# Partial positivity concepts in projective geometry

## Habilitationsschrift

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**Abstract**

Line bundles with good positivity properties form one of the central tools in projective geometry. The fundamental notion of positivity is ampleness, which has geometric, numerical, and cohomological descriptions as well. However, for many purposes — including the study of birational models of varieties — ample line bundles are not sufficiently general. Here we study various concepts that are weaker than amplitude, but still retain some of the useful attributes of it.

## Contents

## Introduction

Our purpose here is to study various phenomena in higher-dimensional complex geometry associated to linear series or line bundles on algebraic varieties. Starting with Kodaira's embedding and vanishing theorems, it has become clear that line bundles satisfying certain positivity conditions play a fundamental role in higher-dimensional complex geometry, and their significance is widely accepted by now.

The original concept of positivity for line bundles is ampleness, which in essence means that global sections of some tensor power of the given bundle give rise to an embedding of the underlying variety into some projective space. Over the years cohomological, geometric, and numerical descriptions have been found; this versatility makes amplitude an extremely useful concept. The fundamental results in this direction are the theorems of Cartan–Serre–Grothendieck (see [61, Theorem 1.2.6]) and Kleiman–Moishezon–Nakai (see [61, Theorem 1.2.23]), which we now recall.

The result of Cartan–Serre–Grothendieck says the following. Let $X$ be a complete projective scheme, and $\mathcal{L}$ a line bundle on $X$. Then the following are equivalent:

1. There exists a positive integer $m_0 = m_0(X,\mathcal{L})$, such that $\mathcal{L}^{\otimes m}$ is very ample for all $m \geq m_0$ (that is, the global sections of $H^0\left(X,\mathcal{L}^{\otimes} m\right)$ give rise to a closed embedding of $X$ into some projective space).

2. For every coherent sheaf $\mathcal{F}$ on $X$, there exists a positive integer $m_1(X,\mathcal{F},\mathcal{L})$ for which $\mathcal{F}\otimes\mathcal{L}^{\otimes m}$ is globally generated for all $m \geq m_1$.

3. For every coherent sheaf $\mathcal{F}$ on $X$ there exists a positive integer $m_2(X,\mathcal{F},\mathcal{L})$ such that
$$H^i\left(X,\mathcal{F}\otimes\mathcal{L}^{\otimes m}\right) = 0$$
for all $i \geq 1$ and all $m \geq m_2$.

The Kleiman–Moishezon–Nakai criterion adds a numerical characterization of ample line bundles: if $\mathcal{L}$ is a line bundle on a projective scheme, then $\mathcal{L}$ is ample exactly if
$$c_1(\mathcal{L}|_V)^d > 0$$
for every irreducible subvariety of dimension $d$, where $0 < d \leq \dim X$.

Extremely important as they are, ample divisors have various shortcomings. For one, there exist many effective line bundles found in nature that possess many sections, but are nevertheless not ample. Such examples become increasingly important in birational geometry, since the pullback of an ample line bundle under a proper birational morphism (which leaves the underlying variety unchanged away from a codimension two locus), is no longer ample.

For these and many other reasons it makes sense to study line bundles, which, although not ample themselves, still retain some of the useful properties of ample ones. There are various ways to make this requirement precise, perhaps the most common one is to study line bundles whose tensor powers create birational maps instead of closed embeddings. Such bundles are called big; as big line bundles correspond to birational models of varieties, it is not surprising that they play a fundamental role in birational geometry. The behaviour and invariants of big line bundles have been a subject of intensive study in the last decades, for overviews, see [33] or [61, Chapter 2], and the references therein.

One of the main objectives of this writing is the study of invariants associated to big line bundles. Of the various invariants that have been defined, we will concentrate on Newton–Okounkov bodies and the volume of line bundles. Newton–Okounkov bodies are convex bodies in Euclidean space that are defined with the help of some auxiliary information; the volume of a line bundle is then simply the Lebesgue measure of an arbitrary Newton–Okounkov body of the bundle. Based on earlier work of Okounkov, these invariants were defined by Lazarsfeld–Mustaţă [63] and Kaveh–Khovanskii [48] in the context of complex geometry. The collection of all Newton–Okounkov bodies of a given line bundle determines the bundle up to numerical equivalence [45].

This area is relatively new and, although Newton–Okounkov bodies promise to be of great significance for the theory of big linear series, as of right now there are more questions than answers. Nevertheless, besides algebraic geometry, they have already found important applications in convex geometry [48], and geometric representation theory [46, 47].

It is quite probable that Newton–Okounkov bodies will soon play a fundamental role in the theory of completely integrable systems. Moment maps associated to smooth projective toric varieties and Guillemin–Sternberg integrable systems are two prominent examples that can be easily explained in the language of Newton–Okounkov bodies. Following this line of thought there is a detailed heuristic due to Harada and Kaveh, which promises to construct completely integrable systems coming from Newton–Okounkov bodies on many smooth projective varieties via toric degenerations [2].

We will focus on two geometric questions related to Newton–Okounkov bodies: on their description on surfaces, and on their polyhedrality properties in the case of higher-dimensional varieties. The elaboration of this circle of ideas takes place in Section 1.

The volume of a line bundle is the common Lebesgue measure of all Okounkov bodies, and as such, it is strongly related to them. In essence, one can think about the theory of Newton–Okounkov bodies as replacing a single number by a geometric object. Considered as a function on the numerical equivalence classes of line bundles, the volume turns out to be a log-concave homogeneous and continuous function. We point out that this fits in very nicely with Okounkov's philosophy (cf. [73]), which postulates that 'good' notions of multiplicity should be log-concave for reasons related to entropy in physics.

Volumes of line bundles have found many applications in birational geometry, one substantial example (cf. [39]) is that the volume of the canonical bundle can be used to bound the number of birational automorphisms of a variety of general type. Beside introducing the fundamental properties of the volume and its analogues for higher cohomology and treating the case of surfaces in detail, we will center our attention on related countability issues and arithmetic properties. In addition, we will touch upon a potential connection between volumes of line bundles and periods in transcendental number theory that has emerged lately. We devote Section 2 to this material.

A fairly new line of research involving a different concept of partial ampleness has been recently initiated by Totaro [77] building on earlier work of Demailly–Peternell–Schneider [28]. The idea is to weaken the cohomological criterion for ampleness in terms of vanishing of higher cohomology groups, and study line bundles where vanishing holds in all degrees above a certain natural number $q$; usual ampleness would be the case $q = 0$.

Vanishing theorems for higher cohomology groups have always been a central tool in projective geometry with an abundance of applications. Section 3 hosts a short explanation of some of these, along with a presentation of the theory of $q$-ample line bundles. As it turns out, some of the classical vanishing theorems (due to Serre, Kawamata–Viehweg, and Fujita) generalize well to partially positive line bundles, more precisely, to line bundles that become ample when restricted to general complete intersection subvarieties. Interestingly enough, although these bundles are often not even pseudo-effective, the vanishing of their higher cohomology can still be controlled. A description of these results will also take place in Section 3.

Section 4 goes in a somewhat different direction; here we study negativity of curves on smooth projective surfaces. Still, this material is in some sense close in nature to the topics studied so far, since positivity of the canonical bundle results in a lower bound of the self-intersection numbers of irreducible curves on a surface. The central question in Section 4 is the so-called Bounded Negativity Conjecture, a folklore conjecture going back to the times of Enriques, which states that on a smooth projective surface over the complex numbers the self-intersection of irreducible curves is bounded from below. The analogous statement is quickly seen to be false in positive characteristics due to a Frobenius argument, nevertheless many examples and results point to the validity of the conjecture.

**Conventions.** We will almost exclusively work over the complex number field, hence a variety is complex unless explicitly mentioned otherwise. Varieties are meant to be irreducible, although this will be emphasized on occasion. We use line bundle and Cartier divisor language interchangeably. For Cartier divisors $D_1$ and $D_2$, $D_1 \equiv D_2$ means numerical equivalence, $D_1 \sim D_2$ denotes linear equivalence.

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## 1 Newton–Okounkov bodies in complex geometry

### 1.A Summary

In the study of projective models of algebraic varieties it proved to be a fruitful idea to regard asymptotic invariants of line bundles with sufficiently many sections. This circle of ideas started with the work of Iitaka, and has undergone a serious development in the last fifteen years. Among the various invariants that can be associated to big divisors, a collection of convex bodies, the so-called Newton–

Okounkov bodies, play a central role.

Newton–Okounkov bodies (or Okounkov bodies for short) are universal among the numerical invariants of big divisors in that two big line bundles sharing the same collection of Okounkov bodies (in a sense that will soon be made precise) are numerically equivalent [45]. As such, Newton–Okounkov bodies contain a vast amount of information about the numerical equivalence class of a given divisor; currently research is at a stage where one is trying to find new means to extract fundamental geometric data from them.

The precise procedure that tells us how to associate a convex body to an ample divisor on a projective variety equipped with a complete flag of subvarieties was worked out by Andrei Okounkov in his influential articles [72] and [73]. In this process he was helped by earlier work of Beilinson and Parshin. His method was later generalized to big divisors by Lazarsfeld–Mustaţă [63] and even to the context of non-complete varieties by Kaveh–Khovanskii [48].

There has been a flurry of activities in recent years around Newton–Okounkov bodies, which goes beyond the territory of complex geometry. By now relationship has been established with representation theory, convex geometry, topology, and the theory of completely integrable systems. In this writing we will focus on certain developments in complex geometry.

To be more specific, let $X$ be an irreducible projective variety of dimension $n$, $Y_\bullet$ a complete flag of irreducible subvarieties, and $D$ a big divisor on $X$. The Newton–Okounkov body associated with $D$ and $Y_\bullet$ is a convex subset $\Delta_{Y_\bullet}(D) \subset \mathbb{R}^n$. Lazarsfeld and Mustaţă in [63] extended this notion to $\mathbb{R}$-divisors, and observed their basic properties, the most important of which we summarize now.

1. The Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ depends only on the numerical equivalence class of $D$.

2. For any non-negative real number $\alpha$, we have
$$\Delta_{Y_\bullet}(\alpha D) = \alpha \cdot \Delta_{Y_\bullet}(D) .$$

3. $\mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D)) = n! \cdot \mathrm{vol}_X(D)$, where the latter is the volume of $D$ as defined in Definition 2.10.

4. As $D$ runs through all big $\mathbb{R}$-divisor classes, the associated Newton–Okounkov bodies fit together into a convex cone, the Okounkov cone $\Delta_{Y_\bullet}(X)$ of the variety $X$.

Our main results center around the question how the behaviour of $D$ and $Y_\bullet$ influence the geometry of the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$. First we give a fairly complete description of Okounkov bodies on surfaces.

By considering the relationship between Okounkov bodies and variation of Zariski decomposition [11], the authors of [63] proved that $\Delta_{Y_\bullet}$ exhibits a locally polygonal behaviour. Pushing this analysis further, we obtained the following results.

**Theorem 1.1.** (*Küronya–Lozovanu–Maclean, Theorem 1.19*) *The Okounkov body of a big divisor $D$ on a smooth projective surface $S$ is a convex polygon with rational slopes.*

*A rational polygon $\Delta \subseteq \mathbb{R}^2$ is up to translation the Okounkov body $\Delta(D)$ of a divisor D on some smooth projective surface S equipped with a complete flag $(C,x)$ if and only if the following set of conditions is met.*

*There exists a rational number $\mu > 0$, and $\alpha, \beta$ piecewise linear functions on $[0,\mu]$ such that*

1. $\alpha \leq \beta$,
2. $\beta$ *is a convex function,*
3. $\alpha$ *is increasing, concave and* $\alpha(0) = 0$*;*

*moreover*

$$\Delta = \left\{(t,y) \in \mathbb{R}^2 \,|\, 0 \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\right\} .$$

**Theorem 1.2.** (*Anderson–Küronya–Lozovanu, Theorem 1.21*) *Let X be a smooth projective surface, D a pseudo-effective divisor on X. Then there exists an admissible flag $Y_\bullet$ on X with respect to which the Newton–Okounkov body of D is a rational polygon.*

In higher dimensions much of the research revolves around finitely generated divisors. A very exiting open question posed by Lazarsfeld and Mustaţă is whether every smooth Fano variety possesses an admissible flag with respect to which the global Okounkov cone of the variety is finite rational polyhedral. Our first result in the direction shows that one needs to exercise serious caution regarding the flag even in the presence of strong global finite generation.

**Theorem 1.3.** (*Küronya–Lozovanu–Maclean, Theorem 1.33*) *There exists a three-dimensional Mori dream space Z and an admissible flag $Y_\bullet$ on Z, such that the Okounkov body of a general ample divisor is non-polyhedral, and remains so after generic deformations of the flag in its linear equivalence class.*

On the other hand, we also prove one of the expected ingredients for an affirmative answer to the Lazarsfeld–Mustaţă question: finite generation of the section ring implies the existence of rational polyhedral Okounkov bodies.

**Theorem 1.4.** (*Anderson–Küronya–Lozovanu, Theorem 1.28*) *Let X be a normal complex projective variety, L a big Cartier divisor on X. Assume that D is either nef, or its section ring is finitely generated. Then there exists an admissible flag $Y_\bullet$ on X such that the Newton–Okounkov body $\Delta_{Y_\bullet}(L)$ is a rational simplex.*

### 1.B Definition and basic properties

The idea of Newton–Okounkov bodies is in essence a bookkeeping process, which keeps track of vanishing properties of global sections of multiples of a certain Cartier divisor. Here is a simple example of how to do this, which already contains all the essential features.

**Example 1.5** ($\mathcal{O}(1)$ on $\mathbb{P}^2$)**.** We consider $\mathbb{P}^2$, and the line bundle $\mathcal{O}(1)$. In order to be able to describe the Newton–Okounkov body associated to $\mathcal{O}(1)$ (and some additional data), we will need to regard global sections of all powers of it. As it is well-known, elements of $H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)\right)$ correspond to degree $d$ homogeneous polynomials in the variables $x_0$,$x_1$, and $x_2$.

The plan is to look at the vanishing behaviour of these global sections along the line $Y_1 \stackrel{\text{def}}{=} V(x_0)$, and at the point $Y_2 \stackrel{\text{def}}{=} V(x_0,x_1)$ considered as a subvariety of $Y_1$. Let $\phi \in H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)\right)$ be a non-zero section. Then

$$\phi = x_0^{s_1}\phi_1 \ ,$$

where $0 \leq s_1 \leq d$ is the order of vanishing of $\phi$ along the line $Y_1$, and $\phi_1$ is a homogeneous polynomial which does not vanish identically along $Y_1$.

As a consequence, we can restrict $\phi_1$ to $Y_1$, where we obtain a homogeneous polynomial of degree $d - s_1$ in the coordinates $x_1$ and $x_2$, denoted by $\phi_1|_{Y_1}$. Next, let the order of vanishing of $\phi_1|_{Y_1}$ at $Y_2$ be $s_2$. We associate to $\phi$ the point

$$\nu_{Y_\bullet}(\phi) \stackrel{\text{def}}{=} \tfrac{1}{d}(s_1,s_2) \in \mathbb{R}^2 \ .$$

By repeating this process for all non-zero global sections of all the bundles $\mathcal{O}_{\mathbb{P}^2}(d)$, we obtain the set

$$\bigcup_{d=1}^{\infty} \bigcup_{\phi \in H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)\right)^\times} \nu_{Y_\bullet}(\phi) = \left\{(a,b) \in \mathbb{Q}^2 \,|\, a,b \geq 0\,, a+b \leq 1\right\} \subseteq \mathbb{R}^2 \ .$$

The set of points in the case $d = 2$ is shown in Figure 1.

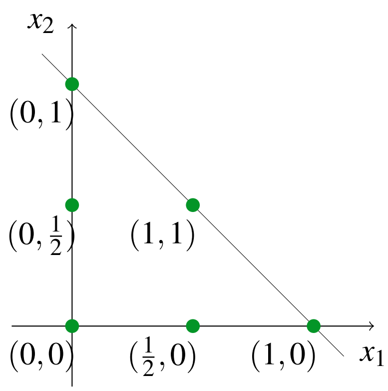



Figure 1: Normalized vanishing vectors for $d = 2$

The closed convex hull of the above is the Newton–Okounkov body $\Delta_{Y_\bullet}(\mathcal{O}(1))$ of $\mathcal{O}(1)$ with respect to the flag of subvarieties $\mathbb{P}^2 = Y_0 \supset Y_1 \supset Y_2$; this is the simplex in $\mathbb{R}^2$ give by the inequalities (see Figure 2.)

$$x \geq 0\,, y \geq 0\,, x+y \leq 1 \ .$$

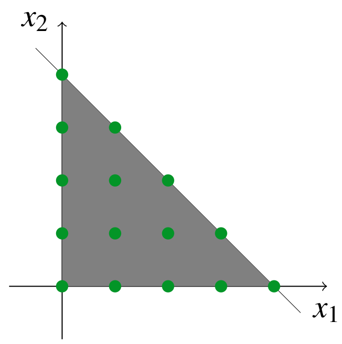



Figure 2: Normalized vanishing vectors and the Newton–Okounkov body of $\mathcal{O}_{\mathbb{P}^2(1)}$

**Remark 1.6.** We would like to use this opportunity to point a simple but vital fact. Let $S$ be a smooth projective surface, $C \subseteq S$ a curve, $p \in C$ a point. If $D$ is an effective divisor, $s \in H^0(S, \mathcal{O}_S(D))$, then in general

$$\mathrm{ord}_p^C s|_C > \mathrm{ord}_p^X s + \mathrm{ord}_C p .$$

In particular, it does matter whether we consider the order of vanishing of $s$ on $S$ or on $C$.

The convex body $\Delta_{Y_\bullet}(D)$ encodes a large amount of information on the asymptotic behaviour of the complete linear system $|D|$. In the landmark paper [63], Lazarsfeld and Mustaţă extend Okounkov's construction to big divisors, which we now briefly summarize, and prove various properties of these convex bodies. Note that Kaveh and Khovanskii [48] obtained similar results in a slightly less restrictive setting.

We start with a projective variety $X$ of dimension $n$ defined over the complex numbers (note that an uncountable algebraically closed field of arbitrary characteristic would suffice). Fix a complete flag of irreducible subvarieties

$$X = Y_0 \supset Y_1 \supset \ldots \supset Y_{n-1} \supset Y_n = \mathrm{pt}$$

with $Y_i$ of codimension $i$ in $X$. In addition we require that all elements of the flag be smooth at the point $Y_n$. We refer to such a flag as admissible.

For a given big Cartier divisor $D$ the choice of the flag determines a rank $n = \dim X$ valuation

$$\begin{aligned} \nu_{Y_\bullet,D}\colon H^0(X, \mathcal{O}_X(D)) \setminus \{0\} &\longrightarrow \mathbb{Z}^n \\ s &\mapsto \nu(s) \stackrel{\mathrm{def}}{=} (\nu_1(s), \ldots, \nu_n(s)) , \end{aligned}$$

where the values of the $\nu_i(s)$'s are defined in the following manner. We set

$$\nu_1(s) \stackrel{\mathrm{def}}{=} \mathrm{ord}_{Y_1}(s) .$$

Dividing $s$ by a local equation of $Y_1$, we obtain a section $\tilde{s_1} \in H^0(X, D - \nu_1(s)Y_1)$ not vanishing identically along $Y_1$. This way, upon restricting to $Y_1$, we arrive at a non-zero section

$$s_1 \in H^0(Y_1, (D - \nu_1(s)Y_1)|_{Y_1}) .$$

Then we write

$$\nu_2(s) \stackrel{\text{def}}{=} \text{ord}_{Y_2}(s_1) \ .$$

Continuing in this fashion, we can define all the integers $\nu_i(s)$. The image of the function $\nu_{Y_\bullet,D}$ in $\mathbb{Z}^n$ is denoted by $\nu(D)$. With this in hand, we define the *Newton–Okounkov body of D with respect to the flag* $Y_\bullet$ to be

$$\Delta_{Y_\bullet}(D) \stackrel{\text{def}}{=} \text{ the convex hull of } \bigcup_{m=1}^{\infty} \frac{1}{m}\cdot \nu(mD) \subseteq \mathbb{R}^n \ .$$

In the context of complex geometry the usual terminology for $\Delta_{Y_\bullet}(D)$ is 'Okounkov body'.

**Remark 1.7.** The above construction is similar to one we meet in toric geometry, where one associates a rational polytope $P_D$ to a torus-invariant Cartier divisor $D$ on the variety $X$. In this setting, working with a flag of torus-invariant subvarieties of the toric variety, one recovers $P_D$ as a translate of $\Delta_{Y_\bullet}(D)$. Analogous polyhedra on spherical varieties have been considered in [17], [71], [1], [46]. For a remarkable relationship with geometric representation theory, where Littelmann's string polytopes are realized as Okounkov bodies, the reader is invited to look at [47].

**Remark 1.8.** The formation of Okounkov bodies is feasible in wider context. As it is shown in [63, Section 2], one can obtain analogous results for graded linear series. In this case one does not need to assume the completeness of the underlying variety.

Let $X$ be an irreducible quasi-projective variety, $D$ a Cartier divisor on $X$. A graded linear series associated to $D$ is a collection of finite-dimensional subspaces

$$V_k \subseteq H^0(X, \mathcal{O}_X(kD))$$

satisfying the following multiplicativity property: for natural numbers $k,l$, let $V_k \cdot V_l$ denote the image of the vector space $V_k \otimes V_l$ under the multiplication maps

$$H^0(X, \mathcal{O}_X(kD)) \otimes H^0(X, \mathcal{O}_X(lD)) \longrightarrow H^0(X, \mathcal{O}_X((k+l)D) \ .$$

In this notation, we require that

$$V_k \cdot V_l \subseteq V_{k+l}$$

for all $k,l \in \mathbb{N}$.

In other words, the direct sum $\oplus_{k\in\mathbb{N}} V_k$ is a graded $\mathbb{C}$-subalgebra of the ring of sections $R(X,D)$.

By replacing the spaces of global section with the $V_k$'s in the definition of the Okounkov body, we arrive at an analogous construction for graded linear series. This we denote by $\Delta_{Y_\bullet}(V_\bullet)$.

**Remark 1.9.** Let $D_1$ and $D_2$ be two Cartier divisors on $X$. The map

$$H^0(X, \mathcal{O}_X(D_1)) \otimes H^0(X, \mathcal{O}_X(D_2)) \longrightarrow H^0(X, \mathcal{O}_X(D_1 + D_2))$$

induced by multiplication of sections is compatible with the valuations $\nu_{Y_\bullet}$. As a consequence,

$$\Delta_{Y_\bullet}(D_1)+\Delta_{Y_\bullet}(D_2) \subseteq \Delta_{Y_\bullet}(D_1+D_2) .$$

The Brunn–Minkowski inequality from convex geometry implies that

$$\Delta_{Y_\bullet}(D_1)^{\frac{1}{n}}+\Delta_{Y_\bullet}(D_2)^{\frac{1}{n}} \leq \Delta_{Y_\bullet}(D_1+D_2)^{\frac{1}{n}} ,$$

in other words, the Euclidean volume of Newton–Okounkov bodies is log-concave.

