 0 \le s < t \le T, t - s \le \delta\}$.

Proposition 8. *For any $m > 3$ and every $T, R > 0$ we have*

(i) $\sup_{\varepsilon\in(0,1)} \mathbb{E}_\mu[\sup_{0<t<T} \|u_t^\varepsilon\|_{-m}^2] < +\infty$,
(ii) $\sup_{\varepsilon\in(0,1)} \mathbb{E}_\mu[\sup_{0<t<T} \|e_t^\varepsilon\|_{-m}^2] < +\infty$,
(iii) $\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \mathbb{P}_\mu[\sup_{(\delta,T)} \|u_t^\varepsilon - u_s^\varepsilon\|_{-m} > R] = 0$,
(iv) $\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \mathbb{P}_\mu[\sup_{(\delta,T)} \|e_t^\varepsilon - e_s^\varepsilon\|_{-m} > R] = 0$.

To prove Proposition 8, we need several technical results, a basic a priori bound first of all. As before, we may, and do assume that $\beta = 1$ and $\rho = 0$.

Lemma 9. *There exists a constant $B = B(T) < \infty$ such that*

$$\mathbb{E}_\mu[\sup\{u_t^\varepsilon(\psi)^2 : t \in [0, T]\}] \le B\,(\|\psi\|^2 + \|\psi'\|^2)$$

and

$$\mathbb{E}_\mu[\sup\{e_t^\varepsilon(\varphi)^2 : t \in [0, T]\}] \le B\,(\|\varphi\|^2 + \|\varphi'\|^2 + \|\varphi''\|^2)$$

for all $\varphi, \psi \in S(\mathbb{R})$ and $0 < \varepsilon < 1$.

Proof: Set $f_\varepsilon(s) := \sqrt{\varepsilon}\sum_{k\in\mathbb{Z}} \nabla_\varepsilon\psi(\varepsilon k) p_{k+1}(s/\varepsilon^2)$, from (1.2) for r_k we have

$$u_t^\varepsilon(\psi) = u_0^\varepsilon(\psi) - \frac{1}{\varepsilon}\int_0^t f_\varepsilon(s)\,ds\,,$$

whence

$$\mathbb{E}_\mu\left[\sup_{0<t<T} u_t^\varepsilon(\psi)^2\right] \le 2\mathbb{E}_\mu u_0^\varepsilon(\psi)^2 + 2\mathbb{E}_\mu\left[\sup_{0<t<T}\left(\frac{1}{\varepsilon}\int_0^t f_\varepsilon(s)\,ds\right)^2\right].$$

Since $\mathcal{S} f_\varepsilon = -\gamma f_\varepsilon$ by Theorem 2.2 of [(10)]

$$\mathbb{E}_\mu\left[\sup_{0<t<T}\left(\frac{1}{\varepsilon}\int_0^t f_\varepsilon(s)\,ds\right)^2\right] \le 8T\varepsilon\mathbb{E}_\mu\left[f_\varepsilon(-\mathcal{S})^{-1} f_\varepsilon\right]$$
$$= \tfrac{8T\varepsilon}{\gamma}\textstyle\sum_{k\in\mathbb{Z}} (\nabla_\varepsilon\psi(\varepsilon k))^2\,,$$

which completes the proof of the first bound because the case of the initial value is trivial, and by Schwarz

$$\varepsilon\sum_{k\in\mathbb{Z}}(\nabla_\varepsilon\psi(\varepsilon k))^2 \le \|\psi'\|^2\,.$$

The energy field is a bit more complicated. For reader's convenience recall (2.5),

$$e_t^\varepsilon(\varphi) = e_0^\varepsilon(\varphi) - \frac{1}{\sqrt{\varepsilon}}\int_0^t \sum_{k\in\mathbb{Z}} \nabla_\varepsilon\varphi(\varepsilon k) J_k(s/\varepsilon^2)\,ds$$
$$+\frac{\gamma\sqrt{\varepsilon}}{2}\int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k)\big(p_k^2(s/\varepsilon^2)-1\big)\,ds - K_t^{e,\varepsilon}(\nabla_\varepsilon\varphi)\,, \quad (3.2)$$

where

$$K_t^{e,\varepsilon}(\varphi) = \sqrt{\gamma\varepsilon}\int_0^t \sum_{k\in\mathbb{Z}} \varphi(\varepsilon k) p_k p_{k+1}\,d\bar{W}_k\,.$$