**Example 1.10.** In dimension one, for a divisor $D$ of degree $d$ on a smooth curve, the Okounkov body $\Delta_{Y_\bullet}(D)$ with respect to an arbitrary flag (which amounts to fixing a point on the curve) is the line segment $[0,d] \subseteq \mathbb{R}$ (see [63, Example 1.13]).

**Remark 1.11.** The shape of an Okounkov body depends a lot on the choice of the flag. The illustration below shows two Okounkov bodies of an ample divisor on an abelian surface.

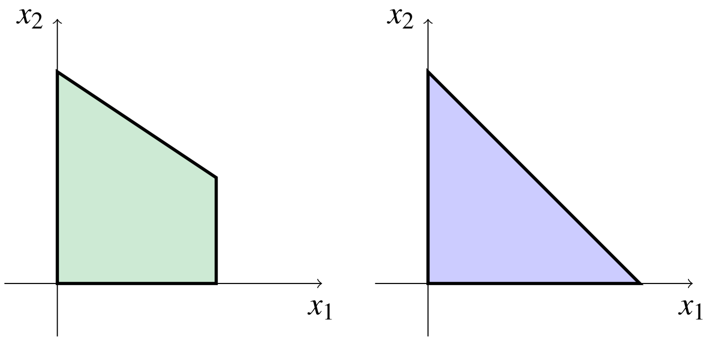



Figure 3: Two Okounkov bodies of an ample divisor on an abelian surface.

It is an interesting question with no answer so far whether one really needs to consider the closure of the convex hull of the set of normalized valuation vectors or it suffices to close up this set. The next result points in the direction that the formation of the convex hull might not be necessary.

**Proposition 1.12.** (*Küronya–Maclean–Szemberg,[58]*) *Let $X$ be a smooth projective variety, $A$ an ample divisor on $X$. Fix an admissible flag $Y_\bullet$ on $X$ with the property that every subvariety $Y_i$ is smooth. Then the set of normalized valuation vectors is dense in $\Delta_{Y_\bullet}(A)$.*

*Proof.* For simplicity we present the argument when $X$ is a curve. In this case the Okounkov body $\Delta_{Y_\bullet}(A)$ is the interval $[0,c]$, with $c=\deg(A)$, as shown in [63, Example 1.13]. As Okounkov bodies scale well when taking multiplies, we may assume that $p \in X$ is not a base point of $A$ ($A$ is very ample for instance). Let $t$ be a section in $H^0(X,\mathcal{O}_X(D))$ non-vanishing at $p$.

Let $\frac{r}{q}$ be a rational number in the open interval $(0,c)$. There exist a positive integer $k$ and a section $s \in H^0(X,\mathcal{O}_X(kA))$ satisfying

$$\frac{\operatorname{ord}_p(s)}{k} > \frac{r}{q}. \tag{1}$$

We claim that there exist positive integers $\alpha$ and $\beta$ such that $\frac{r}{q}$ is the normalized valuation vector of the section $s^{\alpha}t^{\beta} \in H^0(X, \mathcal{O}_X((\alpha k+\beta)A))$. In other words, we claim that

$$\frac{\mathrm{ord}(s^{\alpha}t^{\beta})}{\alpha k+\beta} = \frac{r}{q}\ .$$

This holds for example for $\alpha = r$ and $\beta = (\mathrm{ord}_p(s)q - rk)$. Note that $\beta$ is positive by (1). $\square$

**Remark 1.13.** One can also look for examples when forming the convex hull of the normalized valuation vectors suffices. From the lack of a limiting procedure one would then expect some sort of a finite generation phenomenon (cf. [2, Lemma 2.2]).

**Example 1.14.** A simple concrete example of an Okounkov body is the one associated to $\mathcal{O}(1)$ on $\mathbb{P}^n$ and a complete flag of linear subspaces. In this case the function $\nu$ on the sections of $\mathcal{O}(m)$ gives the lexicographic order with respect to an ordering of the variables; the Okounkov body of a divisor in $\mathcal{O}(1)$ turns out to be the standard $n$-dimensional simplex

$$x_1,\dots,x_n \geq 0\,, x_1+\cdots+x_n \leq 1\ .$$

**Example 1.15** (Okounkov bodies of $\mathbb{P}^2$ and its blow up)**.** Let $\ell$ be a line in $\mathbb{P}^2$ and $P_0 \in \ell$ a point. In what follows we operate with the flag

$$Y_\bullet:\ X_0 = \mathbb{P}^2 \supset \ell \supset \{P_0\}\,.$$

a) Let $\mathcal{O}_{\mathbb{P}^2}(D_0) = \mathcal{O}_{\mathbb{P}^2}(2)$. Then $\Delta_{Y_\bullet}(D_0)$ is twice the standard simplex in $\mathbb{R}^2$ as shown in Figure 4.

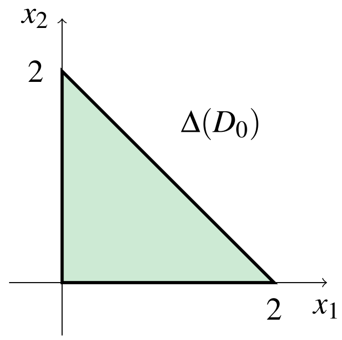



Figure 4: The Okounkov body of $\mathcal{O}_{\mathbb{P}^2}(2)$

b) Let $P_1$ be a point in the plane not lying on the line $\ell$ and let $f_1 : X_1 = \mathrm{Bl}_{P_1}\mathbb{P}^2 \to \mathbb{P}^2$ be the blow up of $P_1$ with exceptional divisor $E_1$. For $D_1 = f_1^*\mathcal{O}_{\mathbb{P}^2}(2) - E_1$, we have

Standard procedures in the theory of asymptotic invariants of line bundles let one extend the construction of Newton–Okounkov bodies to divisors with rational or even real coefficients. It turns out that Okounkov bodies defined for $\mathbb{R}$-Cartier divisors fulfill the usual expectations regarding asymptotic invariants.

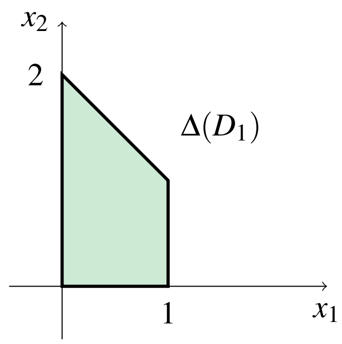



Figure 5: The Okounkov body of $\Delta_{X_1}(D_1)$.

**Theorem 1.16.** *(Lazarsfeld–Mustaţă, [63, Proposition 4.1 and Theorem 4.5]) Let $D$ be a big $\mathbb{R}$-Cartier divisor on an irreducible projective variety of dimension $n$; fix an admissible flag $Y_\bullet$ on $X$. Then*

1. *The Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ depends only on the numerical equivalence class of $D$.*

2. *For any non-negative real number $\alpha$, we have*

$$\Delta_{Y_\bullet}(\alpha D) = \alpha \cdot \Delta_{Y_\bullet}(D) .$$

The Okounkov bodies of big $\mathbb{R}$-divisor classes vary continuously, and fit together nicely to form a 'global Okounkov cone' $\Delta_{Y_\bullet}(X)$, which is an invariant of the underlying variety $X$.

It is important to note that the Okounkov body $\Delta_{Y_\bullet}(D)$ varies wildly with the flag $Y_\bullet$. In the light of this, the following result is particularly interesting.

**Theorem 1.17.** *(Lazarsfeld–Mustaţă,[63, Theorem 2.3]) Let $X$ be an irreducible projective variety with an admissible flag $Y_\bullet$, $D$ a big divisor on $X$. Then*

$$\mathrm{vol}_{\mathbb{R}^n}\left(\Delta_{Y_\bullet}(D)\right) = n! \cdot \mathrm{vol}_X\left(D\right) ,$$

The volume on the left-hand side is the Lebesgue measure on $\mathbb{R}^n$, while on the right-hand side it is the asymptotic rate of growth of the number of global sections of $D$ (see Definition 2.10):

$$\mathrm{vol}_X\left(D\right) \stackrel{\mathrm{def}}{=} \limsup_{m\to\infty} \frac{h^0\left(X, \mathcal{O}_X(mD)\right)}{m^n/n!} .$$

In particular, the volume of a Newton–Okounkov body is an invariant of the divisor, and does not depend on the chosen flag. Perhaps the first application of these invariants is a straightforward proof of the basic properties of volumes of divisors. We will discuss volumes of line bundles and asymptotic cohomology in general in more detail in Section 2.

### 1.C Newton–Okounkov bodies on smooth surfaces

One of the non-trivial illustrations of the new theory in [63] is the case of surfaces, where one can in fact explicitly determine $\Delta_{Y_\bullet}(D)$ in terms of the behaviour of linear series. Zariski decomposition of divisors provides a tool sufficiently strong for making such a concrete description viable. We will now quickly go through this process. For the definition and basic properties of Zariski decomposition, see Subsection 2.C.

A complete flag on a smooth projective surface $S$ consists of a smooth curve $C$ on $S$, and a point $x$ on $C$. Given such a surface $S$ equipped with a flag $Y_\bullet = (X \supset C \supset \{x\})$ and a $\mathbb{Q}$-divisor $D$, Lazarsfeld and Mustaţă define real numbers $\nu$ and $\mu$ by setting

$$\begin{aligned} \nu &= \text{ the coefficient of } C \text{ in the negative part of the Zariski decomposition of } D \\ \mu &= \sup\{t | D - tC \text{ is big }\} . \end{aligned}$$

Equivalently, $\nu$ is the minimal real number for which $C$ is not in the support of the negative part of the Zariski decomposition of $D - \nu C$.

As it turns out, the Okounkov body of $D$ lives over the interval $[\nu, \mu]$, and there it is described by two functions, $\alpha(t)$ and $\beta(t)$ as follows. Let $P_t \stackrel{\text{def}}{=} P_{D-tC}$ and $N_t \stackrel{\text{def}}{=} N_{D-tC}$ be the positive and negative parts of the Zariski decomposition of $D - tC$, respectively. By setting

$$\begin{aligned} \alpha(t) &= \operatorname{ord}_x(N_t|_C) , \\ \beta(t) &= \operatorname{ord}_x(N_t|_C) + (P_t \cdot C) \end{aligned}$$

Lazarsfeld and Mustaţă make the following observation ([63, Theorem 6.4]): the Okounkov body $\Delta(D)$ is given by the inequalities

$$\Delta_{Y_\bullet}(D) = \{(t,y) \in \mathbb{R}^2 | \nu \le t \le \mu, \alpha(t) \le y \le \beta(t)\} .$$

As a consequence of [11, Theorem 1.1], they conclude in particular that $\alpha$ and $\beta$ are both piecewise linear and rational on any interval $[\nu, \mu']$ where $\mu' < \mu$.

**Remark 1.18.** By taking a sufficiently general point $x \in C$, one can arrange for $\alpha$ to be identically zero.

Our first result in the theory of Okounkov bodies is a substantial sharpening of the Lazarsfeld–Mustaţă statement for surfaces; in fact we show that Okounkov bodies on smooth projective surfaces are always polygons. In a way this comes as a surprise, since the Zariski chamber structure given in [11] which constitutes the basis of the piecewise polygonal behaviour established in [63]) is only locally finite, and even that only in the interior of the effective cone. The extra strength comes from Fujita's extension of Zariski decomposition to pseudo-effective divisors that are potentially not effective.

We also give a complete characterization of the rational convex polygons which can appear as Okounkov bodies of surfaces. The precise statement from [55] goes as follows.

**Theorem 1.19.** (*Küronya–Lozovanu–Maclean,[55]*) *The Okounkov body of a big divisor D on a smooth projective surface S is a convex polygon with rational slopes.*

*A rational polygon $\Delta \subseteq \mathbb{R}^2$ is up to translation the Okounkov body $\Delta(D)$ of a divisor D on some smooth projective surface S equipped with a complete flag $(C,x)$ if and only if the following set of conditions is met.*

*There exists a rational number $\mu > 0$, and $\alpha, \beta$ piecewise linear functions on $[0,\mu]$ such that*

1. $\alpha \leq \beta$,

2. $\beta$ *is a convex function,*

3. $\alpha$ *is increasing, concave and* $\alpha(0) = 0$;

*moreover*
$$\Delta = \left\{(t,y) \in \mathbb{R}^2 \,|\, 0 \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\right\} .$$

As mentioned above, the proof relies on a study of the variation of Zariski decomposition of the divisors $D - tC$. In the other direction, we prove the existence of $\mathbb{R}$-divisors with a given Okounkov body by a direct computation on toric varieties.

The existence and uniqueness of Zariski decompositions for pseudo-effective divisors on smooth projective surfaces has the consequence that, from the point of view of asymptotic invariants, all pseudo-effective divisors behave as if they had a finitely generated section ring. Although this is clearly not the case, an arbitrary pseudo-effective divisor certainly has a rational volume for instance (see 2.46).

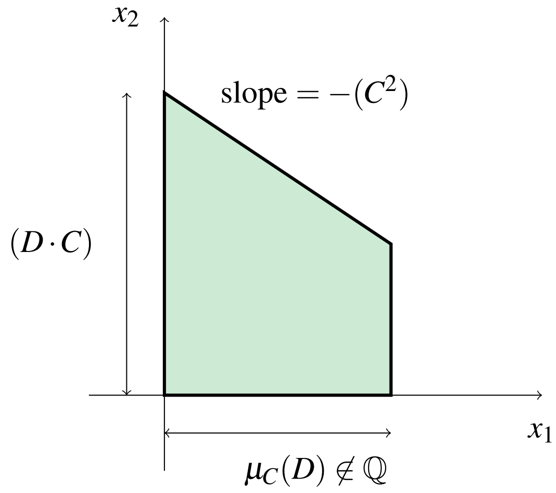
Thus, the rationality of Okounkov bodies on surfaces promises to be a very interesting question. It does not take long to come up with examples where the Okounkov body (with respect to a suitable flag) is not a rational polygon; the prototypical example is given in the Introduction of [63].

**Example 1.20.** Let $X$ be an abelian surface without complex multiplications, thus having Picard number $\rho(X) = 3$, and fix an ample curve $C \subseteq X$ along with a smooth point $x \in C$. This gives rise to an admissible flag

$$X \supseteq C \supseteq \{x\} .$$

For an ample divisor $D$ on $X$, we denote by $\mu = \mu_C(D) \in \mathbb{R}$ the smallest root of the quadratic polynomial $p(t) = (D - tC)^2$ obtained from the self-intersection numbers along the half-line $[D, D - \infty \cdot C]$.

The number $\mu_C(D)$ is irrational for infinitely many choices of $D$ and $C$. The Okounkov body of $D$ is the trapezoidal region in $\mathbb{R}^2$ determined by the two axes and the lines $x_1 = \mu_C(D)$, and $x_2 = (D \cdot C) - (C^2)t$.




The Okounkov body $\Delta_{Y_\bullet}(D)$, although polyhedral, is not rational if $\mu_C(D)$ isn't.

On the other hand, it turns out that for a judicious choice of the flag, it is always possible to obtain a rational polygonal Newton–Okounkov body. Thus, the rationality of the volume of a divisor on a surface has some kind of a structural reason from the point of view of Okounkov bodies.

**Theorem 1.21.** (*Anderson–Küronya–Lozovanu, [3]*) *Let $X$ be a smooth projective surface, $D$ a pseudo-effective divisor on $X$. Then there exists an admissible flag $Y_\bullet$ on $X$ with respect to which the Newton–Okounkov body of $D$ is a rational polygon.*

**Remark 1.22.** In the case of Example 1.20, there is an easy way to make sure that $\Delta_{Y_\bullet}(D)$ is a rational polygon: assuming $D$ is very ample, let $C \in |D|$ be a general element, $x \in C$ a general point. With these choices, $\Delta_{Y_\bullet}(D)$ is a simplex with vertices $(0,0)$, $(0,1)$, and $((D^2),0)$. We will see in Theorem 1.28 that this picture generalizes to all dimensions.

To round off our discussion about Okounkov bodies of divisors on smooth surfaces, we present an application of Okounkov bodies, which gives arithmetic constraints on the possible values of the invariant

$$\mu_C(D) \stackrel{\text{def}}{=} \sup\{t \geq 0 \,|\, D - tC \text{ is big}\} \ .$$

When $C = K_X$ this number is called the Kodaira energy of $D$, and is studied in Mori theory.

**Proposition 1.23.** (*Küronya–Lozovanu–Maclean, [55]*) *Let $X$ be a smooth projective surface, $D$ a big divisor, $X \supset C \supset \{x\}$ an admissible flag on $X$. Then*

1. *$\mu_C(D)$ is either rational or satisfies a quadratic equation over $\mathbb{Q}$.*

2. *If an irrational number $\alpha > 0$ satisfies a quadratic equation over $\mathbb{Q}$ and the conjugate $\overline{\alpha}$ of $\alpha$ over $\mathbb{Q}$ is strictly larger than $\alpha$, then there exists a smooth, projective surface $S$, an ample $\mathbb{Q}$-divisor $D$ and an admissible flag $C \ni x$ on $S$ such that $\mu(D) = \alpha$.*

*Proof.* The number $\nu$ is rational because the positive and negative parts of the Zariski decomposition of a $\mathbb{Q}$-divisor are rational (see Proposition 2.46; consequently, the numbers $\alpha(\nu)$ and $\beta(\nu)$ are also rational. The break-points of $\alpha$ and $\beta$ occur at points $t_i$ which are intersection points between the line $D - tC$ and faces of the Zariski chamber decomposition of the cone of big divisors [11]. However, it is proved in [11, Theorem 1.1] that this decomposition is locally finite rational polyhedral, and hence the break-points of $\alpha$ and $\beta$ occur at rational points.

For (2), let us remind ourselves that the volume $\mathrm{vol}_X(D)$, which is half of the area of the Okounkov polygon $\Delta(D)$, is rational. As all the slopes and intermediate breakpoints of $\Delta(D)$ are rational, the relation computing the area of $\Delta_{Y_\bullet}(D)$ gives a quadratic equation for $\mu_C(D)$ with rational coefficients. Note that if $\mu$ is irrational then one edge of the polygon $\Delta_{Y_\bullet}(D)$ must sit on the vertical line $t = \mu$.

The final part of the Proposition is based on a result of Morrison's [69] which states that any even integral quadratic form $q$ of signature $(1,2)$ occurs as the self-intersection form of a K3 surface $S$ with Picard number 3. An argument of Cutkosky's [21, Section 3] shows that if the coefficients of the form are all divisible by 4, then the pseudo-effective and nef cones of $S$ coincide and are given by

$$\{\alpha \in N^1(S) \,|\, (\alpha^2) \geq 0,\, (h \cdot \alpha) > 0\}$$

for any ample divisor $h$ on $S$. If $D$ is an ample divisor and $C \subseteq S$ an irreducible curve (not in the same class as $D$), then the function $f(t) \stackrel{\text{def}}{=} ((D-tC)^2)$ has two positive roots and $\mu(D)$ with respect to $C$ is equal to the smaller one, i.e.

$$\mu(D) = \frac{(D \cdot C) - \sqrt{(D \cdot C)^2 - (D^2)(C^2)}}{(C^2)} .$$

Since we are only interested in the roots of $f$ we can start with any integral quadratic form of signature $(1,2)$ and multiply it by 4. Hence we can exhibit any number with the required properties as $\mu(D)$ for suitable choices of the quadratic form, $D$, and $C$. □

**Remark 1.24.** It was mentioned in passing in [63] that the knowledge of all Okounkov bodies determines Seshadri constants. In the surface case, there is a link between the irrationality of $\mu$ for certain special forms of the flag and that of Seshadri constants. Let $D$ be an ample divisor on $S$, and let $\pi : \widetilde{S} \to S$ be the blow-up of a point $x \in S$ with exceptional divisor $E$. Then the Seshadri constant of $D$ at $x$ is defined by

$$\varepsilon(D,x) \stackrel{\text{def}}{=} \sup\{t \in \mathbb{R} \,|\, \pi^*(D) - tE \text{ is nef in } \widetilde{S}\}.$$

We note that if $\varepsilon(D,x)$ is irrational, then $\varepsilon(D,x) = \mu(\pi^*(D))$ with respect to any flag of the form $(E,y)$. Indeed, the Nakai–Moishezon criterion implies that either there is a curve $C \subseteq \widetilde{S}$ such that $C \cdot (\pi^*(D) - \varepsilon E) = 0$ or $((\pi^*(D) - \varepsilon E)^2) = 0$. But since $C \cdot \pi^*(D)$ and $C \cdot E$ are both rational, $C \cdot (\pi^*(D) - \varepsilon E) = 0$ is impossible if $\varepsilon$ is irrational. Therefore $(\pi^*(D) - \varepsilon E)^2 = 0$, hence $\pi^*(D) - \varepsilon E$ is not big and therefore $\varepsilon = \mu \notin \mathbb{Q}$.

### 1.D Polyhedral behaviour in higher dimensions

It is in general quite difficult to say anything specific about asymptotic invariants of divisors on higher-dimensional varieties. In particular, there is little in the way of regularity that we can expect for Okounkov bodies with respect to randomly chosen flags. As one would expect, it is established in [63] building on a classical example of Cutkosky [20] that there exist big divisors on higher-dimensional varieties with non-polyhedral Okounkov bodies.

A question that immediately leaps to mind is the relation between divisors with finitely generated section rings, and the associated Okounkov bodies. The volume of such finitely generated divisors must always be rational, which implies that the corresponding Okounkov bodies have rational Euclidean volumes, that is, the arithmetic properties of the volume of $\Delta_{Y_\bullet}(D)$ present no obstruction to the Okounkov body being a rational polyhedron.

The general yoga of Okounkov bodies says that an awful lot depends on the flag that we pick. Again, one can see this in a clear way: on the one hand, we can always change a suitable flag for a finitely generated big divisor to obtain a rational polyhedral Newton–Okounkov body; on the other hand, even on Fano varieties, a not-too-careful choice of a flag might result in a non-polyhedral Okounkov body.

First, let us have a look at what happens in the case of very positive divisors. The following discussion is taken from [3].

**Proposition 1.25.** *Let $L$ be a very ample divisor on $X$ with vanishing higher cohomology, $E_1,\dots,E_n \in |L|$ such that the flag $Y_1 = E_1,\dots,\ Y_k = E_1\cap\cdots\cap E_k,\dots,$ $Y_n =$ a point in $E_1\cap\cdots\cap E_n$ is admissible. Then $\Delta_{Y_\bullet}(L)$ is a simplex.*

*Proof.* We proceed by induction on dimension. By taking the elements $E_i \in |L|$ to be general, the flag $Y_\bullet$ defined above will be admissible.

We observe that

$$\Delta_{Y_\bullet}(L)|_{x_1=t} = \Delta_{Y_\bullet|_{Y_1}}((L-tY_1)|_{Y_1})$$

according to [55, Proposition 3.1]. On the other hand, since $Y_1 \sim L$, we have

$$\begin{aligned}\Delta_{Y_\bullet|_{Y_1}}((L-tY_1)|_{Y_1}) &= \Delta_{Y_\bullet|_{Y_1}}((1-t)L|_{Y_1}) \\ &= (1-t)\cdot\Delta_{Y_\bullet|_{Y_1}}(L|_{Y_1})\end{aligned}$$

for $0<t<1$ by the homogeneity of Okounkov bodies [63, Proposition 4.1]. Consequently, we one obtains

$$\Delta_{Y_\bullet}(L)|_{x_1=t} = (1-t)\cdot\Delta_{Y_\bullet|_{Y_1}}(L|_{Y_1})$$

for all $0\le t\le 1$.

To be able to invoke the induction hypothesis, we need to verify that $L|_{Y_1}$ and the flag $Y_\bullet|_{Y_1}$ satisfy the hypotheses of the Proposition. This is fine, however, since $L|_{Y_1}$ remains very ample,

$$E_2|_{E_1}, E_3|_{E_1}, \dots, E_n|_{E_1}$$

are sections of $L|_{Y_1}$, and

$$Y_\bullet|_{Y_1} = Y_2 \supset Y_3 \supset \dots \supset Y_n$$

remains admissible.

This way we have managed to reduce the Proposition to the case of a curve. Assume now that $\dim X = 1$, $L$ is a very ample line bundle on $X$, and $Y_\bullet = X \supset E_n$, where $E_n$ is a point on $X$, and $L$ is linearly equivalent to $c \cdot E_n$, with $c$ a positive integer. By [63, Example 1.13],

$$\Delta_{Y_\bullet}(L) = [0, c \cdot \deg L] ,$$

which finishes the proof. □

**Remark 1.26.** By looking carefully at the above proof we obtain a more precise statement; namely, $\Delta_{Y_\bullet}(L)$ is the simplex given by the inequalities

$$x_1 \geq 0, \dots, x_n \geq 0, x_1 + \cdots + x_{n-1} + (L^n)x_n \leq 1 .$$

As Okounkov bodies scale well, that is, $\Delta_{Y_\bullet}(p \cdot D) = p \cdot \Delta_{Y_\bullet}(D)$, the above proposition provides an answer for all ample divisors.

**Corollary 1.27.** *Let $X$ be an irreducible projective variety, $L$ an ample divisor on $X$. Then there exists an admissible flag on $X$ with respect to which the Okounkov body of $L$ is a rational simplex.*

Some further effort involving Fujita's vanishing theorem 3.10 takes us to the following result.

**Theorem 1.28.** (*Anderson–Küronya–Lozovanu,[3]*) *Let $X$ be a normal complex projective variety, $L$ a big Cartier divisor on $X$. Assume that $D$ is either nef, or its section ring is finitely generated. Then there exists an admissible flag $Y_\bullet$ on $X$ such that the Newton–Okounkov body $\Delta_{Y_\bullet}(L)$ is a rational simplex.*

**Remark 1.29.** It follows from Theorem 1.21 (there are various ways to see this) that it can easily happen for a divisor with a non-finitely-generated section ring to have a rational polyhedral Okounkov body. The example we have in mind is Zariski's construction of a big and nef but not semi-ample divisor (for a neat account see [61, Section 2.3.A]); which then automatically fails to be finitely generated. On the other hand, by being big and nef, such a divisor will always possess Newton–Okounkov bodies that are rational polygons.

Note that in higher dimensions there exist big divisors $D$ for which $\Delta_{Y_\bullet}(D)$ cannot be a rational polytope. The reason is that the Euclidean volume of a rational polytope is always rational, while one can construct big divisors on varieties of dimension three or more with irrational volume.

Fano varieties enjoy strong finite generation properties, which guarantee that all previously known asymptotic invariants behave in a 'rational polyhedral way' on them. In their breakthrough paper on the finite generation of the canonical ring (see [13], also [19] for a considerably simpler proof) Birkar, Cascini, Hacon, and McKernan also prove that the Cox ring of a Fano variety is finitely generated. Consequently, one could hope that Okounkov bodies associated to divisors on Fano varieties turn out to be rational polytopes. This, however, is not the case, as the following result of [56] shows.

**Theorem 1.30.** (*Küronya–Lozovanu–Maclean, [56]*) *There exists a Fano fourfold $X$ equipped with a flag $X = Y_0 \supset Y_1 \supset Y_2 \supset Y_3 \supset Y_4$ and an ample divisor $D$ on $X$, such that the Okounkov body of $D$ with respect to the flag $Y_\bullet$ is not a polyhedron.*

The proof uses a description of slices of Okounkov bodies, and the structure of these three-dimensional slices. The following two results formulate the necessary results.

**Proposition 1.31.** (*Küronya–Lozovanu–Maclean, [55]*) *Let $X$ be a smooth projective variety of dimension $n$ equipped with an admissible flag $Y_\bullet$. Suppose that $D$ is a divisor such that $D - sY_1$ is ample. Then we have the following lifting property*

$$\Delta_{Y_\bullet}(X;D) \cap \left(\{s\} \times \mathbb{R}^{n-1}\right) = \Delta_{Y_\bullet}(Y_1;(D-sY_1)|_{Y_1}).$$

*In particular, if $\overline{\mathrm{Eff}}(X)_{\mathbb{R}} = \mathrm{Nef}(X)_{\mathbb{R}}$ then on setting $\mu(D;Y_1) = \sup\{\, t > 0 \mid D - tY_1 \text{ ample } \}$ we have that the Okounkov body $\Delta_{Y_\bullet}(X;D)$ is the closure in $\mathbb{R}^n$ of the following set*

$$\{(s,\underline{v}) \mid 0 \leq s < \mu(D;Y_1), \underline{v} \in \Delta_{Y_\bullet}(Y_1;(D-sY_1)|_{Y_1}\}$$

**Proposition 1.32.** (*Küronya–Lozovanu–Maclean, [55]*) *Let $X$ be a smooth threefold and $Y_\bullet = (X,S,C,x)$ an admissible flag on $X$. Suppose that $\overline{\mathrm{Eff}}(X)_{\mathbb{R}} = \mathrm{Nef}(X)_{\mathbb{R}}$ and $\overline{\mathrm{Eff}}(S)_{\mathbb{R}} = \mathrm{Nef}(S)_{\mathbb{R}}$. The Okounkov body of any ample divisor $D$ with respect to the admissible flag $Y_\bullet$ can be described as follows*

$$\Delta_{Y_\bullet}(X;D) \;=\; \{(r,t,y) \in \mathbb{R}^3 \mid 0 \leq r \leq \mu(D;S), 0 \leq t \leq f(r), 0 \leq y \leq g(r,t)\;\}$$

*where $f(r) = \sup\{\, s > 0 \mid (D-rS)|_S - sC \text{ is ample } \}$ and $g(r,t) = (C.(D-rS)|_S) - t(C^2)$. (All intersection numbers in the above formulae are defined with respect to the intersection form on $S$.)*

*Proof of Theorem 1.30.* We set $X = \mathbb{P}^2 \times \mathbb{P}^2$ and let $D$ be a divisor in the linear series $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(3,1)$. We set

$$Y_\bullet \;:\; Y_0 = \mathbb{P}^2 \times \mathbb{P}^2 \;\supseteq\; Y_1 = \mathbb{P}^2 \times E \;\supseteq\; Y_2 = E \times E \;\supseteq\; Y_3 = C \;\supseteq\; Y_4 = \{\mathrm{pt}\}$$

where $E$ is a general elliptic curve. Since $E$ is general we have that

$$\overline{\mathrm{Eff}}(E \times E)_{\mathbb{R}} = \mathrm{Nef}(E \times E)_{\mathbb{R}} = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z \geq 0, xy+xz+yz \geq 0\}$$

under the identification

$$\mathbb{R}^3 \to N^1(E \times E)_{\mathbb{R}},\; (x,y,z) \to xf_1 + yf_2 + z\Delta_E,$$

where $f_1 = \{\mathrm{pt}\} \times E$, $f_2 = E \times \{\mathrm{pt}\}$ and $\Delta_E$ is the diagonal divisor. Let $C \subseteq E \times E$ be a smooth general curve in the complete linear series $|f_1 + f_2 + \Delta_E|$ and let $Y_4$ be a general point on $C$. To prove that the Okounkov body $\Delta_{Y_\bullet}(X;D)$ is not polyhedral it will be enough to prove that the slice $\Delta_{Y_\bullet}(X;D) \cap \{0 \times \mathbb{R}^2\}$ is not polyhedral. Since $\overline{\mathrm{Eff}}(\mathbb{P}^2 \times \mathbb{P}^2)_{\mathbb{R}} = \mathrm{Nef}(\mathbb{P}^2 \times \mathbb{P}^2)_{\mathbb{R}}$, Proposition 1.31 applies and it will be enough to show that $\Delta_{Y_\bullet}(Y_1;\mathcal{O}_{Y_1}(D))$ is not polyhedral.

The threefold $Y_1 = \mathbb{P}^2 \times E$ is homogeneous, so its nef cone is equal to its pseudo-effective cone: this cone is bounded by the rays $\mathbb{R}_+[\text{line} \times E]$ and $\mathbb{R}_+[\mathbb{P}^2 \times \{\text{pt}\}]$. We note that hypotheses of Proposition 1.32 therefore apply to $Y_1$ equipped with the flag $(Y_2, Y_3, Y_4)$.

Using the explicit description given above of $\mathrm{Nef}(Y_1)_{\mathbb{R}}$, we see that

$$\mu(\mathcal{O}_{Y_1}(D), Y_2) = 1 .$$

A simple calculation gives us

$$g(r,t) = (C.(D|_{Y_1} - rY_2)|_{Y_2}) - t(C^2) = 24 - 18r - 6t.$$

Let us now consider

$$\begin{aligned} f(r) &= \sup\{s > 0 \mid (D - rY_2)|_{Y_2} - sC \text{ is ample}\} \\ &= \sup\{s > 0 \mid (9 - 9r - s)f_1 + (3 - s)f_2 - s\Delta_E \text{ is ample }\}. \end{aligned}$$

After calculation, we see that for positive $s$ the divisor $(9 - 9r - s)f_1 + (3 - s)f_2 - s\Delta_E$ is ample if and only if $s < (4 - 3r - \sqrt{9r^2 - 15r + 7})$. Proposition 1.32 therefore tells us that the Okounkov body of $D$ on $Y_1$, $\Delta_{Y_\bullet}(Y_1; D)$, has the following description

$$\{(r,t,y) \in \mathbb{R}^3 | 0 \le r \le 1, 0 \le t \le 4 - 3r - \sqrt{9r^2 - 15r + 7}, 0 \le y \le 24 - 18r - 6t\}.$$

As this body is non-polyhedral, the same can be said about the Okounkov body $\Delta_{Y_\bullet}(X; D)$. $\square$

The above construction describes the behaviour of ample divisors for a specific flag, but does not reveal much about generic choices. Our next result shows, that provided one is willing to weaken the positivity hypothesis somewhat, and settle for $-K_X$ big and nef, we can obtain (see [55]) a specimen with strong generic non-regularity. The example in question is a Mori dream space, which means in particular that it still has a finitely generated Cox ring.

**Theorem 1.33.** (*Küronya–Lozovanu–Maclean, [55]*) *There exists a three-dimensional Mori dream space $Z$ and an admissible flag $Y_\bullet$ on $Z$, with the property that the Okounkov body of a general ample divisor is non-polyhedral and remains so after generic deformations of the flag in its linear equivalence class.*

The construction that leads to this statement is based on a work of Cutkosky [21], where he produces a quartic surface $S \subseteq \mathbb{P}^3$ such that the nef and effective cones of $S$ coincide and are round. The Néron-Severi space $N^1(S)$ of this quartic surfaces is isomorphic to $\mathbb{R}^3$ with the lattice $\mathbb{Z}^3$ and the intersection form $q(x,y,z) = 4x^2 - 4y^2 - 4z^2$. In particular he shows

1. The divisor class on $S$ represented by the vector $(1,0,0)$ corresponds to the class of a very ample line bundle, which embeds $S$ in $\mathbb{P}^3$ as a quartic surface.

2. The nef and pseudo-effective cones of $S$ coincide, and a vector $(x,y,z) \in \mathbb{R}^3$ represents a nef (pseudo-effective) class if it satisfies the inequalities

$$4x^2 - 4y^2 - 4z^2 \ge 0 , \; x \ge 0 .$$

We consider the surface $S \subset \mathbb{P}^3$, and the pseudo-effective classes on $S$ given by $\alpha = (1,1,0)$ and $\beta = (1,0,1)$. By the Riemann–Roch theorem we have that $H^0(S,\alpha) \geq 2$ and $H^0(S,\beta) \geq 2$, so both $\alpha$ and $\beta$, being extremal rays in the effective cone, are classes of irreducible moving curves. Since $\alpha^2 = \beta^2 = 0$, both these families are base-point free, and it follows from the base-point free Bertini theorem that there are smooth irreducible curves $C_1$ and $C_2$ representing $\alpha$ and $\beta$ respectively, which are elliptic by the adjunction formula. We may assume that $C_1$ and $C_2$ meet transversally in $C_1 \cdot C_2 = 4$ points.

Our threefold $Z$ is constructed as follows. Let $\pi_1 : Z_1 \to \mathbb{P}^3$ be the blow-up along the curve $C_1 \subseteq \mathbb{P}^3$. We then define $Z$ to be the blow up of the strict transform $\overline{C}_2 \subseteq Z_1$ of the curve $C_2$. Let $\pi_2 : Z \to Z_1$ be the second blow-up and $\pi$ the composition $\pi_1 \circ \pi_2 : Z \to \mathbb{P}^3$. We denote by $E_2$ the exceptional divisor of $\pi_2$ and by $E_1$ the strict transform of the exceptional divisor of $\pi_1$ under $\pi_2$.

Then the variety $Z$ defined above is a Mori dream space, and $-K_Z$ is effective. Given any two ample divisors on $Z$, $L$ and $D$, such that the classes $[D],[L],[-K_Z]$ are linearly independent in $N^1(Z)_{\mathbb{R}}$, the Okounkov body $\Delta_{Y_\bullet}(X;D)$ is non-polyhedral with respect to any admissible flag $(Y_1,Y_2,Y_3)$ such that $\mathcal{O}_Z(Y_1) = -K_Z$, $\mathrm{Pic}(Y_1) = \langle H, C_1, C_2\rangle$ and $\mathcal{O}_{Y_1}(Y_2) = L|_{Y_1}$, where $H$ is the pullback of a hyperplane section of $\mathbb{P}^3$ by the map $\pi$.

**Remark 1.34.** The advantage of this example over the Fano one is that it does not depend on a choice of flag elements which are exceptional from a Noether-Lefschetz point of view. (Note that by standard Noether-Lefschetz arguments the condition that $\mathrm{Pic}(Y_1) = \langle H, C_1, C_2\rangle$ holds for any very general $Y_1$ in $|-K_Z|$.)

In particular, the previous example depended upon the fact that $Y_2$ had a non-polyhedral nef cone, which in this case was possible only because $Y_2$ had Picard group larger than that of $X$: moreover, it was necessary to take $Y_3$ to be a curve not contained in the image of the Picard group of $Y_1$. It is to a certain extent less surprising that choosing flag elements in $\mathrm{Pic}(Y_i)$ that do not arise by restriction of elements in $\mathrm{Pic}(Y_{i-1})$ should lead to bad behaviour in the Okounkov body. There does not seem to be any reason why the fact that $X$ is Fano should influence the geometry of the boundary of the part of the nef cone of $Y_i$ which does not arise by restriction from $X$.

Moreover, such behaviour cannot be general, so there is little hope of using such examples to construct a counterexample to [63, Problem 7.1].

# 2 Asymptotic cohomology and countability questions regarding linear series

## 2.A Summary

The volume of a Cartier divisor $D$ on an irreducible projective variety $X$ of dimension $n$ quantifies the asymptotic rate of growth of the number of global sections of $mD$ as we take higher and higher multiples; more precisely, it defined as

$$\mathrm{vol}_X(D) \stackrel{\mathrm{def}}{=} \limsup_{m\to\infty} \frac{h^0(X,\mathcal{O}_X(mD))}{m^n/n!} .$$

Volumes of Cartier divisors enjoy many unexpected formal properties, in particular, they give rise to a homogeneous continuous function from $N^1(X)_{\mathbb{R}}$ to the non-negative real numbers. For a complete account, the reader is invited to look at [61] and the recent paper [63].

The notion of the volume first arose implicitly in Cutkoskys work [20], where he used the irrationality of the volume to establish the non-existence of birational Zariski decompositions with rational coefficients. It was then studied subsequently by Demailly, Ein, Fujita, Lazarsfeld, and others, while pioneering efforts regarding other asymptotic invariants of linear systems were made by Nakayama, and Tsuji.

An important application of this circle of ideas is present in the recent work of Hacon–McKernan–Xu [39] where the authors give an estimate

$$|\operatorname{Bir}(X)| \leq C_{\dim X} \cdot \operatorname{vol}_X(K_X)$$

on the size of the birational automorphism group of varieties of general type in terms of the volume of the canonical divisor.

The volume function is log-concave, which – according to the influential paper [73] of Okounkov – is an indication that it is a good notion of multiplicity. In a different direction, Demailly, Ein and Lazarsfeld [29] showed that the volume of a divisor is the normalized upper limit of the moving self-intersection numbers of its multiples. The analogous notion on compact Kähler manifolds has been studied by Boucksom [15].

Asymptotic cohomology functions of divisors are direct generalizations of the volume function: the $i^{\text{th}}$ asymptotic cohomology of an integral Cartier divisor on $X$ is defined to be

$$\widehat{h}^i(X,D) \stackrel{\text{def}}{=} \limsup_{m\to\infty} \frac{h^i(X,\mathcal{O}_X(mD))}{m^n/n!} .$$

This concept has been extended by Demailly [26] to certain Bott–Chern cohomology classes on compact complex manifolds and subsequently applied to obtain a converse to the Andreotti–Grauert vanishing theorem [27] in the surface case. Totaro links the vanishing of higher asymptotic cohomology to $q$-ampleness; this circle of ideas will be discussed further in Section 3.

Our first main result is that asymptotic cohomology functions exhibit almost the same formal behaviour as volumes.

**Theorem 2.1** (Küronya, Proposition 2.14 and Theorem 2.15)**.** *For all $0 \leq i \leq n$ the $i^{th}$ asymptotic cohomology is homogeneous of degree n, invariant with respect to numerical equivalence of divisors, and extends uniquely to a continuous function*

$$\widehat{h}^i(X,\cdot) : N^1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}_{\geq} .$$

Asymptotic cohomology satisfies a simplified version of Serre duality.

**Proposition 2.2** (Küronya, Theorem 2.20)**.** *With notation as above,*

$$\widehat{h}^i(X,D) = \widehat{h}^{n-i}(X,-D) .$$

On certain classes of varieties with additional structure, notable examples being complex abelian varieties and generalized flag varieties, the cohomology of line bundles exhibits an interesting chamber structure. The Néron–Severi space $N^1(X)_{\mathbb{R}}$ is divided into a set of open cones, and for the integral points in each such cone there is a single non-vanishing cohomology group.

Typical examples of this phenomenon are Mumfords index theorem for complex abelian varieties, and the Borel–Weil–Bott theorem on homogeneous spaces. The behaviour of cohomology on these spaces is suggestively similar to the behaviour of the volume function on smooth projective surfaces.

**Theorem 2.3** (Bauer–Küronya–Szemberg, Theorem 2.29)**.** *Let $X$ be smooth projective surface. Then there exists a locally finite rational polyhedral decomposition of the cone of big divisors such that on each of the resulting chambers the support of the negative part of the Zariski decomposition is constant. In particular, on every such region, all asymptotic cohomology functions are given by homogeneous polynomials.*

One obtains a comparable chamber structure on toric varieties (Hering–Küronya–Payne,[42]), by considering the maximal regions in the Néron–Severi space where some of the asymptotic cohomology functions are given by a single polynomial. The decomposition for the volume coincides with the Gelfand–Kapranov–Zelevinsky decomposition of the fan of the variety.

Going one step further, one obtains an analogous finite rational polyhedral decomposition on a Mori dream space via the volume function. As a consequence of the work of Birkar–Cascini–Hacon–McKernan [13] this is the case on Fano varieties as well. Interestingly enough, it is not known in this case what happens for the higher asymptotic cohomology.

Perhaps the first application of higher asymptotic cohomology is an asymptotic converse to Serre's vanishing theorem.

**Theorem 2.4** (de Fernex–Küronya–Lazarsfeld, Theorem 2.21)**.** *A numerical equivalence class $\xi_0 \in N^1(X)_{\mathbb{R}}$ is ample if and only if*

$$\widehat{h}^i(\xi) \;=\; 0$$

*for all $i > 0$ and all $\xi \in N^1(X)_{\mathbb{R}}$ in a small neighborhood of $\xi_0$ with respect to the Hausdorff topology of the real vector space $N^1(X)_{\mathbb{R}}$.*

In the toric case we obtain a somewhat stronger statement.

**Theorem 2.5** (Hering–Küronya–Payne)**.** *Let $D$ be a divisor on an $n$-dimensional complete simplicial toric variety $X$. Then $D$ is ample if and only if*

$$\mathrm{vol}_X(\xi) \;=\; (\xi^n)$$

*for all $\mathbb{R}$-divisor classes $\xi$ in a neighborhood of $D$ in $N^1(X)_{\mathbb{R}}$.*

Next, we will investigate the properties of real numbers that occur as volumes of divisors. It is a simple consequence of Zariski decomposition that the volume of a divisor on a smooth surface is always a rational number. In higher dimension the same observation holds under the assumption that the section ring of the divisor is finitely generated.

In his work on the non-existence of birational Zariski decomposition in higher dimensions, Cutkosky produced a big integral divisor with a non-rational but algebraic volume. Nevertheless, this still leaves a fairly large room as far as the cardinality and arithmetic properties of volumes are concerned.

**Theorem 2.6** (Küronya–Lozovanu–Maclean, Theorem 2.51)**.** *The set of volumes of integral divisors on an $n$-dimensional irreducible projective variety is countable, and contains transcendental numbers.*

It is important to note, that the occurrence of transcendental numbers in the set of volumes is not an isolated incident. In fact, one can construct examples, where, in the absence of finite generation, the volume function is given by a transcendental function over an open subset of the Néron–Severi space. As far as countability goes, one can even obtain a stronger statement.

**Theorem 2.7** (Küronya–Lozovanu–Maclean, Theorem 2.51)**.** *Let $V_{\mathbb{Z}} = \mathbb{Z}^{\rho}$ be a lattice inside the vector space $V_{\mathbb{R}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. Then there exist countably many functions $f_j : V_{\mathbb{R}} \to \mathbb{R}_+$ with $j \in \mathbb{N}$, so that for any irreducible projective variety $X$ of dimension $n$ and Picard number $\rho$, we can construct an integral linear isomorphism*

$$\rho_X : V_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$$

*with the property that* $\mathrm{vol}_X \circ \rho_X = f_j$ *for some* $j \in \mathbb{N}$.

It then follows that up to integral linear equivalence the cardinality of possible ample cones/big cones/effective cones is also countable.