The initial value is easy to treat, the second integral on the right hand side is estimated directly by the Schwarz inequality:

$$\mathbb{E}_\mu\Big[\sup_{0<t<T}\Big(\gamma\sqrt{\varepsilon}\int_0^t \sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k)\big(p_k^2(s/\varepsilon^2)-1\big)\,ds\Big)^2\Big]$$
$$\le \frac{\varepsilon\gamma^2 T}{2}\mathbb{E}_\mu\int_0^T\Big(\sum_{k\in\mathbb{Z}} \Delta_\varepsilon\varphi(\varepsilon k)\big(p_k^2(s/\varepsilon^2)-1\big)\Big)^2\,ds = O(\|\psi''\|^2)\,.$$

For brevity set $g_\varepsilon := \sum_{k\in\mathbb{Z}} \nabla_\varepsilon\varphi(\varepsilon k) J_k$. Since $\mathcal{S}J_k = -\gamma J_k$, by Theorem 2.2 of [10] on the critical first integral we get

$$\mathbb{E}_\mu\left[\sup_{0<t<T}\left(\frac{1}{\sqrt{\varepsilon}}\int_0^t \sum_{k\in\mathbb{Z}} \nabla_\varepsilon\varphi(\varepsilon k) J_k(s/\varepsilon^2)\,ds\right)^2\right] \le \frac{8T\varepsilon}{\gamma}\mathbb{E}_\mu[g_\varepsilon(-\mathcal{S})^{-1}g_\varepsilon]$$
$$= \frac{16T\varepsilon}{\gamma}\sum_{k\in\mathbb{Z}}(\nabla_\varepsilon\varphi(\varepsilon k))^2 = O(\|\varphi'\|^2)\,.$$

Finally, the martingale is treated by means of Doob's inequality:

$$\mathbb{E}_\mu\Big[\sup_{0<t<T} K_t^{e,\varepsilon}(\nabla_\varepsilon\varphi)^2\Big] \le 4\mathbb{E}_\mu K_T^{e,\varepsilon}(\nabla_\varepsilon\varphi)^2$$
$$= 4\gamma\varepsilon T\sum_{k\in\mathbb{Z}}(\nabla_\varepsilon\varphi(\varepsilon k))^2 = O(\|\varphi'\|^2)\,.$$

Combining the estimates above, we obtain the second bound. □

Proof of (i) and (ii) of Proposition 8: From (3.1) and Lemma 9 we obtain

$$\mathbb{E}_\mu\Big[\sup_{0<t<T}\|u_t^\varepsilon\|_{-m}^2\Big] \le \sum_{n\in\mathbb{N}}(2n+1)^{-m}\mathbb{E}_\mu\Big[\sup_{0<t<T} u_t^\varepsilon(h_n)^2\Big]$$
$$\le B\sum_{n\in\mathbb{N}}(2n+1)^{-m}(\|h_n\|^2+\|h_n'\|^2) \le B\sum_{n\in\mathbb{N}}(2n+1)^{-m}\big(1+\|h_n\|_1^2\big)\,,$$

and the last series converges if $m > 2$ because $\|h_n\|_1^2 \le 2n+1$.

The case of energy field is similar,

$$\mathbb{E}_\mu\Big[\sup_{0<t<T} \|e_t^\varepsilon\|_{-m}^2\Big] \le \sum_{n\in\mathbb{N}}(2n+1)^{-m}\mathbb{E}_\mu\Big[\sup_{0<t<T} e_t^\varepsilon(h_n)^2\Big]$$
$$\le B\sum_{n\in\mathbb{N}}(2n+1)^{-m}(\|h_n\|^2+\|h_n'\|^2+\|h_n''\|^2)$$
$$\le B\sum_{n\in\mathbb{N}}(2n+1)^{-m}(2+\|h_n\|_1^2+\|h_n\|_2^2)$$

Since $\|h_n\|_2^2 \le (2n+1)^2$, both bounds are finite if $m > 3$. □

By means of the Hermite expansion we reduce the problem of equicontinuity as follows. From 3.1

$$\mathbb{E}_\mu\Big[\sup_{(\delta,T)} \|u_t^\varepsilon - u_s^\varepsilon\|_{-m}^2\Big] \le \sum_{n\in\mathbb{N}}(2n+1)^{-m}\mathbb{E}_\mu\Big[\sup_{(\delta,T)}(u_t^\varepsilon(h_n)-u_s^\varepsilon(h_n))^2\Big],$$

however

$$\mathbb{E}_\mu\Big[\sup_{(\delta,T)}(u_t^\varepsilon(h_n)-u_s^\varepsilon(h_n))^2\Big] \le 4\mathbb{E}_\mu\Big[\sup_{(\delta,T)} u_t^\varepsilon(h_n)^2\Big],$$

thus in view of Lemma 9, the series above is uniformly convergent if $m > 2$. Therefore **(iii)** is implied by