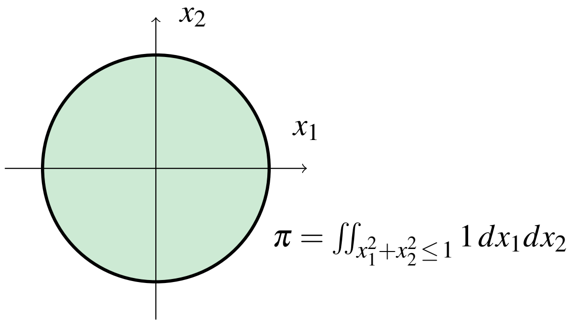
We point out that the above countability statements are not formal, in other words, they only hold in the complete case. In the non-complete setting, mostly anything satisfying a few formal requirements can occur.

**Theorem 2.8** (Küronya–Lozovanu–Maclean, Theorem 2.55)**.** *Let $K \subseteq \mathbb{R}^{\rho}_{+}$ be a closed convex cone with non-empty interior, $f : K \to \mathbb{R}_+$ a continuous function, which is non-zero, homogeneous, and log-concave of degree $n$ in the interior of $K$. Then there exists a smooth, projective variety $X$ of dimension $n$ and Picard number $\rho$, a multigraded linear series $W_\bullet$ on $X$ and a positive constant $c > 0$ such that* $\mathrm{vol}_{W_\bullet} \equiv c \cdot f$ *on the interior of $K$. Moreover we have* $\mathrm{supp}(W_\bullet) = K$.

In order to be able to say more about the arithmetic properties of volumes, we will focus on a certain important type of transcendental numbers. Recall that in number theory, a period is a real or complex number that can be realized as the integral of a rational function with rational coefficients over a semi-algebraic domain in Euclidean space.

Periods form a countable $\mathbb{Q}$-subalgebra, which includes many famous transcendental numbers like $\pi$ or the base $e$ logarithms of natural numbers.

$$\iint_{x_1^2+x_2^2 \leq 1} 1\, dx_1 dx_2 = \pi \quad , \quad \int_1^n \frac{1}{x}\, dx = \log_e n \,.$$




All real numbers that have been confirmed to be volumes of divisors are periods. In the light of Theorem 1.17, which says

$$\operatorname{vol}_X(D) \stackrel{\text{def}}{=} \limsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}$$

this is not necessarily surprising. The connection between volumes and Okounkov bodies supported by computational evidence raises the following exciting question.

**Question 2.9.** Are all volumes periods?

## 2.B Definition and basic properties of asymptotic cohomology

Let $X$ be an irreducible projective variety, $D$ a divisor on $X$. The idea of looking at the way the number of global sections of $\mathcal{O}_X(mD)$ grows with $m$ first arose in a work of Cutkosky as a tool for creating an obstacle for the existence of birational Zariski decomposition in higher dimensions. The plan was simple: the existence of a birational Zariski decomposition implies that the asymptotic rate of growth of the number of global section of $\mathcal{O}_X(mD)$ (which is now called to volume of $D$) must be a rational number; hence, to create a counterexample to the existence problem it suffices to manufacture a big divisor with irrational volume. This was in turn achieved by Cutkosky in [20].

To be more concrete, here is the formal definition.

**Definition 2.10.** The *volume of a Cartier divisor $D$* on an $n$-dimensional projective variety $X$ is defined as

$$\operatorname{vol}_X(D) \stackrel{\text{def}}{=} \limsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!} \ .$$

The same notion can be introduced for cohomology groups of arbitrary degrees.

**Definition 2.11.** Let $X$ be an $n$-dimensional irreducible projective variety, $D$ a Cartier divisor on $X$. Then the $i^{th}$ *asymptotic cohomology of $D$* is given by

$$\widehat{h}^i(X, D) \stackrel{\text{def}}{=} \limsup_{m\to\infty} \frac{h^i(X, \mathcal{O}_X(mD))}{m^n/n!} \ .$$

**Remark 2.12** (Riemann–Roch problem)**.** The so-called Riemann–Roch problem is one of the oldest and most difficult questions in algebraic geometry. It asks to describe the function

$$m \mapsto h^0(X, \mathcal{O}_X(mD))$$

for a Cartier divisor on a smooth projective variety.

It has a very satisfactory answer in dimensions one and two; for curves it comes in the form of the Riemann–Roch theorem: for a divisor $D$ on a smooth curve $C$, one has

$$h^0(X, \mathcal{O}_X(mD)) = \begin{cases} m \cdot \deg_C D + 1 - g(C) & \text{if } m \cdot \deg_C D > 2g(C) - 2\ , \\ \text{a periodic function of } m & \text{if } \deg_C D = 0\ , \\ 0 & \text{if } \deg_C D < 0\ . \end{cases}$$

Building on earlier work of Castelnuovo, in his seminal paper [79] Zariski proved an analogue of the above statement for surfaces. He showed that for an effective divisor on a smooth projective surface one has a quadratic polynomial $P(x)$ such that

$$h^0(X, \mathcal{O}_X(mD)) = P(m) + \lambda(m) \quad \text{for } m \gg 0\ ,$$

where $\lambda(x)$ is a bounded function.

Zariski even made the surprising conjecture that $\lambda(x)$ should always be a periodic function. This turn of events is somewhat unexpected, since the section ring

$$R(X,D) \stackrel{\text{def}}{=} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mD))$$

can often turn out to be a non-finitely-generated $\mathbb{C}$-algebra, as had been shown by himself.

Finally, Cutkosky and Srinivas (see [22]) verified Zariski's conjecture. The proof is difficult, it uses some ideas from Zariski and non-trivial material about group schemes.

In higher dimensions we are not aware that even a conjectural answer would exist. In some sense, the study of the volume of divisors is a first order approximation of the Riemann–Roch problem.

**Remark 2.13.** By now it is a well-known fact that all volumes are limits, that is,

$$\mathrm{vol}_X(D) = \lim_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}\ .$$

The first proof used Fujita's approximation theorem ([61, 11.4.4], a non-trivial piece of birational geometry. As a consequence, for a while, the general impression was that this result is of projective geometric nature.

One of the important feature of the theory of Newton–Okounkov bodies (see [48, 63]) is the realization that the convergence of the sequence

$$\frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}$$

is a purely formal fact, which has mainly to do with properties of semigroups of lattice points.

Note that it is an interesting open question whether the sequences

$$\frac{h^i(X, \mathcal{O}_X(mD))}{m^n/n!}$$

converge for $1 \leq i \leq n = \dim X$.

As the Riemann–Roch function $m \mapsto h^0(X, \mathcal{O}_X(mD))$ is extremely difficult to handle, it is a pleasant surprise that asymptotic cohomology of divisors behaves in a much more controlled manner.

**Proposition 2.14** (Küronya [60], Lazarsfeld [61] for $i = 0$)**.** *Let X be an irreducible projective variety of dimension n, D an arbitrary Cartier divisor on X, $0 \leq i \leq n$ an integer. Then*

1. $\widehat{h}^i(X, aD) = a^n \cdot \widehat{h}^i(X, D)$ *for all natural numbers a;*
2. *if $D'$ is numerically equivalent to D, then* $\widehat{h}^i(X, D') = \widehat{h}^i(X, D)$*;*

It follows from the Proposition that one can define the asymptotic cohomology of a numerical equivalence class $\alpha \in N^1(X)$ by setting it equal to the common value at all representatives. Also, using the homogeneity of asymptotic cohomology, if $\alpha \in N^1(X)_{\mathbb{Q}}$, and $m \cdot \alpha$ is integral for some $m \in \mathbb{N}$, then

$$\widehat{h}^i(X, \alpha) \stackrel{\text{def}}{=} \frac{1}{m^n} \cdot \widehat{h}^i(X, m \cdot \alpha)$$

defines the asymptotic cohomology of $\alpha$ in a way which does not depend on $m$.

As we have seen, one can make sense of the asymptotic cohomology as a function $\widehat{h}^i : N^1(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ while retaining the properties obtained in 2.14. In fact, we can go one step further.

**Theorem 2.15** (Küronya [60], Lazarsfeld [61] for $i = 0$)**.** *With notation as above, the functions $\widehat{h}^i : N^1(X)_{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ extend uniquely to a locally Lipschitz continuous function on $N^1(X)_{\mathbb{R}}$. This latter is the $i^{th}$ asymptotic cohomology function of the variety X.*

There is a pronounced difference between the case of the volume function and higher asymptotic cohomology functions: for a non- big divisor, the volume is zero, while this is not so in general for the higher asymptotic cohomology functions. Therefore, higher asymptotic cohomology carries information about non-effective divisors as well.

**Remark 2.16** (Asymptotic cohomology of ample/nef/big divisors)**.** We determine the asymptotic cohomology of nef divisors. In doing so we rely on the asymptotic version of the Riemann–Roch Theorem [51, Theorem 2.15], which says in our case that for a nef Cartier divisor $D$

$$h^i(X, \mathcal{O}_X(mD)) = \begin{cases} \frac{(D^n)}{n!} \cdot m^n + O(m^{n-1}) & \text{if } i = 0 \\ C \cdot m^{n-1} & \text{if } i > 0\,. \end{cases}$$

Consequently, if $D$ is nef, then

$$\widehat{h}^i(X,D) = \begin{cases} (D^n) & \text{if } i = 0 , \\ 0 & \text{if } i > 0 \end{cases} .$$

By definition a Cartier divisor $D$ is big if $\kappa(X,D) = \dim X$. Once we know that volumes are limits (for example from Theorem 1.17), it follows that

$$\widehat{h}^0(X,D) = \mathrm{vol}_X(D) > 0 .$$

Since the volume is a continuous function, it makes sense to define an $\mathbb{R}$-Cartier divisor to be big if its volume is positive.

The higher asymptotic cohomology of a big divisor $D$ reveals a fair amount of information about its geometry (see for instance Proposition 2.17 or Theorem 2.21). Not surprisingly, this also means that it is difficult to give general statements about it. Nevertheless, it is true that

$$\widehat{h}^n(X,D) = 0$$

whenever $D$ is big. This follows for example from the asymptotic version of Serre duality (Theorem 2.20) and the fact the the cone of big divisors on a projective variety is pointed.

**Proposition 2.17** (Stable base loci and vanishing of asymptotic cohomological functions)**.** *Let $D$ be a big divisor on a smooth variety $X$ with a $d$-dimensional stable base locus. Then*

$$\widehat{h}^i(X,\mathcal{O}_X(D)) = 0$$

*for all $i > d$.*

*Proof.* Let $Z_m \subseteq X$ denote the subscheme defined by the asymptotic multiplier ideal sheaf $\mathcal{J}(\|mD\|)$. Then $\dim Z_m \leq d$ for $m$ large. Consider the exact sequence

$$0 \to \mathcal{J}(\|mD\|) \otimes \mathcal{O}_X(K_X+mD) \to \mathcal{O}_X(K_X+mD) \to \mathcal{O}_{Z_m}(K_X+mD) \to 0 .$$

A form of Nadel vanishing for asymptotic multiplier ideals (cf. [61, Section 11.2.B]) says that for $i > 0$

$$H^i(X,\mathcal{J}(\|mD\|) \otimes \mathcal{O}_X(K_X+mD)) = 0 .$$

Therefore, if $i > 0$ then

$$H^i(X,\mathcal{O}_X(K_X+mD)) = H^i(X,\mathcal{O}_{Z_m}(K_X+mD)) .$$

But $H^i(X,\mathcal{O}_{Z_m}(K_X+mD)) = 0$ for $m \gg 0$, as $Z_m$ is a $d$-dimensional scheme in this case. Hence

$$h^i(X,\mathcal{O}_X(K_X+mD)) = 0$$

for $m \gg 0$. This implies

$$\widehat{h}^i(X,\mathcal{O}_X(D)) = \limsup_m \frac{h^i(X,\mathcal{O}_X(K_X+mD))}{m^n/n!} = 0 ,$$

as required. $\square$

**Remark 2.18** (Künneth formulas for asymptotic cohomology, [11])**.** Let $X_1, X_2$ be irreducible projective varieties of dimensions $n_1$ and $n_2$, $D_1, D_2$ Cartier divisors on $X_1$ and $X_2$, respectively. Then

$$h^i\left(X_1 \times X_2, \pi_1^* D_1 \otimes \pi_2^* D_2\right) \leq \binom{n_1+n_2}{n_1} \sum_{i=j+k} \widehat{h}^j\left(X_1, D_1\right) \cdot \widehat{h}^k\left(X_2, D_2\right).$$

for all $i$'s, where $\pi_l$ denotes the projection map to $X_l$ ($l = 1,2$). Furthermore, we have equality in the case $i = 0$, as volumes are limits.

Asymptotic cohomology behaves well with respect to pullbacks.

**Proposition 2.19** (Asymptotic cohomology of pullbacks)**.** *Let $f: Y \to X$ be a proper surjective map of irreducible projective varieties with* $\dim X = n$*, $D$ a divisor on $X$.*

1. *If $f$ is generically finite with* $\deg f = d$*, then*
$$\widehat{h}^i\left(Y, f^* \mathcal{O}_X(D)\right) = d \cdot \widehat{h}^i\left(X, \mathcal{O}_X(D)\right).$$
2. *If* $\dim Y > \dim X = n$*, then*
$$\widehat{h}^i\left(Y, f^* \mathcal{O}_X(D)\right) = 0$$
*for all $i$'s.*

A useful application is the version of Serre duality for asymptotic cohomology. Interesting features are the disappearance of the canonical bundle, and the complete lack of regularity assumptions on $X$.

**Theorem 2.20** (Asymptotic Serre duality)**.** *Let $X$ be an irreducible projective variety of dimension $n$, $D$ a divisor on $X$. Then for every* $0 \leq i \leq n$

$$\widehat{h}^i(X, D) = \widehat{h}^{n-i}(X, -D).$$

*Proof.* Let $f: Y \to X$ a resolution of singularities of $X$. Then by Proposition 2.19 we have

$$\widehat{h}^i\left(Y, f^* \mathcal{O}_X(D)\right) = \widehat{h}^i\left(X, \mathcal{O}_X(D)\right),$$

and

$$\widehat{h}^{n-i}\left(Y, f^* \mathcal{O}_X(-D)\right) = \widehat{h}^{n-i}\left(X, f^* \mathcal{O}_X(-D)\right).$$

Serre duality on the smooth variety $Y$ gives

$$h^i\left(Y, f^* \mathcal{O}_X(mD)\right) = h^{n-i}\left(Y, K_Y \otimes f^* \mathcal{O}_X(-mD)\right)$$

for every $m \geq 1$. Then

$$\limsup_m \frac{h^{n-i}\left(Y, K_Y \otimes f^* \mathcal{O}_X(-mD)\right)}{m^n/n!} = \widehat{h}^{n-i}\left(Y, f^* \mathcal{O}_X(-D)\right),$$

therefore,

$$\widehat{h}^i\left(Y, f^* \mathcal{O}_X(D)\right) = \widehat{h}^{n-i}\left(Y, f^* \mathcal{O}_X(-D)\right).$$

This implies

$$\widehat{h}^i\left(X, \mathcal{O}_X(D)\right) = \widehat{h}^{n-i}\left(X, \mathcal{O}_X(-D)\right).$$

□

Serre's vanishing theorem is a fundamental building block of projective algebraic geometry. It is known that it converse does not hold. However, one can use asymptotic cohomology to establish a result, which serves as a converse asymptotically.

**Theorem 2.21** (de Fernex–Küronya–Lazarsfeld)**.** *Let $X$ be a projective variety, $D$ be a Cartier divisor on $X$. Assume that there exists a very ample divisor $A$ on $X$ and a number $\varepsilon > 0$ such that*

$$\widehat{h}^i(X, D - tA) \;=\; 0 \quad \text{for all} \;\; i > 0\,,\; 0 \leq t < \varepsilon.$$

*Then $D$ is ample.*

Here we outline the rough idea of the proof. First, one can quickly reduce to the normal case, where we have access to more geometric methods (cf. 2.22). The plan of the proof is to study asymptotic cohomology via restrictions to general complete intersection subvarieties, and use induction on dimension.

Specifically, we choose a sequence of very general divisors

$$E_1\,, E_2\,, \ldots \;\in\; |A|\;;$$

we will restrict to various intersections of these. Given $m, p > 0$ we take the first $p$ of the $E_\alpha$ and form the complex $K^\bullet_{m,p}$ :

$$\mathcal{O}_X(mL) \longrightarrow \overset{p}{\oplus}\, \mathcal{O}_{E_\alpha}(mL) \longrightarrow \overset{\binom{p}{2}}{\oplus}\, \mathcal{O}_{E_\alpha \cap E_\beta}(mL) \longrightarrow \ldots\,, \tag{2}$$

obtained as a twist of the $p$-fold tensor product of the one-step complexes

$$\mathcal{O}_X \longrightarrow \mathcal{O}_{E_\alpha}\,.$$

It is established in [60, Corollary 4.2] that $K^\bullet_{m,p}$ is acyclic, and hence resolves $\mathcal{O}_X(mL - pA)$. In particular,

$$H^r\left(X, \mathcal{O}_X(mL - pA)\right) \;=\; \mathbb{H}^r\big(K^\bullet_{m,p}\big). \tag{3}$$

The hypercohomology group on the right in (3) is in turn computed by a first-quadrant spectral sequence with

$$E_1^{i,j} \;=\; \begin{cases} H^j\Big(\mathcal{O}_X(mL)\Big) & i = 0 \\ \overset{\binom{p}{i}}{\oplus} H^j\big(\mathcal{O}_{E_{\alpha_1} \cap \ldots \cap E_{\alpha_i}}(mL)\big) & i > 0. \end{cases} \tag{4}$$

As shown in [61, 2.2.37] or [60, Section 5], for very general choices the dimensions of all the groups appearing on the right in (4) are independent of the particular divisors $E_\alpha$ that occur.

The actual proof consists of a careful analysis of the above spectral sequence and and the following geometric observation.

**Proposition 2.22** (de Fernex–Küronya–Lazarsfeld)**.** *Let $D$ be a divisor on a normal projective variety $X$, and denote by*

$$\mathfrak{b}(|pD|) \subseteq \mathcal{O}_X$$

*the base-ideal of the linear series in question. If $D$ is not nef, then there exist positive integers $q$ and $c$, and an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ vanishing on a set of dimension $\geq 1$, such that*

$$\mathfrak{b}(|mqD|) \ \subseteq \ \mathfrak{a}^{m-c}$$

*for all $m > c$.*

One of the interesting points leading to the Theorem is to check that the vanishing hypothesis of the theorem descends to very general divisors in $|A|$. Since the proof of this statement is characteristic to the arguments that we use, we present it in detail.

**Lemma 2.23.** *With notation as above, assume that there is a positive real number $\varepsilon > 0$ such that*

$$\widehat{h}^i\left(X, L-tA, \right) = \ 0 \ \ \textit{for all} \ \ i > 0 \ , \ 0 \leq t < \varepsilon.$$

*Let $E \in |A|$ be a very general divisor. Then*

$$\widehat{h}^i(E, (L-tA)|_E) \ = \ 0 \ \ \textit{for all} \ \ i > 0 \ , \ 0 \leq t < \varepsilon.$$

*Proof.* Let us assume that

$$\widehat{h}^i\left(X, L-tA, \right) = \ 0 \ \ \text{for all} \ \ i > 0 \ , \ 0 \leq t < \varepsilon$$

for certain $\varepsilon > 0$. Then it suffices to show

$$\widehat{h}^i\left(E, L_E, \right) = \ 0 \ \ \text{for all} \ \ i > 0. \tag{5}$$

For then the more general statement of the Lemma follows (via the homogeneity and continuity of the higher asymptotic cohomology functions on $X$ and on $E$) upon replacing $L$ by $L - \delta A$ for a rational number $0 < \delta < \varepsilon$.

Suppose then that (5) fails, and consider the complex $K^{\bullet}_{m,p}$. We compute a lower bound on the dimension of the group $E^{1,i}_{\infty}$ in the hypercohomology spectral sequence. Specifically, by looking at the possible maps coming into and going out from the $E^{1,i}_r$, one sees that

$$\begin{aligned} &h^{i+1}\big(X, mL - pA\big) \ + \ h^i\big(X, mL\big) \ \geq \\ &p \cdot h^i\big(\mathcal{O}_{E_1}(mL)\big) \ - \ \binom{p}{2} \cdot h^i\big(\mathcal{O}_{E_1 \cap E_2}(mL)\big) \ - \ \binom{p}{3} \cdot h^{i-1}\big(\mathcal{O}_{E_1 \cap E_2 \cap E_3}(mL)\big) \ - \ \ldots . \end{aligned}$$

Now we can find some fixed constant $C_1 > 0$ such that for all $m \gg 0$:

$$\begin{aligned} h^i\big(\mathcal{O}_{E_1 \cap E_2}(mL)\big) \ &\leq \ C_1 \cdot m^{d-2}, \\ h^{i-1}\big(\mathcal{O}_{E_1 \cap E_2 \cap E_3}(mL)\big) \ &\leq \ C_1 \cdot m^{d-3}, \text{ etc.} \end{aligned}$$

Moreover, since we are assuming for a contradiction that $\widehat{h^i}(E, L_E, >)\, 0$, we can find a constant $C_2 > 0$, together with a sequence of arbitrarily large integers $m$, such that

$$h^i\big(\mathcal{O}_{E_1}(mL)\big) \;\geq\; C_2 \cdot m^{d-1}. \tag{6}$$

Putting this together, we find that there are arbitrarily large integers $m$ such that

$$h^{i+1}\big(X, mL - pA\big) + h^i\big(X, mL\big) \;\geq\; C_3 \cdot \Big(pm^{d-1} - p^2 m^{d-2} - p^3 m^{d-3} - \dots\Big) \tag{7}$$

for suitable $C_3 > 0$. Note that this constant $C_3$ is independent of $p$. At this point, we fix a very small rational number $0 < \delta \ll 1$. By the homogeneity of $\widehat{h^i}$ on $E_1$, we can assume that the sequence of arbitrarily large values of $m$ for which (6) and (7) hold is taken among multiples of the denominator of $\delta$ (see 0.3). Then, restricting $m$ to this sequence and taking $p = \delta m$, the first term on the RHS of (7) dominates provided that $\delta$ is sufficiently small. Hence

$$h^{i+1}\big(X, mL - pA\big) + h^i\big(X, mL\big) \;\geq\; C_4 \cdot \delta m^d$$

for a sequence of arbitrarily large $m$, and some $C_4 > 0$. But this implies that

$$\widehat{h}^{i+1}(X, L - \delta A, \,) + \widehat{h^i}(X, L, \,) > \; 0,$$

contradicting the hypothesis. $\square$

The volume function can be often described in terms of some additional structure on the underlying variety. For instance, it can be explicitly determined on toric varieties in terms of the fan describing the toric variety (Hering–Küronya–Payne, see [42]); as we will see in the next subsection, on surfaces the volume function can be computed in terms of the variation of Zariski decomposition in the Néron–Severi space. In every case, the volume reveals a substantial amount of the underlying geometry.

**Remark 2.24.** (Abelian varieties) Let $X$ be a $g$-dimensional complex abelian variety, expressed as a quotient of a $g$-dimensional complex vector space $V$ by a lattice $L \subseteq V$. Line bundles on $X$ are given in terms of Appel–Humbert data, that is, pairs $(\alpha, H)$, where $H$ is a Hermitian form on $V$ such that its imaginary part $E$ is integral on $L \times L$, and

$$\begin{aligned} \alpha \;\; : \;\; & L \to U(1) \text{ is a function for which} \\ & \alpha(l_1 + l_2) = \alpha(l_1) \cdot \alpha(l_2) \cdot (-1)^{E(l_1, l_2)}\,. \end{aligned}$$

By the Appel–Humbert theorem any pair $(\alpha, H)$ determines a unique line bundle $\mathcal{L}(\alpha, H)$ on $X$, and every line bundle on $X$ is isomorphic to one of the form $\mathcal{L}(\alpha, H)$ for some $(\alpha, H)$. The Hermitian form $H$ on $V$ is the invariant two-form associated to $c_1(\mathcal{L}(\alpha, H))$.

Fix a line bundle $\mathcal{L} = \mathcal{L}(\alpha, H)$. It is called *non-degenerate*, if 0 is not an eigenvalue of $H$. For a non-degenerate line bundle $\mathcal{L}$, the number of negative eigenvalues of $H$ is referred to as the *index* of $\mathcal{L}$, which we denote by $\operatorname{ind}\mathcal{L}$. The case of a positive definite matrix $H$ corresponds to the line bundle $\mathcal{L}(\alpha, H)$ being ample.

Mumford's Index Theorem says the following: let $\mathcal{L}$ be a non-degenerate line bundle on $X$. Then

$$h^i(X,\mathcal{L}) = \begin{cases} (-1)^i\chi(\mathcal{L}) \;=\; \sqrt{\det_L E} & \text{if } i = \operatorname{ind}(\mathcal{L}) \\ 0 & \text{otherwise.} \end{cases}$$

The first Chern class of line bundles is additive, therefore $\operatorname{ind}(\mathcal{L}^{\otimes m}) = \operatorname{ind}(\mathcal{L})$ for all $m \geq 1$.

Consider $\operatorname{NonDeg}(X) \subseteq \operatorname{Pic}(X)_{\mathbb{R}}$, the cone generated by all non-degenerate line bundles. Then $\operatorname{NonDeg}(X)$ is an open cone in $\operatorname{Pic}(X)_{\mathbb{R}}$, and its complement has Lebesgue measure zero. For every $1 \leq j \leq g$, define $\mathcal{C}_j$ to be the cone spanned by non-degenerate line bundles of index $j$. Then (apart from the origin) $\operatorname{NonDeg}(X)$ is the disjoint union of $\mathcal{C}_1, \ldots, \mathcal{C}_g$. On each $\mathcal{C}_j$, we have

$$\widehat{h}^i(X,\xi) \;=\; \begin{cases} (-1)^i(\xi^n) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

This way, we obtain a finite decomposition of $N^1(X)_{\mathbb{R}}$ into a set of cones, such that on each cone, the asymptotic cohomological functions are homogeneous polynomials of degree $g$.

**Example 2.25.** We will consider a concrete example in more detail. Consider $X \stackrel{\text{def}}{=} E \times E$, the product of an elliptic curve with itself. We assume that $E$ has no complex multiplication. Fix a point $P \in E$. Then the three classes

$$e_1 = [\{P\} \times E] \;,\; e_2 = [E \times \{P\}] \;,\; \delta = [\Delta]$$

in $N^1(X)_{\mathbb{R}}$ are independent ($\Delta \subseteq E \times E$ is the diagonal) and generate $N^1(X)_{\mathbb{R}}$. The various intersection numbers among them are as follows:

$$\delta \cdot e_1 = \delta \cdot e_2 = e_1 \cdot e_2 = 1 \text{ and } e_1^2 = e_2^2 = \delta^2 = 0 \;.$$

Any effective curve on $X$ is nef, $\overline{\mathrm{NE}}(X) = \operatorname{Nef}(X)$, furthermore, a class $\alpha \in N^1(X)_{\mathbb{R}}$ is nef if and only if $\alpha^2 \geq 0$ and $\alpha \cdot h \geq 0$ for some (any) ample class $h$.

In particular, if $\alpha = x \cdot e_1 + y \cdot e_2 + z \cdot \delta$ then $\alpha$ is nef if and only if

$$xy + xz + yz \;\;\geq\;\; 0 \text{ and } x + y + z \geq 0 \;.$$

As a reference for these statements see [61], Section 1.5.B. One can see that $\operatorname{Nef}(X)$ is a circular cone inside $N^1(X)_{\mathbb{R}} \simeq \mathbb{R}^3$. By continuity, define the index of a real divisor class $\alpha \in \operatorname{NonDeg}(X)$ to be 0 if it is ample, 2 if $-\alpha$ is ample, and 1 otherwise. Then

$$\widehat{h}^i(X,\alpha) = \begin{cases} (-1)^{\operatorname{ind}(\alpha)}(\alpha^2) = (-1)^{\operatorname{ind}(\alpha)}(xy + xz + yz) & \text{if } i = \operatorname{ind}(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 2.26.** (Fano varieties) The results of Birkar–Cascini–Hacon–McKernan [13] give rise to a finite rational polyhedral chamber structure on $\operatorname{Eff}(X)$, such that each of these chambers can be realized as the (rational) pullback of a birational model of $X$. In particular, on each of these chambers the volume function is given by a homogeneous polynomial of degree $n$. More generally, it holds on all Mori dream spaces that the effective cone has a finite rational polyhedral decomposition with respect to which the volume function is piecewise polynomial.

The analogous question for $\widehat{h}^i$ is open for $i \geq 1$.

## 2.C Smooth projective surfaces

One model case for the various decompositions of the effective or big cone of a projective variety is the example of smooth projective surfaces. This has been first explored in [11]. The upshot is that by looking at the way the support of the negative part of the Zariski decomposition changes as we move through the cone of big divisors, we obtain a locally finite decomposition of $\mathrm{Big}(X)$ into locally rational polyhedral chambers. On the individual chambers the support of the stable base loci remains constant, and asymptotic cohomology functions (in particular the volume) are given by a single polynomial.

The decomposition obtained this way is quite close to the one discussed in [43], in fact, when the surface under consideration is a Mori dream space, the open parts of the chambers in the respective decompositions agree. However, it is a significance difference that here we do not rely on any kind of finite generation hypothesis, the decomposition we present always exists. The self-duality between curves and and divisors enables us to carry out this analysis with the help of linear algebra in hyperbolic vector spaces. It is this property of surfaces that makes the proofs particularly transparent.

Since we consider variation of Zariski decomposition on surfaces to be very important, we give a fairly detailed account. Most of the material presented here is borrowed freely from [11]. For reference, here is the statement of Zariski in the version generalized by Fujita.

**Theorem 2.27** (Existence and uniqueness of Zariski decompositions for $\mathbb{R}$-divisors, [50], Theorem 7.3.1)**.** *Let $D$ be a pseudo-effective $\mathbb{R}$-divisor on a smooth projective surface. Then there exists a unique effective $\mathbb{R}$-divisor*

$$N_D = \sum_{i=1}^{m} a_i N_i$$

*such that*

(i) $P_D = D - N_D$ *is nef,*

(ii) $N_D$ *is either zero or its intersection matrix* $(N_i \cdot N_j)$ *is negative definite,*

(iii) $P_D \cdot N_i = 0$ *for* $i = 1, \ldots, m$.

*Furthermore, $N_D$ is uniquely determined by the numerical equivalence class of $D$, and if $D$ is a $\mathbb{Q}$-divisor, then so are $P_D$ and $N_D$. The decomposition*

$$D = P_D + N_D$$

*is called the* Zariski decomposition *of $D$.*

First we give an example to illustrate the kind of picture we have in mind.

**Example 2.28** (Blow-up of two points in the plane)**.** Let $X$ be the blow-up of the projective plane at two points; the corresponding exceptional divisors will be denoted by $E_1$ and $E_2$. We denote the pullback of the hyperplane class on $\mathbb{P}^2$ by $L$. These divisor classes generate the Picard group of $X$ and their intersection numbers are: $L^2 = 1$, $(L.E_i) = 0$ and $(E_i.E_j) = -\delta_{ij}$ for $1 \leq i, j \leq 2$.

There are three irreducible negative curves: the two exceptional divisors, $E_1, E_2$ and the strict transform of the line through the two blown-up points, $L - E_1 - E_2$, and the corresponding hyperplanes determine the chamber structure on the big cone. They divide the big cone into five regions on each of which the support of the negative part of the Zariski decomposition remains constant.

In this particular case the chambers are simply described as the set of divisors that intersect negatively the same set of negative curves [1].

We will parametrize the chambers with big and nef divisors (in actual fact with faces of the nef cone containing big divisors). In our case, we pick big and nef divisors $A, Q_1, Q_2, L, P$ based on the following criteria:

$$\begin{array}{rcc} A & : & \text{ample} \\ Q_1 & : & (Q_1 \cdot E_1) = 0 \ , \ (Q_1 \cdot C) > 0 \text{ for all other curves} \\ Q_2 & : & (Q_2 \cdot E_2) = 0 \ , \ (Q_2 \cdot C) > 0 \text{ for all other curves} \\ L & : & (L \cdot E_1) = 0 \ , \ (L \cdot E_2) = 0 \ , \ (L \cdot C) > 0 \text{ for all other curves} \\ P & : & (P \cdot L - E_1 - E_2) = 0 \ , \ (P \cdot C) > 0 \text{ for all other curves .} \end{array}$$

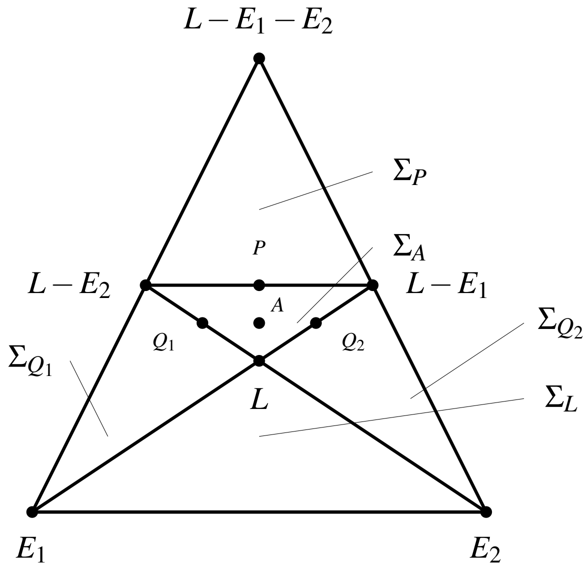
The divisors $L, P, Q_1, Q_2$ are big and nef divisors in the nef boundary (hence necessarily non-ample) which are in the relative interiors of the indicated faces. For one of these divisors, say $P$, the corresponding chamber consists of all $\mathbb{R}$-divisor classes whose negative part consists of curves orthogonal to $P$. This is listed in the following table. Observe that apart from the nef cone, the chambers do not contain the nef divisors they are associated to.

Note that not all possible combinations of negative divisors occur. This in part is accounted for by the fact that certain faces of the nef cone do not contain big divisors.

| Chamber | Supp Neg$(D)$ | $(D \cdot C) < 0$ |
|---|---|---|
| $\Sigma_A$ | $\emptyset$ | none |
| $\Sigma_{Q_1}$ | $E_1$ | $E_1$ |
| $\Sigma_{Q_2}$ | $E_2$ | $E_2$ |
| $\Sigma_L$ | $E_1, E_2$ | $E_1, E_2$ |
| $\Sigma_P$ | $L - E_1 - E_2$ | $L - E_1 - E_2$ |

The following picture describe a cross section of the effective cone of $X$ with the chamber structure indicated.

[1] It happens for instance on K3 surfaces that the negative part of a divisor $D$ contains irreducible curves intersecting $D$ non-negatively.




Let $D = aL - b_1E_i - b_2E_2$ be a big $\mathbb{R}$-divisor. Then one can express the volume of $D$ in terms of the coordinates $a, b_1, b_2$ as follows:

$$\operatorname{vol}(D) = \begin{cases} D^2 = a^2 - b_1^2 - b_2^2 & \text{if } D \text{ is nef, i.e. } D \in \Sigma_A \\ a^2 - b_2^2 & \text{if } D \cdot E_1 < 0 \text{ and } D \cdot E_2 \geq 0 \text{ i.e. } D \in \Sigma_{Q_1} \\ a^2 - b_1^2 & \text{if } D \cdot E_2 < 0 \text{ and } D \cdot E_1 \geq 0 \text{ i.e. } D \in \Sigma_{Q_2} \\ a^2 & \text{if } D \cdot E_1 < 0 \text{ and } D \cdot E_2 < 0 \text{ i.e. } D \in \Sigma_L \\ 2a^2 - 2ab_1 - 2ab_2 + 2b_1b_2 & \text{if } D \cdot (L - E_1 - E_2) < 0 \ \text{ i.e. } D \in \Sigma_P. \end{cases}$$

On an arbitrary smooth projective surface we obtain the following statement.

**Theorem 2.29.** (*Bauer–Küronya–Szemberg,[11]*) *Let $X$ be smooth projective surface. Then there exists a locally finite rational polyhedral decomposition of the cone of big divisors such that on each of the resulting chambers that support of the negative part of the Zariski decomposition is constant.*

*In particular, on every such region, all asymptotic cohomology functions are given by homogeneous polynomials, and the stable base locus is constant.*

It turns out that the existence of a locally finite polyhedral decomposition is relatively simple; most of the work is required for the rationality, and the explicit description of the chambers. For this reason, we give a separate proof for local finiteness.

First, we establish some notation. If $D$ is an $\mathbb{R}$-divisor, we will write

$$D = P_D + N_D$$

for its Zariski decomposition, and we let

$$\operatorname{Null}(D) = \{ C \,|\, C \text{ irreducible curve with } D \cdot C = 0 \}$$

and

$$\operatorname{Neg}(D) = \{ C \,|\, C \text{ irreducible component of } N_D \} .$$

It is immediate that $\operatorname{Neg}(D) \subset \operatorname{Null}(P_D)$.

Given a big and nef $\mathbb{R}$-divisor $P$, the Zariski chamber associated to $P$ is defined as

$$\Sigma_P \stackrel{\text{def}}{=} \{ D \in \mathrm{Big}(X) \mid \mathrm{Neg}(D) = \mathrm{Null}(P) \} \ .$$

**Theorem 2.30.** *The subsets $\Sigma_P \subseteq \mathrm{Big}(X)$ are convex cones. As $P$ runs through all big and nef divisors, they form a locally finite polyhedral decomposition of* $\mathrm{Big}(X)$.

*Proof.* Once we check that the decomposition of $\mathrm{Big}(X)$ into the convex $\Sigma_P$'s is locally finite, the locally polyhedral property comes for free from an observation in convex geometry (Lemma 2.31).

The convexity of the chambers $\Sigma_P$ follows from the uniqueness of Zariski decomposition: if

$$D = P + \sum_{i=1}^{r} a_i E_i \quad \text{and} \quad D' = P' + \sum_{i=1}^{r} a'_i E_i$$

are the respective Zariski decompositions of the big divisors $D$ and $D'$, then

$$D + D' = (P + P') + \sum_{i=1}^{r} (a_i + a'_i) E_i$$

is the unique way to decompose $D + D'$ satisfying all the necessary requirements of 2.27.

Next, we need to show that given a big divisor $D$, there exists a big and nef divisor $P$ with

$$\mathrm{Neg}(D) = \mathrm{Null}(P) \ .$$

Assume that $\mathrm{Neg}(D) = \{E_1, \ldots, E_r\}$, and let $A$ be an arbitrary ample divisor on $X$. We claim that a divisor $P$ as required can be constructed explicitly in the form $A + \sum_{i=1}^{k} \lambda_i C_i$ with suitable non-negative rational numbers $\lambda_i$. In fact, the conditions to be fulfilled are

$$\left( A + \sum_{i=1}^{k} \lambda_i C_i \right) \cdot C_j = 0 \qquad \text{for } j = 1, \ldots, k \ .$$

This is a system of linear equations with negative definite coefficient matrix $(C_i \cdot C_j)$, and [11, Lemma 4.1] guarantees that all components $\lambda_i$ of its solution are non-negative. In fact all $\lambda_i$'s must be positive as $A$ is ample.

Last, we verify that the decomposition is locally finite. Every big divisor has an open neighborhood in $\mathrm{Big}(X)$ of the form

$$D + \mathrm{Amp}(X)$$

for some big divisor $D$. Since

$$\mathrm{Neg}(D + A) \subseteq \mathrm{Neg}(D)$$

for all ample divisors $A$ by [11, Lemma 1.12], only finitely many chambers $\Sigma_P$ can meet this neighborhood. $\square$

**Lemma 2.31.** *Let $U \subseteq \mathbb{R}^n$ be a convex open subset, and let $\mathcal{S} = \{S_i \,|\, i \in I\}$ be a locally finite decomposition of $U$ into convex subsets. Then every point $x \in U$ has an open neighbourhood $V \subseteq U$ such that each $S_i \cap V$ is either empty, or given as the intersection of finitely many half-spaces.*

*Proof.* It is known that any two disjoint convex subsets of $\mathbb{R}^n$ can be separated by a hyperplane. Let $x \in U$ be an arbitrary point, $x \in V \subseteq U$ a convex open neighbourhood such that only finitely many of the $S_i$'s intersect $V$, call them $S_1, \dots, S_r$. For any pair $1 \leq i < j \leq r$ let $H_{ij}$ be a hyperplane separating $S_i \cap V$ and $S_j \in V$.

Then for every $1 \leq i \leq r$, $S_i \cap V$ equals the the intersection of all half-spaces $H_{ij}^+$ and $V$, as required. □

**Corollary 2.32.** *With notation as above, the part of the nef cone, which sits inside the cone of big divisors is locally polyhedral.*

**Example 2.33.** The question whether the chambers are indeed only locally finite polyhedral comes up naturally. Here we present an example with a non-polyhedral chamber. Take a surface $X$ with infinitely many $(-1)$-curves $C_1, C_2, \dots$, and blow it up at a point that is not contained in any of the curves $C_i$. On the blow-up consider the exceptional divisor $E$ and the proper transforms $C_i'$. Since the divisor $E + C_i'$ is negative definite, we can proceed to construct for every index $i$ a big and nef divisor $P_i$ with $\mathrm{Null}(P_i) = \{E, C_i'\}$, and also a divisor $P$ such that $\mathrm{Null}(P) = \{E\}$. But then $\mathrm{Face}(P)$ meets contains all faces $\mathrm{Face}(P_i)$, and therefore it is not polyhedral.

We now move on to explaining why the chamber decomposition is rational. This depends on two facts:

1. The intersection of the nef cone with the big cone is locally finite rational polyhedral.

2. If $F$ is a face of the nef cone containing a big divisor $P$, then

$$\mathrm{Big}(X) \cap \overline{\Sigma_P} = (\mathrm{Big}(X) \cap F) + \text{ the cone generated by } \mathrm{Null}(P)\ .$$

Of these, the proof of the second statement involves lengthy linear algebra computations with the help of Zariski decomposition. We will not go down this road here, a detailed proof can be found in [11, Proof of Proposition 1.8].

We will prove the first one here; it gives a strong generalization of the Campana–Peternell theorem on the structure of the nef boundary.

Denote by $\mathcal{I}(X)$ the set of all irreducible curves on $X$ with negative self-intersection. Note that if $D$ is a big divisor, then by the Hodge index theorem we have $\mathrm{Null}(D) \subset \mathcal{I}(X)$. For $C \in \mathcal{I}(X)$ denote

$$C^{\geqslant 0} \stackrel{\text{def}}{=} \left\{ D \in N^1_{\mathbb{R}}(X) \,\middle|\, D \cdot C \geq 0 \right\}\ ,$$

and

$$C^{\perp} \stackrel{\text{def}}{=} \left\{ D \in N^1_{\mathbb{R}}(X) \,\middle|\, D \cdot C = 0 \right\}\ .$$

**Proposition 2.34.** *(Bauer–Küronya–Szemberg,[11]) The intersection of the nef cone and the big cone is locally rational polyhedral, that is, for every $\mathbb{R}$-divisor $P \in \mathrm{Nef}(X) \cap \mathrm{Big}(X)$ there exists a neighborhood $U$ and curves $C_1, \ldots, C_k \in \mathcal{I}(X)$ such that*

$$U \cap \mathrm{Nef}(X) = U \cap \left( C_1^{\geqslant 0} \cap \cdots \cap C_k^{\geqslant 0} \right)$$

*Proof.* The key observation is that given a big and nef $\mathbb{R}$-divisor $P$ on $X$, there exists a neighborhood $\mathcal{U}$ of $P$ in $N^1_{\mathbb{R}}(X)$ such that for all divisors $D \in \mathcal{U}$ one has

$$\mathrm{Null}(D) \subset \mathrm{Null}(P) .$$

As the big cone is open, we may choose big (and effective) $\mathbb{R}$-divisors $D_1, \ldots, D_r$ such that $P$ lies in the interior of the cone $\sum_{i=1}^r \mathbb{R}^+ D_i$.

We can have $D_i \cdot C < 0$ only for finitely many curves $C$. Therefore, after possibly replacing $D_i$ with $\eta D_i$ for some small $\eta > 0$, we can assume that

$$(P + D_i) \cdot C > 0 \tag{$*$}$$

for all curves $C$ with $P \cdot C > 0$. We conclude then from $(*)$ that

$$\mathrm{Null}\left( \sum_{i=1}^r \alpha_i (P + D_i) \right) \subset \mathrm{Null}(P)$$

for any $\alpha_i > 0$. So the cone

$$\mathcal{U} = \sum_{i=1}^r \mathbb{R}^+ (P + D_i)$$

is a neighborhood of $P$ with the desired property.

Let now $\mathcal{U}$ be a neighborhood of $P$ as above . Observe that

$$\mathrm{Big}(X) \cap \mathrm{Nef}(X) = \mathrm{Big}(X) \cap \bigcap_{C \in \mathcal{I}(X)} C^{\geqslant 0}$$

and therefore

$$\mathcal{U} \cap \mathrm{Nef}(X) = \mathcal{U} \cap \bigcap_{C \in \mathcal{I}(X)} C^{\geqslant 0} . \tag{$*$}$$

For every $C \in \mathcal{I}(X)$ we have either $\mathcal{U} \subset C^{\geqslant 0}$, in which case we may safely omit $C^{\geqslant 0}$ from the intersection in $(*)$, or else $\mathcal{U} \cap C^{\perp} \neq \emptyset$. By our choice of $\mathcal{U}$, the second option can only happen for finitely many curves $C$.

In fact,

$$\mathcal{U} \cap \mathrm{Nef}(X) = \mathcal{U} \cap \bigcap_{C \in \mathrm{Null}(P)} C^{\geqslant 0} .$$

□

A fun application of Theorem 2.29 is the continuity of Zariski decomposition in side the big cone.