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{E}_\mu\Big[\sup_{(\delta,T)}(u_t^\varepsilon(h_n)-u_s^\varepsilon(h_n))^2\Big] = 0. \tag{3.3}$$

Similarly, to prove **(iv)** we have to show that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{E}_\mu\Big[\sup_{(\delta,T)}(e_t^\varepsilon(h_n)-e_s^\varepsilon(h_n))^2\Big] = 0 \tag{3.4}$$

for each $n \in \mathbb{Z}_+$, at this point it is important that $m > 3$. However, we need not worry about dependence of the rates of convergence on n any more. The structure of the forthcoming calculations resembles those in Section 2.

Proof of (iii) of Proposition 8: Using notation introduced there, from (2.1) we get

$$u_t^\varepsilon(h_n) - u_s^\varepsilon(h_n) = \frac{1}{\gamma}\int_s^t u_\tau^\varepsilon(\Delta_\varepsilon h_n)\,d\tau + \frac{\varepsilon}{\gamma}(\pi_t^\varepsilon(\nabla_\varepsilon h_n) - \pi_s^\varepsilon(\nabla_\varepsilon h_n))$$
$$+M_s^{u,\varepsilon}(\nabla_\varepsilon h_n) - M_t^{u,\varepsilon}(\nabla_\varepsilon h_n)\,.$$

In view of Lemma 9, the integral of u_τ^ε can directly be estimated by Schwarz,

$$\mathbb{E}_\mu\left[\sup_{(\delta,T)}\left(\int_s^t u_\tau^\varepsilon(\Delta_\varepsilon h_n)\,d\tau\right)^2\right] \le B(n,T)\,\delta\,.$$

The singularity appearing in the stochastic differential of $\varepsilon\pi^\varepsilon$ is suppressed by the damping and the factor ε in front of it. In fact, we prove that

$$\sup_{0<\varepsilon<1} \mathbb{E}_\mu\Big[\sup_{0<t<T} \pi_t^\varepsilon(\nabla_\varepsilon h_n)^2\Big] < +\infty\,. \tag{3.5}$$

Indeed, from 1.2

$$p_k(t) = e^{-\gamma t} p_k(0) + \int_0^t e^{-\gamma(t-s)}(r_k - r_{k-1})\,ds + m_k(t)\,,$$

where

$$m_k(t) := \sqrt{\gamma}\int_0^t e^{-\gamma(t-s)}(p_{k+1}\,dW_k - p_{k-1}\,dW_{k-1})\,,$$

whence

$$\pi_t^\varepsilon = e^{-\gamma t/\varepsilon^2}\pi_0^\varepsilon - \frac{1}{\varepsilon}\int_0^t e^{-\gamma(t-s)/\varepsilon^2} u_s^\varepsilon(\Delta_\varepsilon h_n)\,ds + \sqrt{\varepsilon}\sum_{k\in\mathbb{Z}} \nabla_\varepsilon h_n(\varepsilon k)\, m_k(t/\varepsilon^2)\,.$$

Due to the exponential factor, the first term obviously vanishes. Separating u_s^ε from $e^{-\gamma(t-s)/\varepsilon^2}$ by means of the Schwarz inequality, we obtain a factor ε, which implies immediately that the contribution of the deterministic integral is bounded. The case of the stochastic integral is similar. Since

$$m_k(t/\varepsilon^2) := \frac{\sqrt{\gamma}}{\varepsilon}\int_0^t e^{-\gamma(t-s)/\varepsilon^2}(p_{k+1}\,d\bar{W}_k - p_{k-1}\,d\bar{W}_{k-1})\,,$$

where $\bar{W}_k$ are independent standard Wiener process, just as before, the quadratic variation of the stochastic integral in the above decomposition of π^ε can easily be estimated by Schwarz, and the proof of (3.5) is completed by Doob's maximal inequality.