**Corollary 2.35.** (*Bauer–Küronya–Szemberg,[11, Proposition 1.14]*) *Let* $(D_n)$ *be a sequence of big divisors converging in* $N^1_{\mathbb{R}}(X)$ *to a big divisor* $D$*. If* $D_n = P_n + N_n$ *is the Zariski decomposition of* $D_n$*, and if* $D = P + N$ *is the Zariski decomposition of* $D$*, then the sequences* $(P_n)$ *and* $(N_n)$ *converge to* $P$ *and* $N$ *respectively.*

*Proof.* We consider first the case where all $D_n$ lie in a fixed chamber $\Sigma_P$. In that case we have by definition $\mathrm{Neg}(D_n) = \mathrm{Null}(P)$ for all $n$, so that

$$N_n \in \langle \mathrm{Null}(P) \rangle$$

and hence $P_n \in \mathrm{Null}(P)^\perp$. As

$$N^1_{\mathbb{R}}(X) = \mathrm{Null}(P)^\perp \oplus \langle \mathrm{Null}(P) \rangle$$

we find that both sequences $(P_n)$ and $(N_n)$ are convergent. The limit class $\lim P_n$ is certainly nef. Let $E_1, \ldots, E_m$ be the curves in $\mathrm{Null}(P)$. Then every $N_n$ is of the form $\sum_{i=1}^m a_i^{(n)} E_i$ with $a_i^{(n)} > 0$. Since the $E_i$ are numerically independent, it follows that $\lim N_n$ is of the form $\sum_{i=1}^m a_i E_i$ with $a_i \geq 0$, and hence is either negative definite or zero. Therefore $D = \lim P_n + \lim N_n$ is actually the Zariski decomposition of $D$, and by uniqueness the claim is proved.

Consider now the general case where the $D_n$ might lie in various chambers. Since the decomposition into chambers is locally finite, there is a neighborhood of $D$ meeting only finitely many of them. Thus there are finitely many big and nef divisors $P_1, \ldots, P_r$ such that

$$D_n \in \bigcup_{i=1}^r \Sigma_{P_i}$$

for all $n$. So we may decompose the sequence $(D_n)$ into finitely many subsequences to which the case above applies. $\square$

The connection between Zariski decompositions and asymptotic cohomological functions comes from the following result.

**Proposition 2.36** (Section 2.3.C., [61]). *Let* $D$ *be a big integral divisor,* $D = P_D + N_D$ *the Zariski decomposition of* $D$*. Then*

(i) $H^0(X, kD) = H^0(X, kP_D)$ *for all* $k \geq 1$ *such that* $kP_D$ *is integral, and*

(ii) $\mathrm{vol}(D) = \mathrm{vol}(P_D) = \left(P_D^2\right)$.

By homogeneity and continuity of the volume we obtain that for an arbitrary big $\mathbb{R}$-divisor $D$ with Zariski decomposition $D = P_D + N_D$ we have $\mathrm{vol}(D) = \left(P_D^2\right) = (D - N_D)^2$.

Let $D$ be an $\mathbb{R}$-divisor on $X$. In determining the asymptotic cohomological functions on $X$, we distinguish three cases, according to whether $D$ is pseudo-effective, $-D$ is pseudo-effective or none.

**Proposition 2.37.** *With notation as above, if* $D$ *is pseudo-effective then*

$$\widehat{h}^i(X, D) = \begin{cases} (P_D^2) & \text{if } i = 0 \\ -(N_D^2) & \text{if } i = 1 \\ 0 & \text{if } i = 2\,. \end{cases}$$

*If $-D$ is pseudo-effective with Zariski decomposition $-D = P_D + N_D$ then*

$$\widehat{h^i}(X,D) = \begin{cases} 0 & \text{if } i=0 \\ -(N^2_{-D}) & \text{if } i=1 \\ (P^2_{-D}) & \text{if } i=2\ . \end{cases}$$

*When neither $D$ nor $-D$ are pseudo-effective, one has*

$$\widehat{h^i}(X,D) = \begin{cases} 0 & \text{if } i=0 \\ -(D^2) & \text{if } i=1 \\ 0 & \text{if } i=2\ . \end{cases}$$

*Proof.* We treat the case of pseudo-effective divisors in detail, the case $-D$ pseudo-effective follows from Serre duality for asymptotic cohomology, while the third instance is immediate from Serre duality, asymptotic Riemann–Roch, and the fact that non-big divisors have zero volume.

Let us henceforth assume that $D$ is pseudo-effective. If $D = P_D + N_D$ is the Zariski decomposition of the pseudo-effective divisor $D$, then $\widehat{h^0}(X,D) = \left(P_D^2\right)$. Furthermore, if $D$ is pseudo-effective then $\widehat{h^2}(X,D) = 0$. In order to compute $\hat{h}^1$, consider the equality

$$h^1(X,mD) = h^0(X,mD) + h^2(X,mD) - \chi(X,mD)\ .$$

This implies that

$$\widehat{h^1}(X,D) = \limsup_m \left( \frac{h^0(X,mD)}{m^2/2} + \frac{h^2(X,mD)}{m^2/2} - \frac{\chi(X,mD)}{m^2/2} \right)\ .$$

All three sequences on the right-hand side are convergent. The $h^0$ sequence by the fact that the volume function is in general a limit. The $h^2$ sequence converges by $\widehat{h^2}(X,D) = 0$. Finally, the convergence of the sequence of Euler characteristics follows from the Asymptotic Riemann–Roch theorem. Therefore the lim sup on the right-hand side is a limit, and $\widehat{h^1}(X,D) = -(N_D^2)$. □

Applying Theorem 2.29 we arrive at the following.

**Theorem 2.38.** *With notation as above, there exists a locally finite decomposition of* $\mathrm{Big}(X)$ *into rational locally polyhedral subcones such that on each of those the asymptotic cohomological functions are given by a single homogeneous quadratic polynomial.*

We illustrate the theory outlined above by determining the volume functions of del Pezzo surfaces. Note that by Proposition 2.37 this will then provide an answer for all asymptotic cohomology functions.

Let us establish some notation. We denote by $X = Bl_\Sigma(\mathbb{P}^2)$ the blow-up of the projective plane at $\Sigma \subseteq \mathbb{P}^2$ where $\Sigma$ consists of at most eight points in general position. The exceptional divisors corresponding to the points in $\Sigma$ are denoted by $E_1, \ldots, E_r$ $(r \leq 8)$. We denote the pullback of the hyperplane class on $\mathbb{P}^2$ by $L$. These divisor classes generate the Picard group of $X$ and their intersection numbers are $L^2 = 1$, $(L.E_i) = 0$ and $(E_i.E_j) = -\delta_{ij}$ for $1 \leq i,j \leq r$. For each $1 \leq r \leq 8$ one can describe explicitly all extremal rays on $X$ (see [30]).

**Proposition 2.39.** *With notation as above, the set $\{E^{\perp} | E \in \mathcal{I}\}$ determines the chambers for the volume function. More precisely, we obtain the chambers by dividing the big cone into finitely many parts by the hyperplanes $E^{\perp}$.*

*Proof.* Observe that as the only negative curves on a del Pezzo surface are $(-1)$-curves, the support of every negative divisor consists of pairwise orthogonal curves. This can be seen as follows. Take a negative divisor $N = \sum_{i=1}^{m} a_i N_i$. Then, as the self-intersection matrix of $N$ is negative definite, for any $1 \leq i < j \leq m$ one has

$$0 > (N_i + N_j)^2 = N_i^2 + 2(N_i \cdot N_j) + N_j^2 = 2(N_i \cdot N_j) - 2 .$$

As $N_i \cdot N_j \geq 0$, this can only hold if $N_i \cdot N_j = 0$.

According to [11, Proposition 1.5], a big divisor $D$ is in the boundary of a Zariski chamber if and only if

$$\operatorname{Neg}(D) \neq \operatorname{Null}(P_D) .$$

From [11, 4.3] we see that if $C \in \operatorname{Null}(P_D) - \operatorname{Neg}(D)$ for an irreducible negative curve $C$ then $N_D + C$ forms a negative divisor. By the previous reasoning, this implies that $N_D \cdot C = 0$ hence $D \cdot C = 0$, that is, $D \in C^{\perp}$ as required.

Going the other way, if $D \in C^{\perp}$ for an irreducible negative curve $C$ then either $P_D \cdot C = 0$, that is, $C \in \operatorname{Null}(P_D)$ or $P_D \cdot C > 0$.

In the first case, $C \notin \operatorname{Neg}(D)$, as otherwise we would have $N_D \cdot C < 0$ and consequently $D \cdot C < 0$ contradicting $D \in C^{\perp}$. Therefore $C \in \operatorname{Null}(P_D) - \operatorname{Neg}(D)$ and $D$ is in the boundary of some Zariski chamber.

In the second case, $D \cdot C = 0$ and $P_D \cdot C > 0$ imply $N_D \cdot C < 0$. From this we see that $C \in \operatorname{Neg}(D)$ but this would mean $P_D \cdot C = 0$ which is again a contradiction.

The conclusion is that on a surface on which the only negative curves are $(-1)$-curves, a big divisor $D$ is in the boundary of a Zariski chamber if and only if there exists an $(-1)$-curve $C$ with $D \in C^{\perp}$. □

One can in fact ask for more, and enumerate all the Zariski chambers on a given del Pezzo surface. The number of Zariski chambers (defined in [9])

$$z(X) = \#\{\text{Zariski chambers on } X\} \in \mathbb{N} \cup \{\infty\}$$

on a surface $X$ measures how complicated $X$ is from the point of view of linear series. At the same time it tells us

- the number of different stable base loci of big divisors on $X$;
- the number of different possibilities for the support of the negative part of the Zariski decomposition of a big divisor;
- the number of maximal regions of polynomiality for the volume function.

Bauer–Funke–Neumann in [9] calculate $z(X)$ when $X$ is a del Pezzo surface.

**Theorem 2.40.** *Let $X_r$ be the blow-up of $\mathbb{P}^2$ in $r$ general points with $1 \leq r \leq 8$.*

*(i) The number $z(X_r)$ of Zariski chambers on $X_r$ is given by the following table:*

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $z(X_r)$ | 2 | 5 | 18 | 76 | 393 | 2764 | 33645 | 1501681 |

*(ii) The maximal number of curves that occur in the support of a Zariski chamber on $X_r$ is $r$.*

## 2.D Countability and transcendence

We study the set of non-negative real numbers consisting of volumes of integral divisors on projective varieties. More specifically, we will be interested in the arithmetic properties of volumes. First, a definition.

**Definition 2.41.** Let $n$ be a positive integer. We define

$$\mathcal{V}_n \stackrel{\text{def}}{=} \{\alpha \in \mathbb{R}_{\geq 0} \mid \text{there exists } (X,D) \text{ of dimension } n \text{ with } \mathrm{vol}_X(D) = \alpha\} \ ,$$

where $X$ denotes an irreducible projective variety, $D$ an integral Cartier divisor on $X$. Also, we set

$$\mathcal{V} \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \mathcal{V}_n \ ,$$

and call it the *set of volumes*.

**Remark 2.42.** It is a consequence of the Künneth formula for the volume function (see Remark 2.18)

$$\mathrm{vol}_{X_1 \times X_2}(\pi_1^* \mathcal{O}_{X_1}(L_1) \otimes \pi_2^* \mathcal{O}_{X_2}(L_2)) = \binom{n_1+n_2}{n_1} \mathrm{vol}_{X_1}(L_1) \cdot \mathrm{vol}_{X_2}(L_2)$$

that $\mathcal{V}$ is a multiplicative sub-semigroup of the non-negative real numbers.

Observe however, that the right-hand side contains the constant $\binom{n_1+n_2}{n_1} \neq 1$. To circumvent this problem, note that any non-negative rational number can be displayed as the volume of a line bundle on a surface. Hence, if $L_3$ is a divisor on a smooth surface $X_3$ with

$$\mathrm{vol}_{X_3}(L_3) = \frac{n_1!n_2!2!}{(n_1+n_2+2)!} \ ,$$

then

$$\mathrm{vol}_{X_1}(L_1) \cdot \mathrm{vol}_{X_2}(L_2) = \mathrm{vol}_{X_1 \times X_2 \times X_3}\left(\pi_1^* \mathcal{O}_{X_1}(L^1) \otimes \pi_2^* \mathcal{O}_{X_2}(L_2) \otimes \pi_3^* \mathcal{O}_{X_3}(L_3)\right) \ .$$

Also, by taking $X_2 = \mathbb{P}^1$, and $L_1 = \mathcal{O}(1)$, we arrive at

$$\mathcal{V}_n \subseteq \mathcal{V}_{n+1} \ .$$

The structure of $\mathcal{V}$ appears to be very mysterious, not much is known about it. In particular, it is a wide open (and most likely quite difficult) question whether the above chain of inclusions stabilizes.

**Remark 2.43.** Because asymptotic cohomology is a birational invariant, $\mathrm{vol}_X(D)$ for any $X$ and $D$ can be realized as the volume of a divisor on a smooth projective variety by taking a resolution of singularities. Therefore, when discussing properties of $\mathcal{V}$, we can rightfully restrict our attention to smooth varieties.

We take the following fact as our starting point.

**Proposition 2.44.** *Let $D$ be a Cartier divisor with a finitely generated section ring on a normal projective variety $X$. Then* $\mathrm{vol}_X(D) \in \mathbb{Q}$.

*Proof.* We proceed as in [61, Example 2.1.31]. Because $D$ is finitely generated, there exists a natural number $p > 0$ and a projective birational map

$$\pi : Y \longrightarrow X$$

such that

$$R(X,D)^{(p)} = R(Y,D') ,$$

where $D' = \pi^*(pD) - N$ is semi-ample for some effective integral Cartier divisor $N$ on $Y$.

By the definition of Veronese subrings

$$H^0\left(X, \mathcal{O}_X(mpD)\right) = H^0\left(Y, \mathcal{O}_Y(mD')\right) ,$$

hence

$$p^n \cdot \mathrm{vol}_X(D) = \mathrm{vol}_X(pD) = \mathrm{vol}_Y\left(D'\right) = (D'^n) \in \mathbb{Z} ,$$

whence the volume of $D$ is indeed a rational number. $\square$

**Remark 2.45.** By looking at the proof of [61, Example 2.1.31] one can extract some more information on the volume of $D$. Namely, if $R(X,D)$ is generated in degree $d$, then

$$d^n \cdot \mathrm{vol}_X(D) \in \mathbb{Z} .$$

In other words, the denominator of $\mathrm{vol}_X(D)$ (after simplifying the fraction) divides $d^n$.

On a curve, the volume of a divisor equals its degree whenever this is non-negative, thus $\mathcal{V}_1 = \mathbb{N}$. Next, we take a look at what happens on surfaces.

**Proposition 2.46.** *$\mathcal{V}_2$ equals the set of non-negative rational numbers.*

*Proof.* First we prove that $\mathcal{V}_2 \subseteq \mathbb{Q}$. Since the volume of a non-big divisor is always zero, we will only care about big ones. Given a big integral divisor $D$ with Zariski decomposition $D = P_D + N_D$, we have

$$\mathrm{vol}_X(D) = \mathrm{vol}_X(P_D) = (P_D^2)$$

since $P_D$ is nef. The positive part $P_D$ of $D$ is a $\mathbb{Q}$-divisor, therefore $(P_D^2)$ is a rational number.

For the converse, we invoke [61, Example 2.3.6]. Let $Y$ be a smooth projective curve of genus one, $y \in Y$ a point, $a$ a positive integer. Consider the divisors

$$A_0 = (1-a) \cdot y , \ A_1 = y ,$$

and let $X \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{O}_Y(A_0) \oplus \mathcal{O}_Y(A_1))$. We will compute the volume of $\mathcal{O}_X(1)$.

To this end, we observe that

$$H^0\left(X, \mathcal{O}_X(k)\right) = \bigoplus_{a_0+a_1=k} H^0\left(Y, \mathcal{O}_Y(a_0A_0 + a_1A_1)\right) ,$$

and so

$$h^0(X, \mathcal{O}_X(k)) = h^0(Y, k \cdot y) + h^0(Y, (k-a) \cdot y) + \cdots + h^0\left(Y, (k - \lfloor \frac{k}{a} \rfloor) \cdot y)\right)$$
$$= \frac{k^2}{2a} .$$

Therefore,

$$\mathrm{vol}_X(\mathcal{O}_X(1)) = \frac{1}{a} .$$

Exploiting the homogeneity of volumes we see that

$$\mathrm{vol}_X(\mathcal{O}_X(m)) = \frac{m^2}{a} ,$$

which, for suitable $a$ and $m$, can be any positive rational number. □

Since the above construction plays an important role for us, we describe it in full generality following the exposition in [61, Section 2.3.B].

Let $Y$ be an irreducible projective variety, $r$ a positive integer, $A_0, \ldots, A_r$ integral Cartier divisors on $Y$. Let

$$X \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{O}_Y(A_0) \oplus \cdots \oplus \mathcal{O}_Y(A_r)) .$$

It turns out, that there is a close relation between the spaces of global section of $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(a_0A_0 + \cdots + a_rA_r)$.

**Lemma 2.47.** *([61, Lemma 2.3.2]) With notation as above,*

1. *we have*

$$H^0(X, \mathcal{O}_X(k)) = \bigoplus_{a_0+\cdots+a_r} H^0(Y, \mathcal{O}_Y(a_0A_0 + \cdots + a_rA_r)) .$$

2. *The line bundle $\mathcal{O}_X(1)$ is ample/nef precisely if each of the divisors $A_i$ is ample/nef.*

3. *The line bundle $\mathcal{O}_X(1)$ is big if and only if there exists some $\mathbb{Z}$-linear combination of the $A_i$'s which is big.*

This tool has found abundant applications in higher-dimensional geometry; what is of interest to us is the following.

**Example 2.48.** (A divisor with irrational volume) This is the original application from [21] (see also [61, Example 2.3.6]). Let $Y = E \times E$, with $E$ an elliptic curve without complex multiplication. Then $\rho(Y) = 3$, and the Néron–Severi space has a basis consisting of the two fibres of the respective projections $F_1$ and $F_2$, and the diagonal $\Delta$. The intersection numbers are determined by

$$(F_1^2) = (F_2^2) = (\Delta^2) = 0 ,$$

and

$$(F_1 \cdot F_2) = (F_1 \cdot \Delta) = (F_2 \cdot \Delta) = 1 .$$

The nef cone and the effective cone coincide and are equal to

$$\mathrm{Nef}(Y) = \overline{\mathrm{Eff}(Y)} = \left\{\alpha \in N^1(Y)_{\mathbb{R}} \,|\, (\alpha^2) \geq 0\,,\, (\alpha \cdot H) \geq 0\right\}$$

for an arbitrary ample divisor $H$.

The source of the irrationality is the shape of the nef cone; it is 'circular', that is, it is given by a quadratic inequality. Let $D$ and $H$ be ample integral divisors on $Y$, and consider the number

$$t_0 \stackrel{\text{def}}{=} \sup\{t \geq 0 \,|\, D - tH \text{ is ample/big }\}\ .$$

As it turns out, for most choices of $D$ and $H$ the number $t_0$ is irrational. A concrete example (see [61, Example 2.3.8]) would be $D = F_1 + F_2$ and $H = 3(F_2 + \Delta)$, which gives

$$t_0 = \frac{3-\sqrt{5}}{6}\ .$$

We are going to apply Cutkosky's construction with

$$A_0 = D \quad \text{and} \quad A_1 = -H\ .$$

As a result, we obtain

$$h^0\left(X, \mathcal{O}_X(k)\right) = \bigoplus_{a_0+a_1=k} H^0\left(Y, \mathcal{O}_Y(a_0A_0 + a_1A_1)\right)\ ,$$

where

$$h^0\left(Y, \mathcal{O}_Y(a_0D - a_1H)\right) = \begin{cases} \frac{1}{2}((a_0D - a_1H)^2) & \text{if } \frac{a_1}{a_0} < t_0\ , \\ 0 & \text{if } \frac{a_1}{a_0} > t_0\ . \end{cases}$$

Putting all this together, we conclude that

$$\mathrm{vol}_X\left(\mathcal{O}_X(1)\right) = 3 \cdot \int_{\frac{1}{1+t_0}}^{1} ((tD - (1-t)H)^2)dt\ .$$

In computing the integral, we feed an element of a quadratic number field into a cubic polynomial with rational coefficients; in most cases this will result in an irrational number.

Moving on to higher dimensions, we have seen above that the volume of an integral divisor need not be rational. The example Cutkosky obtains is still algebraic though, which leaves a considerable gap, and the question whether an arbitrary non-negative real number can be realized as the volume of a line bundle. This is the issue that we intend to address from two somewhat complimentary directions.

**Theorem 2.49.** (*Küronya–Lozovanu–Maclean, [57]*) *Let $\mathcal{V}$ denote the set of non-negative real numbers that occur as the volume of a line bundle. Then*

1. *$\mathcal{V}$ is countable;*
2. *$\mathcal{V}$ contains transcendental elements.*

Let us briefly discuss why the above results hold. The transcendency of volumes of integral divisors is again an application of Cutkosky's principle. We will prove a stronger statement, namely, that there exists an open subset in the Néron–Severi space on which the volume is given by a transcendental function.

We return to the surface $Y$ used in Example 2.48, and set $A_0 = F_1 + F_2 + \Delta$, $A_1 = -F_1$ and $A_2 = -F_2$. Then

$$X = \mathbb{P}_Y(\mathcal{O}_Y(F_1 + F_2 + \Delta) \oplus \mathcal{O}_Y(-F_1) \oplus \mathcal{O}_Y(-F_2)) .$$

**Proposition 2.50.** *With notation as above, there exists a non-empty open set in* $\mathrm{Big}(X)$*, on which the volume is given by a transcendental formula.*

*Proof.* The characterization of line bundles on projective space bundles, and the fact that the function $\mathrm{vol}_X$ is continuous, and homogeneous on $\mathrm{Big}(X)_{\mathbb{R}}$, imply that it is enough to handle line bundles of the form

$$A = \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \pi^*(\mathcal{O}_Y(L'))$$

with $L' = c_1F_1 + c_2F_2 + c_3\Delta$ a Cartier divisor on $Y$ with $(c_1, c_2, c_3) \in \mathbb{N}^3$. Using the projection formula the volume of $A$ is given by

$$\mathrm{vol}_X(A) = \limsup_{m\to\infty} \frac{\sum_{a_1+a_2+a_3=m} h^0(mL' + a_0A_0 + a_1A_1 + a_2A_2)}{m^4/24} ,$$

where the sum is taken over all natural $a_i$'s. Riemann-Roch on the abelian surface $Y$ has the simple form

$$h^0(Y, \mathcal{L}) = \frac{1}{2}(\mathcal{L}^2)$$

for an ample line bundle $\mathcal{L}$. It is not hard to show that in the sum above only the contribution of ample divisors $L = L' + a_0A_0 + a_1A_1 + a_2A_2$ counts, hence one can write the volume as

$$\mathrm{vol}_X(A) = \lim_{m\to\infty} \frac{4!}{2m^4} \sum_{a_0+a_1+a_2=m} ((mc_1 + a_0 - a_1)F_1 + (mc_2 + a_0 - a_2)F_2 + (mc_3 + a_0)\Delta))^2 .$$

Upon writing $x_i = a_i/m$ and making a change of coordinates $T : \mathbb{R}^3 \to \mathbb{R}^3$,

$$T(c_1) = c_1 + x_0 - x_1,\ T(c_2) = c_2 + x_0 - x_2,\ T(c_3) = c_3 + x_0 ,$$

we can write

$$\mathrm{vol}_X(A) = \int_{\Gamma(c_1,c_2,c_3)} (T(c_1)f_1 + T(c_2)f_2 + T(c_3)\Delta)^2 ,$$

where the set $\Gamma(c_1, c_2, c_3)$ is the intersection of the image of the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$ by the map $T$ and the nef cone $\mathrm{Nef}(Y)_{\mathbb{R}}$. As a more concrete example, when $c_1 = c_2 = c_3 = 1/4$ it looks like Figure 1. Indeed, one can use Maple to find that $\mathrm{vol}(A)$ is given by a transcendental formula in the $c_i$'s. □

As far as the cardinality of $\mathcal{V}$ is concerned, we obtain it as a consequence of a much stronger result: the countability of asymptotic cohomology functions. By building upon the existence of multigraded Hilbert schemes proved by Haiman and Sturmfels [40], one can in fact conclude that for many non-discrete invariants of varieties, like the ample cone for instance, there exist altogether countably many choices only.