The quadratic variation of $M^{u,\varepsilon}$ is easily controlled, its intensity is just $Q^{u,\varepsilon}$, where

$$Q_t^{u,\varepsilon}(\psi) = \frac{\varepsilon}{\gamma}\sum_{k\in\mathbb{Z}}(\psi(\varepsilon k - \varepsilon)p_{k+1}(t/\varepsilon^2) - \psi(\varepsilon k)p_k(t/\varepsilon^2))^2\,. \tag{3.6}$$

Like in many other cases, the equicontinuity of $M^{u,\varepsilon}$ is controlled by its fourth moment. An easy stochastic calculus exploiting stationarity of the underlying process yields

$$\mathbb{E}_\mu M_t^{u,\varepsilon}(\nabla\varepsilon h_n)^4 = O(t^2)\,, \tag{3.7}$$

where the bound does not depend on ε. To verify this we use a shorthand notation $M_t := M_t^{u,\varepsilon}$, $Q_t := Q_t^{u,\varepsilon}$, $m_t := \mathbb{E}_\mu M_t^4$, $q_t := \mathbb{E}_\mu Q_t^2$, and all we need to exploit is the fact that M_t is a continuous martingale with $M_0 = 0$ and $\int_0^t q_s ds \le Kt$ for all $t, \varepsilon > 0$ with a universal constant K. In fact

$$dM_t^4 = 4M_t^3 dM_t + 6M_t^2 Q_t dt$$

and Schwartz inequality yields

$$m_t \leq 6\sqrt{\int_0^t m_s ds}\sqrt{\int_0^t q_s ds} \leq 6\sqrt{\int_0^t m_s ds}\sqrt{Kt} \tag{3.8}$$

whence $\partial_t \sqrt{\int_0^t m_s ds} \leq 3\sqrt{Kt}$ which results in $\sqrt{\int_0^t m_s ds} \leq 2\sqrt{K}t^{3/2}$. Using (3.8) again we get $m_t \leq 12Kt^2$ and thus (3.7) is varified. From (3.7) we obtain

$$\mathbb{P}_\mu\Big[\sup_{0\leq t\leq\delta} M_t^{u,\varepsilon}(\nabla_\varepsilon h_n)^2 > R\Big] = O(\delta^2/R^2).$$

Now we can divide the interval $(0, T)$ into small pieces of size δ, and exploit stationarity to conclude that

$$\mathbb{P}_\mu[\Gamma_{\delta,T}^\varepsilon > R] = O(\delta/R^2),$$

where

$$\Gamma_{\delta,T}^\varepsilon := \sup_{(\delta,T)}(M_t^{u,\varepsilon}(\nabla_\varepsilon h_n) - M_s^{u,\varepsilon}(\nabla_\varepsilon h_n))^2.$$

From here using the fact that $\mathbb{E}_\mu \Gamma_{\delta,T}^\varepsilon \leq R(1 + \mathbb{P}_\mu[\Gamma_{\delta,T}^\varepsilon > R])$ and taking supremum in R, we can deduce that the variables $\Gamma_{\delta,T}^\varepsilon$ are uniformly integrable when ε and δ go to 0, consequently

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{E}_\mu \Gamma_{\delta,T}^\varepsilon = 0,$$

which completes the proof. □

The identities we have established in the proof of Proposition 3. are most useful for equicontinuity of the energy field. We start with a decomposition

$$e_t^\varepsilon(h_n) = e_0^\varepsilon(h_n) + A_t^\varepsilon(\Delta_\varepsilon h_n) + \varepsilon J_0^\varepsilon(\nabla_\varepsilon h_n) - \varepsilon J_t^\varepsilon(\nabla_\varepsilon h_n) + M_t^{e,\varepsilon}(\nabla_\varepsilon p), \tag{3.9}$$

where $M^{e,\varepsilon} := -K_t^{e,\varepsilon} - N_t^{e,\varepsilon}$, and

$$J_t^\varepsilon := \frac{\sqrt{\varepsilon}}{\gamma}\sum_{k\in\mathbb{Z}}\nabla_\varepsilon h_n(\varepsilon k)J_k(t/\varepsilon^2).$$

Remember that $J^\varepsilon = r_k(p_k + p_{k+1})/2$ and $\mathcal{L}J_k = (1/2\gamma)\nabla_1(p_k^2 + r_{k-1}r_k) - \gamma J_k$, thus all previous tricks are available also here.