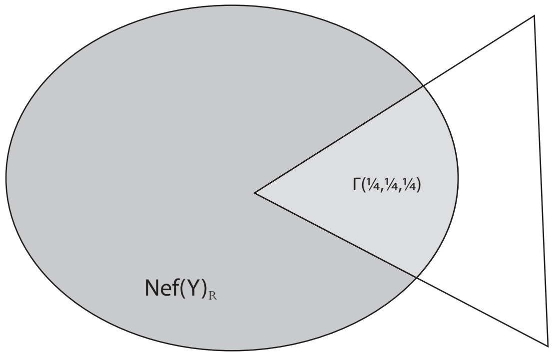



Figure 1

**Theorem 2.51.** (*Küronya–Lozovanu–Maclean, [57]*) *Let $V_{\mathbb{Z}} = \mathbb{Z}^{\rho}$ be a lattice inside the vector space $V_{\mathbb{R}}$. Then there exist countably many closed convex cones $A_i \subseteq V_{\mathbb{R}}$ and functions $f_j : V_{\mathbb{R}} \to \mathbb{R}$ with $i, j \in \mathbb{N}$, such that for any smooth projective variety $X$ of dimension $n$ and Picard number $\rho$, and any integer $1 \leq d \leq n$, we can construct an integral linear isomorphism*

$$\rho_X : V_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$$

*with the properties that*

$$\rho_X^{-1}(\mathrm{Nef}) = A_i, \text{ and } \widehat{h}^d(X, \cdot) \circ \rho_X = f_j$$

*for some $i, j \in \mathbb{N}$.*

**Remark 2.52.** Note that the above result immediately implies the case of irreducible varieties. Let $X$ be an irreducible projective variety and let $\mu : X' \to X$ be a resolution of singularities of $X$. The pullback map

$$\mu^* : N^1(X)_{\mathbb{R}} \to N^1(X')_{\mathbb{R}}$$

is linear, injective, and $\mathrm{vol}_X (=) \mathrm{vol}_{X'} \circ \mu^*$ by [61, Example 2.2.49]. Since the map $\mu^*$ is defined by choosing $\dim(N^1(X)_{\mathbb{R}})$ integral vectors, the countability of the volume functions in the smooth case implies that the same statement is valid for the collection of irreducible varieties. As $\mathrm{Nef}(X)_{\mathbb{R}} = (\mu^*)^{-1}(\mathrm{Nef}(X')_{\mathbb{R}})$, the same statement holds for nef cones.

**Remark 2.53.** From the fact that $\mathrm{Amp}(X) = \mathrm{int}\,\mathrm{Nef}(X)$, we observe that countability remains valid for ample cones as well. Much the same way, the cone of big divisors can be described as

$$\mathrm{Big}(X) = \{D \in N^1(X)_{\mathbb{R}} \mid \mathrm{vol}_X(D) > 0\} ;$$

and the closure of the right-hand side is known to be equal to the pseudo-effective cone $\overline{\mathrm{Eff}(X)}_{\mathbb{R}}$. Hence we conclude that our countability result is also valid for the big and pseudoeffective cones.

*Idea of proof of Theorem 2.51.* The first step is to construct a variety $Y_{n,\rho}$ such that every $n$-dimensional smooth projective variety $X$ with $\rho(X) = \rho$ can be embedded into $Y_{n,\rho}$ such that the induced map

$$\rho_X : V_{\mathbb{R}} \stackrel{\text{def}}{=} N^1(Y_{n,\rho}) \longrightarrow N^1(X)_{\mathbb{R}}$$

is an integral linear isomorphism.

As $X$ is smooth, $X$ embeds into $\mathbb{P}^{2n+1}$. Combining this with the diagonal embedding we arrive at

$$X \stackrel{\text{diag}}{\hookrightarrow} \underbrace{X \times \cdots \times X}_{\rho \text{ times}} \hookrightarrow \underbrace{\mathbb{P}^{2n+1} \times \cdots \times \mathbb{P}^{2n+1}}_{\rho \text{ times}} \stackrel{\text{def}}{=} Y_{n,\rho} ,$$

which satisfies our conditions.

Next, we build countably many families of subschemes of $Y_{n,\rho}$, such that each smooth subvariety $X \hookrightarrow Y_{n,\rho}$ embedded as above appears as a fibre in at least one of them. This is where we use multigraded Hilbert schemes of subvarieties of $Y_{n,\rho}$.

At this point, it suffices to check our countability statements for one of the flat families constructed above. For this purpose we apply induction on dimension, and use the semicontinuity theorem and the behaviour of nefness in families, respectively, to conclude[2]. $\square$

The same principle gives the countability of global Okounkov bodies.

**Theorem 2.54.** (*Küronya–Lozovanu–Maclean, [55]*) *There exists a countable set of closed convex cones $\Delta_i \subseteq \mathbb{R}^n \times \mathbb{R}^\rho$ with $i \in \mathbb{N}$ with the property that for any smooth, irreducible, projective variety $X$ of dimension $n$ and Picard number $\rho$, and any admissible flag $Y_\bullet$ on $X$, there is an integral linear isomorphism*

$$\psi_X : \mathbb{R}^\rho \longrightarrow N^1(X)_{\mathbb{R}} ,$$

*depending only on $X$, such that*

$$(\mathrm{id}_X \times \psi_X)^{-1}(\Delta_{Y_\bullet}(X) = \Delta_i$$

*for some $i \in \mathbb{N}$.*

It must be pointed out that the above countability statements only hold in the complete case. In the non-complete setting, mostly anything satisfying a few formal requirements can occur. To see this we take a look at volume functions of multigraded linear series.

In [63], Lazarsfeld and Mustaţă explain how the notion of the volume can be extended to the non-complete multigraded setting. Specifically, fix Cartier divisors $D_1, \ldots, D_\rho$ on $X$ (where $\rho$ is an arbitrary positive integer for the time being, but soon it will be $\dim_{\mathbb{R}} \mathbb{N}^1(X)_{\mathbb{R}}$), and write $\mathbf{m}D = m_1 D_1 + \ldots + m_\rho D_\rho$ for $\mathbf{m} = (m_1, \ldots, m_\rho) \in \mathbb{N}^\rho$. A multigraded linear series $W_\bullet$ on $X$ associated to $D_1, \ldots, D_\rho$ consists of subspaces

$$W_{\mathbf{m}} \subseteq H^0(X, \mathcal{O}_X(\mathbf{m}D)) ,$$

[2] Although the statement and the proof in [57] only mention volumes, the proof applies to asymptotic cohomology functions verbatim .

such that $R(W_\bullet) = \oplus W_{\mathbf{m}}$ is a graded $\mathbb{C}$-subalgebra of the section ring

$$R(D_1,\ldots,D_\rho) = \oplus_{\mathbf{m}\in\mathbb{N}^\rho} H^0(X,\mathcal{O}_X(\mathbf{m}D)) .$$

The support of $W_\bullet$ is defined to be the closed convex cone in $\mathbb{R}^\rho_+$ spanned by all multi-indices $\mathbf{m}\in\mathbb{N}^\rho$ satisfying $W_{\mathbf{m}} \neq 0$.

Given $\mathbf{a}\in\mathbb{N}^\rho$, we set

$$\mathrm{vol}_X\,(W_\bullet)\,(\mathbf{a}) \stackrel{\mathrm{def}}{=} \limsup_{k\to\infty} \frac{\dim_{\mathbb{C}}(W_{k\cdot\mathbf{a}})}{k^n/n!} .$$

Exactly as in the complete case, the above assignment defines the volume function of $W_\bullet$

$$\mathrm{vol}_X\,(W_\bullet)\,:\,\mathbb{N}^\rho \longrightarrow \mathbb{R}_+ .$$

It is also shown in [63] how to associate an Okounkov cone to a multigraded linear series on a projective variety. Lazarsfeld and Mustaţă verify that under very mild hypotheses the formal properties of the global volume function persist in the multigraded setting.

Precisely as in the global case, the function $\mathbf{m}\mapsto \mathrm{vol}_{W_\bullet}(\mathbf{m})$ extends uniquely to a continuous function

$$\mathrm{vol}_{W_\bullet}\,:\,\mathrm{int}(\mathrm{supp}(W_\bullet)) \longrightarrow \mathbb{R}_+ ,$$

which is homogeneous, log-concave of degree $n$, and extends continuously to the entire $\mathrm{supp}(W_\bullet)$. The construction generalizes the classical case: if $X$ is an irreducible projective variety, then the cone of big divisors $\mathrm{Big}(X)$ is pointed, and $\mathrm{vol}_X$ vanishes outside of it.

Pick Cartier divisors $D_1,\ldots,D_\rho$ on $X$, whose classes in $N^1(X)_{\mathbb{R}}$ generate a cone containing $\mathrm{Big}(X)$. Then $\mathrm{vol}_X = \mathrm{vol}_{W_\bullet}$ on $\mathrm{Big}(X)$, where $W_\bullet = R(D_1,\ldots,D_\rho)$.

Here we prove a converse statement to the effect that a function satisfying the formal properties of volume function is can in fact be realized as the volume function of a multi-graded linear series on a smooth projective (in fact toric) variety.

**Theorem 2.55.** (*Küronya–Lozovanu–Maclean, [57]*) *Let $K\subseteq\mathbb{R}^\rho_+$ be a closed convex cone with non-empty interior, $f: K\to\mathbb{R}_+$ a continuous function, which is nonzero, homogeneous, and log-concave of degree $n$ in the interior of $K$. Then there exists a smooth, projective variety $X$ of dimension $n$ and Picard number $\rho$, a multigraded linear series $W_\bullet$ on $X$ and a positive constant $c>0$ such that $\mathrm{vol}_{W_\bullet}\equiv c\cdot f$ on the interior of $K$. Moreover we have $\mathrm{supp}(W_\bullet)=K$.*

Getting back to the issue of transcendental volumes, we observe the interesting fact that the irregular values obtained so far by Cutkosky's construction have all been produced by evaluating integrals of polynomials over algebraic domains. In fact, all volumes computed to date can be put in such a form quite easily. Such numbers are called periods; here is the definition.

**Definition 2.56.** A complex number $z$ is a *period* if its real and imaginary parts are both values of absolutely convergent integral of rational functions with rational coefficients over subsets of $\mathbb{R}^n$ (for some $n$) that are given by finitely many polynomial inequalities with rational coefficients.

Periods are studied extensively in various branches of mathematics, including number theory, modular forms, and partial differential equations. An enjoyable account of periods can be found in [53], while for more technical questions and a comparison with other definitions [35] is a very good source.

**Remark 2.57.** The set of periods $\mathcal{P}$ is a countable $\mathbb{Q}$-subalgebra of $\mathbb{C}$ containing all algebraic numbers. Needless to say, the set of real periods has the same property in $\mathbb{R}$. Notable transcendental examples are $\pi$, special values of zeta functions, or the natural logarithms of positive integers

$$\log_e n = \int_1^n \frac{1}{x}\, dx \ .$$

**Remark 2.58.** Up until recently it has been a major open question to find explicit real numbers that are not periods. The prime candidate was (and still is) $e$, the base of the natural logarithm. It is not known whether this is true or not. However, in 2008 Yoshinaga [78] constructed a real number that is not a period using the theory of computability[3]. Just for fun, here is Yoshinaga's number to twelve digits

$$0,777664443548...$$

At this point it is natural to wonder about the following.

**Question 2.59.**

1. Is it true that all volumes are periods?
2. Are all algebraic numbers volumes of divisors?

Both questions seem to be extremely difficult. A more refined attempt would be to try and link various geometric properties of a given divisor to various measures of transcendence of its volume.

To some degree the phenomenon that all known volumes are periods is explained and accounted for by the construction of Okounkov bodies. Let us recall Theorem 1.17 of Lazarsfeld and Mustaţă to the effect that

$$\mathrm{vol}_{\mathbb{R}^n}\left(\Delta_{Y_\bullet}(D)\right) = n! \cdot \mathrm{vol}_X\left(D\right) \ .$$

This implies that

$$\mathrm{vol}_X\left(D\right) = \frac{1}{n!} \cdot \int_{\Delta_{Y_\bullet}(D)} 1 \ .$$

Consequently, whenever the Okounkov body of a divisor with respect to a judiciously chosen flag is a semi-algebraic domain, the volume of $D$ will be a period, which indeed happens in all known cases.

This gives rise to the following interesting question.

**Question 2.60.** Let $X$ be an irreducible projective variety, $D$ a big Cartier divisor on $X$. Does there exist an admissible flag $Y_\bullet$ on $X$ with respect to which the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ is a semi-algebraic domain?

[3]Thanks are due to Duco van Straten for bringing the work of Yoshinaga to our attention

# 3 Partial positivity of line bundles

## 3.A Summary

Ampleness is one of the central notions of projective algebraic geometry. An extremely useful feature of ampleness is that it has geometric, numerical and cohomological characterizations. Here we will concentrate on its cohomological side. The fundamental result in this direction is the theorem of Cartan–Serre–Grothendieck (see [61, Theorem 1.2.6]): let $X$ be a complete projective scheme, $\mathcal{L}$ a line bundle on $X$. Then the following are equivalent:

1. There exists a positive integer $m_0 = m_0(X,\mathcal{L})$, such that $\mathcal{L}^{\otimes m}$ is very ample for all $m \geq m_0$ (that is, the global sections of $H^0\left(X,\mathcal{L}^{\otimes} m\right)$ give rise to a closed embedding of $X$ into some projective space).

2. For every coherent sheaf $\mathcal{F}$ on $X$, there exists a positive integer $m_1(X,\mathcal{F},\mathcal{L})$ for which $\mathcal{F}\otimes\mathcal{L}^{\otimes m}$ is globally generated for all $m \geq m_1$.

3. For every coherent sheaf $\mathcal{F}$ on $X$ there exists a positive integer $m_2(X,\mathcal{F},\mathcal{L})$ such that
$$H^i\left(X,\mathcal{F}\otimes\mathcal{L}^{\otimes m}\right) = 0$$
for all $i \geq 1$ and all $m \geq m_2$.

We will focus on line bundles with vanishing cohomology above a certain degree. Historically, the first result in this direction is due to Andreotti and Grauert [4]. They prove that given a compact complex manifold $X$ of dimension $n$, and a holomorphic line bundle $\mathcal{L}$ on $X$ equipped with a Hermitian metric whose curvature is a $(1,1)$-form with at least $n-q$ positive eigenvalues at every point of $X$, then for every coherent sheaf $\mathcal{F}$ on $X$

$$H^i\left(X,\mathcal{F}\otimes\mathcal{L}^{\otimes m}\right) = 0$$

for $m \overset{\text{def}}{=} m_0(\mathcal{L},\mathcal{F})$ and $i > q$.

In [28] the authors posed the question under what circumstances does the converse hold for $q > 0$. In dimension two Demailly [27] gave an asymptotic version relying on asymptotic cohomology; subsequently, Matsumura [66] gave an confirmed the conjecture for surfaces. In higher dimensions, Ottem [74] gave counterexamples to the converse Andreotti–Grauert problem in the range $\frac{n}{2} - 1 < q < n-2$.

Nevertheless, line bundles with the property of the conclusion of the Andreotti–Grauert theorem are interesting to study.

**Definition 3.1.** Let $X$ be a complete scheme of dimension $n$, $0 \leq q \leq n$ an integer. A line bundle $\mathcal{L}$ is called *naively q-ample*, if for every coherent sheaf $\mathcal{F}$ on $X$ there exists an integer $m_0 = m_0(\mathcal{L},\mathcal{F})$ for which

$$H^i\left(X,\mathcal{F}\otimes\mathcal{L}^{\otimes m}\right) = 0$$

for $m \geq m_0(\mathcal{L},\mathcal{F})$ and $i > q$.

It is a consequence of the Cartan–Grothendieck–Serre result that 0-ampleness reduces to the usual notion of ampleness. Demailly–Peternell–Schneider in [28] studied naive $q$-ampleness along with various other notions of partial cohomological positivity. Part of their approach is to look at positivity of restrictions in elements of a complete flag and use it to give a partial vanishing theorem similar to that of Andreotti–Grauert.

In a beautiful paper Totaro [77] proves that the competing partial positivity concepts are in fact all equivalent in characteristic zero, thus laying the foundations of a very satisfactory theory.

**Theorem 3.2** (Totaro,[77])**.** *3.27 Let $X$ be a projective scheme of dimension $n$, $\mathcal{A}$ an arbitrary ample line bundle on $X$, $0 \leq q \leq n$ a natural number. Then there exists a positive integer $m_0$ with the property that for all line bundles $\mathcal{L}$ on $X$ the following are equivalent:*

1. *$\mathcal{L}$ is naively $q$-ample*
2. *$\mathcal{L}$ is uniformly $q$-ample, that is, there exists a constant $\lambda > 0$ such that*
$$H^i\left(X, \mathcal{L}^{\otimes m} \otimes \mathcal{A}^{\otimes -j}\right) = 0$$
*for all $i > q$, $j > 0$, and $m \geq \lambda j$.*
3. *There exists a positive integer $m_1$ such that*
$$H^i\left(X, \mathcal{L}^{\otimes m_1} \otimes \mathcal{A}^{\otimes -j}\right) = 0$$
*for all $i > q$ and $1 \leq j \leq m_0$.*

As an outcome, Totaro proves that on the one hand, for a given $q$, the set of $q$-ample line bundles forms a convex open cone in the Néron–Severi space, on the other hand $q$-ampleness is also an open property in families.

By its very definition, $q$-ampleness is closely linked to Serre-type vanishing results. An important statement along these lines is Fujita's improvement over Serre's vanishing theorem for ample divisors. The essence of Fujita's result is that to some extent it can do away with the major drawback of Serre vanishing, its non-uniform nature. Not surprisingly, it has many significant consequences, for instance it implies the boundedness of numerically trivial line bundles.

**Theorem 3.3** (Fujita,Theorem 3.10)**.** *Let $X$ be a complex projective scheme, $L$ an ample divisor, $\mathcal{F}$ a coherent sheaf on $X$. Then there exists a positive number $m_0 = m_0(L, \mathcal{F})$ such that*
$$H^i\left(X, \mathcal{O}_X(mL+N) \otimes \mathcal{F}\right) = 0$$
*for all $i > 0$, all $m \geq m_0$, and all nef divisors $N$.*

Our first main result is a generalization of Fujita's vanishing theorem in the direction investigated by Demailly–Peternell–Schneider and Totaro.

**Theorem 3.4** (Küronya, Theorem 3.49)**.** *Let $X$ be a complex projective scheme, $L$ a Cartier divisor, $A_1,\dots,A_q$ very ample Cartier divisors on $X$ such that $L|_{E_1\cap\cdots\cap E_q}$ is ample for general $E_j \in |A_j|$. Then for any coherent sheaf $\mathcal{F}$ on $X$ there exists an integer $m(L,A_1,\dots,A_q,\mathcal{F})$ such that*

$$H^i(X,\mathcal{F}\otimes\mathcal{O}_X(mL+N)) = 0$$

*for all $i > q$, $m \geq m(L,A_1,\dots,A_q,\mathcal{F})$, $k_j \geq 0$, and all nef divisors $N$.*

In the very special case $N = 0$ we recover a slightly weaker version of the main result of [28]. For big line bundles one can interpret the above theorem in terms of base loci.

**Corollary 3.5** (Küronya, Corollary 3.54)**.** *Let $X$ be a complex projective scheme, $L$ a Cartier divisor, $\mathcal{F}$ a coherent sheaf on $X$. Then there exists a positive integer $m_0(L,\mathcal{F})$ such that*

$$H^i(X,\mathcal{F}\otimes\mathcal{O}_X(mL+D)) = 0$$

*for all $i > \dim \mathbf{B}_+(L)$, $m \geq m_0(L,\mathcal{F})$, and all nef divisors $D$ on $X$.*

In a recent paper Brown [18] uses this statement to prove a result on restrictions of $q$-ample line bundles.

**Theorem 3.6** (Brown,[18])**.** *Let $\mathcal{L}$ be a big line bundle on a complex projective scheme $X$, denote by $\mathbf{B}_+(\mathcal{L})$ the augmented base locus of $\mathcal{L}$. For a given integer $0 \leq q \leq n$, $\mathcal{L}$ is $q$-ample if and only if $\mathcal{L}|_{\mathbf{B}_+(\mathcal{L})}$ is $q$-ample.*

It turns out that the philosophy of considering positivity of restrictions to general complete intersection subvarieties works well in other cases of importance as well. Adjoint line bundles (line bundles of the form $\omega_X \otimes \mathcal{A}$ with $\mathcal{A}$ ample) have always played a significant role in projective geometry; not in the least because of their vanishing properties first observed by Kodaira.

**Theorem 3.7** (Kodaira's vanishing theorem)**.** *Let $\mathcal{L}$ be an ample line bundle on a smooth projective variety $X$. Then*

$$H^i(X,\omega_X\otimes\mathcal{L}) = 0$$

*for all $i > 0$.*

There is a multitude of generalizations in existence, with arguably the most important one being the version of Kawamata and Viehweg for big and nef $\mathbb{Q}$-divisors because of its role in the minimal model program. We show that Kawamata–Viehweg vanishing also generalizes to bundles that are positive on restrictions to general complete intersection subvarieties.

**Theorem 3.8** (Küronya, Theorem 3.57)**.** *Let $X$ be a smooth projective variety, $\mathcal{L}$ a line bundle, $A$ a very ample divisor on $X$. If $\mathcal{L}|_{E_1\cap\cdots\cap E_q}$ is big and nef for a general choice of $E_1,\dots,E_q \in |A|$, then*

$$H^i(X,\omega_X\otimes\mathcal{L}) = 0 \quad \text{for } i > q.$$

Our last result in this section is an equivalent characterization of big line bundles in the style of Cartan–Serre–Grothendieck.