Proof of (iv) of Proposition 8: By the Schwarz inequality it follows immediately that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{E}_\mu\Big[\sup_{(\delta,T)}(A_t^\varepsilon(\Delta_\varepsilon h_n) - A_s^\varepsilon(\Delta_\varepsilon h_n))^2\Big] = 0.$$

Due to the damping of the microscopic current J_k ,

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{E}_\mu\Big[\sup_{(\delta,T)}(\varepsilon J_t^\varepsilon(\nabla_\varepsilon h_n)-\varepsilon J_s^\varepsilon(\nabla_\varepsilon h_n))^2\Big]=0$$

follows in exactly the same way as we did for the velocity field π^ε in the proof of **(iii).**

Finally, let $Q^{ek,\varepsilon}$ and $Q^{en,\varepsilon}$ denote the intensities of the quadratic variations of $K^{e,\varepsilon}$ and $N^{e,\varepsilon}$, respectively. We have

$$Q_t^{ek,\varepsilon}(\varphi)=\gamma\varepsilon\sum_{k\in\mathbb{Z}}\varphi^2(\varepsilon k)p_k^2(t/\varepsilon^2)p_{k+1}^2(t/\varepsilon^2)\,, \tag{3.10}$$

$$Q_t^{en,\varepsilon}(\varphi)=\frac{\varepsilon}{4\gamma}\sum_{k\in\mathbb{Z}}G_k^2(t/\varepsilon^2)\,, \tag{3.11}$$

see (2.11) for the definition of G_k . These martingales can not be exponentiated, but their fourth moments can again be estimated, thus from $\mathbb{E}_\mu M_t^{ek,\varepsilon}(\nabla\varepsilon h_n)^4=O(t^2)$ and $\mathbb{E}_\mu M_t^{en,\varepsilon}(\nabla\varepsilon h_n)^4=O(t^2)$ we obtain

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{E}_\mu\Big[\sup_{(\delta,T)}(M_t^{e,\varepsilon}(\nabla_\varepsilon h_n)-M_s^{e,\varepsilon}(\nabla_\varepsilon h_n))^2\Big]=0$$

in the same way as for $M^{u,\varepsilon}$ in the proof of **(iii).** □

4. THE MACROSCOPIC EQUATIONS

Having proven both the Boltzmann–Gibbs principle and tightness of the fluctuation fields, the proof of Theorem 1. is almost complete. First we identify the quadratic variations of the martingales M^u and M^e in (1.9) when $\beta=1$ and $\rho=0$. From (3.6), (3.10) and (3.11) by the law of large numbers

$$\lim_{\varepsilon\to 0}Q_t^{u,\varepsilon}(\psi')=\frac{2}{\gamma}\|\psi'\|^2\,,\quad \lim_{\varepsilon\to 0}Q_t^{ek,\varepsilon}(\varphi')=\gamma\|\varphi'\|^2\,,\quad \lim_{\varepsilon\to 0}Q_t^{en,\varepsilon}(\varphi')=\frac{1}{\gamma}\|\varphi'\|^2$$

in L^1 . Moreover, all cross variations vanish in the limit, and the bounds on fourth moments of these martingales imply uniform integrability, consequently

$$\lim_{\varepsilon\to 0}\langle M_t^{u,\varepsilon},M_t^{u,\varepsilon}\rangle=\frac{2t}{\gamma}\|\psi'\|^2\,,\quad \lim_{\varepsilon\to 0}\langle M_t^{e,\varepsilon},M_t^{e,\varepsilon}\rangle=\frac{t+t\gamma^2}{\gamma}\|\varphi'\|^2\,, \tag{4.1}$$

and the fluctuation equations read as

$$\partial_t u=\frac{1}{\gamma}\,\Delta u+\sqrt{\frac{2}{\gamma}}\,\nabla j_1\,,\quad \partial_t e=\frac{1+\gamma^2}{2\gamma}\,\Delta e+\sqrt{\frac{1+\gamma^2}{\gamma}}\,\nabla j_2 \tag{4.2}$$

in the particular case of $\beta = 1$ and $\rho = 0$. The general case follows from here by a linear transformation $\tilde{p}_k = p_k/\sqrt{\beta}$, $\tilde{r}_k = r_k/\sqrt{\beta} + \rho$, i.e.

$$\frac{1}{2}(\tilde{p}_k^2 + \tilde{r}_k^2) = \frac{1}{2\beta}(p_k^2 + r_k^2) + \frac{r_k \rho}{\sqrt{\beta}} + \frac{\rho^2}{2},$$

consequently convergence of the transformed process to the solution of (4.2) implies convergence of the original to the solution of (1.8).

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