**Proposition 3.9** (Küronya, Proposition 3.56)**.** *Let $X$ be an irreducible projective variety, $\mathcal{L}$ a line bundle on $X$. Then $\mathcal{L}$ is big precisely if there exists a proper Zariski-closed subset $V \subsetneq X$ such that for all coherent sheaves $\mathcal{F}$ on $X$ there exists a possibly infinite sequence of sheaves of the form*

$$\cdots \to \bigoplus_{i=1}^{r_i} \mathcal{L}^{\otimes -m_i} \to \cdots \to \bigoplus_{i=1}^{r_1} \mathcal{L}^{\otimes -m_1} \to \mathcal{F} \to 0\,,$$

*which is exact off $V$.*

## 3.B Vanishing theorems

Much of higher-dimensional algebraic geometry rests on variants of two celebrated theorems about the vanishing of higher cohomology of line bundles. Serre's vanishing theorem cited above claims that for an ample line bundle $\mathcal{L}$ and a coherent sheaf $\mathcal{F}$ on a complete scheme $X$ there exists an integer $m_0 = m_0(\mathcal{L},\mathcal{F})$ for which

$$H^i\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}\right) = 0 \quad \text{for all } i > 0 \text{ and } m \geq m_0\,.$$

From a different angle, Kodaira vanishing says that whenever we have an ample line bundle on a smooth complex projective variety $X$,

$$H^i\left(X, \mathcal{L} \otimes \omega_X\right) = 0 \quad \text{whenever } i > 0$$

automatically follows.

Putting the issue of smoothness aside, the important features of the two theorems are: in Kodaira's statement one has vanishing for the higher cohomology of a given line bundle at the expense of tensoring by the canonical sheaf, while Serre's result has a much wider range of applicability, but one loses control of the actual line bundle whose higher cohomology vanishes.

A powerful interpolation between the two is Fujita's vanishing theorem (for a proof see [37] or [61, Theorem 1.4.35]), which to a certain extent eliminates the non-uniformity present in Serre's statement.

**Theorem 3.10.** *Let $X$ be a projective scheme, $\mathcal{L}$ an ample line bundle, $\mathcal{F}$ a coherent sheaf on $X$. Then there exists an integer $m_0 = m_0(\mathcal{L},\mathcal{F})$ for which*

$$H^i\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{N}\right) = 0$$

*for all $i > 0$, $m \geq m_0$, and nef line bundles $\mathcal{N}$ on $X$*

Note that $m_0$ is independent of the nef bundle $\mathcal{N}$; this is where the strength of this result comes from.

The usefulness of vanishing theorems in algebraic geometry cannot be overemphasized. Their effect becomes the most visible when they are applied to exact sequences. The following is a toy application, which, in its most sophisticated form plays a crucial role in the recent proof of the finite generation of the canonical ring [13] (see also [19]).

Let $X$ be a complete scheme, $\mathcal{L}$ a line bundle, $H$ an effective Cartier divisor on $X$. Consider the short exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{O}_X(-H) \to \mathcal{L} \to \mathcal{L}|_H \to 0 \, .$$

From the associated long exact sequence in cohomology, we obtain the exact fragment

$$H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}|_H) \longrightarrow H^1(X, \mathcal{L} \otimes \mathcal{O}_X(-H)) \, .$$

If for whatever reason we know that

$$H^1(X, \mathcal{L} \otimes \mathcal{O}_X(-H)) = 0 \, ,$$

then

$$H^0(X, \mathcal{L}) \twoheadrightarrow H^0(X, \mathcal{L}|_H) \, ,$$

that is, all global section of $\mathcal{L}|_H$ extend to global sections of $\mathcal{L}$ on $X$.

Next, we present a bit more concrete variant.

**Proposition 3.11.** *Let $X$ be a smooth projective variety, $\mathcal{L}$ a line bundle on $X$. Then $\omega_X \otimes \mathcal{L}$ is globally generated if and only if*

$$H^1(X, \omega_X \otimes \mathcal{L} \otimes \mathfrak{m}_x) = 0$$

*for all points $x \in X$.*

In the proposition $\mathfrak{m}_x$ denotes the ideal sheaf of the point $x$.

*Proof.* By the definition of global generation, we need to show that the map

$$H^0(X, \omega \otimes \mathcal{L}) \longrightarrow H^0(x, \mathcal{O}_x \otimes \omega_X \otimes \mathcal{L}) \simeq \mathbb{C}$$

is surjective for all points $x \in X$.

We start with the short exact sequence

$$0 \to \omega_X \otimes \mathcal{L} \otimes \mathfrak{m}_x \to \omega_X \otimes \mathcal{L} \to \mathcal{O}_x \otimes \omega \otimes \mathcal{L} \to 0 \, .$$

From the corresponding long exact sequence we obtain

$$H^0(X, \omega \otimes \mathcal{L}) \longrightarrow H^0(x, \mathcal{O}_x \otimes \omega_X \otimes \mathcal{L}) \longrightarrow H^1(X, \omega_X \otimes \mathcal{L} \otimes \mathfrak{m}_x) \longrightarrow 0 \, ,$$

since

$$H^1(X, \omega_X \otimes \mathcal{L}) = 0$$

by Kodaira. Given a point $x \in X$,

$$H^0(X, \omega \otimes \mathcal{L}) \longrightarrow H^0(x, \mathcal{O}_x \otimes \omega_X \otimes \mathcal{L})$$

is surjective exactly if

$$H^1(X, \omega_X \otimes \mathcal{L} \otimes \mathfrak{m}_x) = 0 \, ,$$

as claimed. $\square$

Coupled with Castelnuovo–Mumford regularity, vanishing theorems give rise to useful sufficient criteria to very ampleness as well.

**Definition 3.12.** Let $X$ be an irreducible projective variety, $\mathcal{F}$ a coherent sheaf, $\mathcal{A}$ an ample and globally generated line bundle on $X$. We say that $\mathcal{F}$ is $m$-regular with respect to $\mathcal{A}$ for an integer $m$, if

$$H^i\left(X, \mathcal{F} \otimes \mathcal{A}^{\otimes(m-i)}\right) = 0$$

for all $i \geq 0$.

The smallest number $m$ for which $\mathcal{F}$ is $m$-regular is called that *regularity of* $\mathcal{F}$ *with respect to* $\mathcal{A}$.

The main result in this direction is Mumford's theorem (for a simple proof see [61, Theorem 1.8.5]).

**Theorem 3.13** (Mumford)**.** *With notation as above, assume that the coherent sheaf* $\mathcal{F}$ *is* $m$*-regular. Then we have the following statements for all* $k \geq 0$*.*

1. $\mathcal{F}$ *is* $(m+k)$*-regular with respect to* $\mathcal{A}$*.*
2. *The sheaf* $\mathcal{F} \otimes \mathcal{A}^{\otimes(m+k)}$ *is globally generated.*
3. *The multiplication maps*

$$H^0\left(X, \mathcal{F} \otimes \mathcal{A}^{\otimes m}\right) \otimes H^0\left(X, \mathcal{A}^{\otimes k}\right) \to H^0\left(X, \mathcal{F} \otimes \mathcal{A}^{\otimes(m+k)}\right)$$

*are all surjective.*

**Corollary 3.14.** *([61, Example 1.8.22]) Keeping the notation let* $\mathcal{L}$ *be a* $0$*-regular line bundle with respect to* $\mathcal{A}$ *on* $X$*. Then* $\mathcal{L} \otimes \mathcal{A}$ *is very ample.*

We can use this to give a criterion for the very ampleness of globally generated ample bundles resembling Fujita's conjecture. This application is taken from [61, Example 1.8.23]

**Proposition 3.15.** *Let* $X$ *be a smooth projective variety,* $\mathcal{N}$ *a nef,* $\mathcal{A}$ *a globally generated ample line bundle on* $X$*. Then* $\omega_X \otimes \mathcal{A}^{\otimes k} \otimes \mathcal{N}$ *is*

$$\begin{aligned} \textit{globally generated} \quad \textit{if} \quad & k \geq \dim X + 1 \ , \\ \textit{very ample} \quad \textit{if} \quad & k \geq \dim X + 2 \ . \end{aligned}$$

*Proof.* The starting point is Kodaira's vanishing theorem saying that

$$H^i\left(X, \mathcal{L} \otimes \omega_X\right) = 0 \quad \text{for } i > 0$$

for any ample line bundle $\mathcal{L}$.

In proving that $\omega_X \otimes \mathcal{A}^{\otimes k} \otimes \mathcal{N}$ is globally generated whenever $k \geq \dim X + 1$, we will rely on Mumford's theorem (Theorem 3.13). In order for this to apply it suffices to check that $\omega_X \otimes \mathcal{A}^{\otimes k} \otimes \mathcal{N}$ is 0-regular with respect to the ample and globally generated line bundle $\mathcal{A}$, that is,

$$H^i\left(X, (\omega_X \otimes \mathcal{A}^{\otimes k} \otimes \mathcal{N}) \otimes \mathcal{A}^{\otimes -i}\right) = 0 \quad \text{for all } 1 \leq i \leq n = \dim X.$$

Observe that

$$(\omega_X \otimes \mathcal{A}^{\otimes k} \otimes \mathcal{N}) \otimes \mathcal{A}^{\otimes -i} = \omega_X \otimes (\mathcal{A}^{(k-1-i)} \otimes \mathcal{A} \otimes \mathcal{N}) \ .$$

If $k \geq n+1$, then $\mathcal{A}^{\otimes(k-1-i)}$ is nef, hence $(\mathcal{A}^{(k-1-i)} \otimes \mathcal{A} \otimes \mathcal{N})$ is ample, and Kodaira indeed applies.

For the statement about very ampleness one uses Corollary 3.14 much the same way. □

**Remark 3.16** (Fujita's conjecture)**.** Extrapolating from the case of curves, Fujita posed a conjecture about geometric properties of adjoint line bundles. The precise statement goes as follows. Let $X$ be a smooth projective variety of dimension $n$, $\mathcal{A}$ an ample line bundle on $X$. Then $\omega_X \otimes \mathcal{A}^{\otimes k}$ is

$$\begin{aligned} \text{globally generated} \quad & \text{if} \quad k \geq \dim X + 1 \ , \\ \text{very ample} \quad & \text{if} \quad k \geq \dim X + 2 \ . \end{aligned}$$

The statement is classical for curves; it translates to the result that $\deg_C \mathcal{A} \geq 2g$ implies that $\mathcal{A}$ is globally generated, while $\deg_C \mathcal{A} \geq 2g+1$ implies that $\mathcal{A}$ is very ample. In dimension two, the conjecture was proved by Reider via vector bundle methods. The global generation question has been proved in dimension 3 (Ein–Lazarsfeld [31]) and Helmke and Kawamata [49].

The first result valid in all dimensions is due to Demailly who used techniques inspired by physics to conclude that $\omega_X^{\otimes}2 \otimes \mathcal{A}^{\otimes 12n^n}$ is very ample. The best general result to date has been obtained by Angehrn–Siu. They prove that Fujita's conjecture for global generation holds for

$$k \geq \binom{n+1}{2} + 1 \ .$$

**Remark 3.17** (Kollár–Matsusaka)**.** Such considerations play a role in the Demailly–Siu approach to the theorem of Kollár and Matsusaka (for details see [61, Section 10.2], the original work of Kollár–Matsusaka is [52]). Their result says that for an ample line bundle $\mathcal{A}$ and a nef line bundle $\mathcal{N}$ on a smooth projective variety $X$ of dimension $n$, there exists a positive integer

$$m_0 = m_0(c_1(\mathcal{A})^n, c_1(\omega_X) \cdot c_1(\mathcal{A})^{n-1}, c_1(\mathcal{N}) \cdot c_1(\mathcal{A})^{n-1})$$

depending only on the given intersection numbers such that

$$\mathcal{A}^{\otimes m} \otimes \mathcal{N}^{\otimes -1}$$

is very ample whenever $m \geq m_0$.

**Proposition 3.18** (Upper-semicontinuity of Castelnuovo–Mumford regularity)**.** *Let $X$ an irreducible projective variety, $\mathcal{A}$ an ample and globally generated line bundle on $X$. Given a flat family of line bundles $\mathcal{L}$ on $X$ parametrized by a quasi-projective variety $T$, the function*

$$T \ni t \mapsto \operatorname{reg}_{\mathcal{A}}(\mathcal{L}_t)$$

*is upper-semicontinuous.*

*Proof.* Since Castelnuovo–Mumford regularity is checked by the vanishing of finitely many line bundles, the statement follows from the semicontinuity theorem for cohomology. □

**Remark 3.19.** One classical application of vanishing theorems is a solution to the Riemann–Roch problem in an important particular case. We recall from Remark 2.12 that this means the determination of the function

$$m \mapsto h^0\left(X, \mathcal{L}^{\otimes m}\right)$$

for a given line bundle $\mathcal{L}$. It is a simple but really useful observation that whenever

$$H^i\left(X, \mathcal{L}^{\otimes m}\right) = 0 \quad \text{for all } i > 0$$

for a given natural number $m$, then

$$h^0\left(X, \mathcal{L}^{\otimes m}\right) = \sum_{i=1}^{n} (-1)^i h^i\left(X, \mathcal{L}^{\otimes m}\right) = \chi(X, \mathcal{L}^{\otimes m}) ,$$

where this latter can sometimes be computed from the Riemann–Roch theorem. Interestingly enough, this means in particular that under the conditions $h^0\left(X, \mathcal{L}^{\otimes m}\right)$ is invariant with respect to numerical equivalence of line bundles.

For this approach to be successful, most of the time it suffices to have vanishing for the higher cohomology for $m \gg 0$.

Next we treat some applications of Fujita's theorem (Theorem 3.10). This extension of Serre vanishing combines the generality offered by Serre vanishing and the uniform nature of Kodaira's result. It is an interesting fact that unlike Kodaira's theorem, it is valid in positive characteristic. As a first application (taken from [61, Theorem 1.4.40]), we present an estimate on the growth of higher cohomology of nef divisors.

**Proposition 3.20.** *Let $\mathcal{N}$ be a nef line bundle, $\mathcal{F}$ a coherent sheaf on a projective scheme[4] of dimension $n$. Then*

$$h^i\left(X, \mathcal{F} \otimes \mathcal{N}^{\otimes m}\right) \leq C_{\mathcal{F},\mathcal{L}} \cdot m^{n-i}$$

*for all $1 \leq i \leq n$.*

[4] We require schemes to be separated and of finite type over $\mathbb{C}$

*Proof.* We will use induction on $n = \dim X$; the claim is clear in dimensions zero and one. Let $X$ have dimension $n$, and assume that the statement holds for schemes of dimension at most $n-1$.

Fujita's theorem implies that there exists a very ample line bundle $\mathcal{A}$ on $X$ satisfying
$$H^i\left(X, \mathcal{F}\otimes\mathcal{N}^{\otimes m}\otimes\mathcal{A}\right) = 0 \quad \text{for all } i > 0 \text{ and } m \geq 0\ ,$$
in fact, a suitable tensor power of any given ample line bundle will do.

A general divisor $H$ in the linear series associated to $\mathcal{A}$ will not contain any of the associated subschemes of $\mathcal{F}$, hence we have the short exact sequence
$$0 \to \mathcal{F}\otimes\mathcal{N}^{\otimes m} \to \mathcal{F}\otimes\mathcal{N}^{\otimes m}\otimes\mathcal{A} \to \mathcal{F}\otimes\mathcal{N}^{\otimes m}\otimes\mathcal{A}\otimes\mathcal{O}_H \to 0\ .$$

As a consequence, we obtain from the corresponding long exact sequence that
$$h^i\left(X, \mathcal{F}\otimes\mathcal{N}^{\otimes m}\right) \leq h^{i-1}\left(H, \mathcal{F}\otimes\mathcal{N}^{\otimes m}\otimes\mathcal{A}\otimes\mathcal{O}_H\right) \leq C'\cdot m^{(n-1)-(i-1)}\ ,$$
just as we wanted. □

Part of the significance of Fujita's statement comes from its corollary to the effect that numerically trivial line bundles form a bounded family. Here is the precise formulation from [61, 1.4.37].

**Theorem 3.21.** *Let $X$ be a projective scheme. Then there exists a scheme $T$ (necessarily of finite type) along with a line bundle $\mathcal{L}$ on $X\times T$ such that any numerically trivial line bundle $\mathcal{N}$ on $X$ arises as a restriction $\mathcal{L}|_{X_t}$ for some $t\in T$.*

This phenomenon is responsible for the numerical invariance of asymptotic cohomology. For the invariance of the volume a simple version suffices (see [61, Lemma 2.2.42]), for the higher cohomology one needs the following more complicated version.

**Proposition 3.22.** *Let $X$ be an irreducible projective variety of dimension $n$. Then there exists a family*
$$\begin{array}{ccc} V & \subseteq & X\times T \\ \downarrow{\scriptstyle\phi} & & \\ T & & \end{array}$$
*with $T$ a quasi-projective variety (not necessarily irreducible), $V\subseteq X\times T$ a closed subscheme and $\phi$ flat, together with a very ample divisor $A$ on $X$, such that*

1. *$A+N$ is very ample for every numerically trivial divisor $N$ on $X$,*
2. *if $D\in|A+N|$ for some $N\equiv 0$, then $D = V_t$ for some $t\in T$.*

*Proof.* We start by showing that there exists a very ample divisor $A$ on $X$ such that $A+N$ is very ample for every numerically trivial divisor $N$.

According to Fujita's vanishing theorem, for any fixed ample divisor $B$ there exists $m_0 > 0$ such that
$$H^i\left(X, \mathcal{O}_X(mB+E)\right) = 0$$

for all $m \geq m_0$, $i \geq 1$ and all nef divisors $E$. In particular, vanishing holds for every numerically trivial divisor. Take any very ample divisor $B$, let $m_0$ be as in Theorem 3.10.

Consider $A' = (m_0+n)B+N$ where $N$ is an arbitrary numerically trivial divisor. Then

$$H^i\left(X, \mathcal{O}_X(A'-iB)\right) = H^i\left(X, \mathcal{O}_X(N+(m_0+n-i)B)\right) = 0$$

for all $i \geq 0$ hence $A$ is 0-regular with respect to $B$. By Mumford's theorem $A'$ is globally generated. But then $A'+B = (m_0+n+1)B+N$ is very ample. Observe that the coefficient of $B$ is independent of $N$ hence the choice

$$A \stackrel{\text{def}}{=} (m_0+n+1)B$$

will satisfy the requirements for $A$.

To prove the Proposition, pick $A$ as above to begin with. Thanks to Theorem 3.21 there exists a scheme $Q$ of finite type over $\mathbb{C}$, and a line bundle $\mathcal{L}$ on $X \times Q$, with the property that for every line bundle $\mathcal{O}_X(A+N)$ with $N$ numerically trivial, there exists $q \in Q$ for which

$$\mathcal{O}_{X\times q}(A+N) = \mathcal{L}|_{X\times\{q\}} \ .$$

Let $p$,$\pi$ denote that projection maps from $Q \times X$ to $Q$ and $X$, respectively. We can arrange by possibly twisting $\mathcal{L}$ further by $\pi^*\mathcal{O}_X(mA)$, that $R^j p_*\mathcal{L} = 0$ for $j > 0$, and that the natural map $\rho : p^*p_*\mathcal{L} \to \mathcal{L}$ is surjective.

By the theorem on cohomology and base change [41, Section III.12],

$$\mathcal{E} \stackrel{\text{def}}{=} p_*\mathcal{L}$$

is a vector bundle on $Q$ whose formation commutes with base change over $Q$. As $\mathcal{E}(q) = H^0\left(X, \mathcal{L}|_{X\times q}\right)$, its projectivization $\mathbb{P}(\mathcal{E})$ will parametrize the divisors $D \in |N+A|$, with $N$ numerically trivial.

The universal divisor over $\mathbb{P}(\mathcal{E})$ is constructed as follows. The kernel $\mathcal{M}$ of the natural map $\rho$ (which is surjective in our case) is a vector bundle whose formation respects base change on $Q$. By restricting to the fibre of $p$ over $q$, one has an exact sequence

$$0 \to (M_X)_q \to H^0\left(\{q\}\times X, L\right) \otimes_{\mathbb{C}} \mathcal{O}_X \to L \to 0$$

where $L = \mathcal{L}|_{\{q\}\times X}$ and $(M_X)_q = H^0\left(X, \mathcal{M}|_{\{q\}\times X}\right)$. Hence we see that

$$\mathbb{P}_{\text{sub}}(M_X) = \left\{ (s,x) \,|\, x \in X,\ s \in H^0\left(X, L\right),\ s(x) = 0 \right\} \ ,$$

ie. $V = \mathbb{P}_{\text{sub}}(\mathcal{M})$ and $T = \mathbb{P}(\mathcal{E})$ will satisfy the requirements of the proposition. □

From the point of view of birational geometry ample divisors have a major drawback: the pullback of an ample divisor via a proper birational morphism is never ample (unless of course the morphism is an isomorphism). The traditional remedy for this situation is to consider big and nef divisors instead of ample ones. This point of view is satisfactory on two counts:

1. The pullback of an ample line bundle with respect to a proper birational morphism is big and nef.

2. The pullback of a big and nef line bundle with respect to a proper birational morphism is again big and nef.

Hence, for the purposes of birational geometry one can often replace ample divisors with big and nef ones. To get the most mileage out of this idea, there is one very important issue: big and nef bundles should possess at least some of the vanishing properties of ample line bundles.

The vanishing theorem more significant for birational geometry and Mori theory in particular is Kodaira's. Luckily, it extends to big and nef divisors. Although one can even afford certain divisors with rational coefficients, we restrict our attention to line bundles.

**Theorem 3.23** (Kawamata–Viehweg)**.** *Let $X$ be a smooth projective variety, $\mathcal{L}$ a big and nef line bundle on $X$. Then*

$$H^i(X, \omega_X \otimes \mathcal{L}) = 0 \quad \textit{for all } i > 0.$$

This result is one of the fundamental building blocks of the minimal model program, and has a multitude of applications. Since we do not intend to discuss Mori theory here, we will not explore this direction further.

As far as a possible extension of Serre vanishing to big and nef divisors goes, a direct generalization fails. Nevertheless, as explained in Remark 2.16, one at least has an asymptotic statement: for a nef line bundle $\mathcal{L}$ and a coherent sheaf $\mathcal{F}$ on a projective variety of dimension $n$, one has

$$h^i\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}\right) \leq C_{\mathcal{F},\mathcal{L}} \cdot m^{n-1}$$

for all $i > 0$.

### 3.C q-ample divisors

Starting with the pioneering work [28] of Demailly–Peternell–Schneider related to the Andreotti–Grauert problem, there has been a certain interest in studying line bundles with vanishing cohomology above a given degree. Just as big line bundles are a generalization of ample ones along its geometric side, these so-called $q$-ample bundles focus on a weakening of the cohomological characterization of ampleness. In general, there are various competing definitions, which are shown to be equivalent in characteristic zero in an influential paper by Totaro [77].

**Definition 3.24** (Definitions of partial ampleness)**.** Let $X$ be a complete scheme over an algebraically closed field of arbitrary characteristic, $\mathcal{L}$ an invertible sheaf on $X$, $q$ a natural number.

1. The invertible sheaf $\mathcal{L}$ is called *naively $q$-ample*, for every coherent sheaf $\mathcal{F}$ on $X$ there exists a natural number $m_0 = m_0(\mathcal{L}, \mathcal{F})$ having the property that

$$H^i\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}\right) = 0 \quad \text{for all } i > q \text{ and } m \geq m_0.$$

2. Fix a very ample invertible sheaf $\mathcal{A}$ on $X$. We call $\mathcal{L}$ *uniformly q-ample* if there exists a constant $\lambda = \lambda(\mathcal{A},\mathcal{L})$ such that

$$H^i\left(X,\mathcal{L}^{\otimes m}\otimes\mathcal{A}^{\otimes -j}\right) = 0 \quad \text{for all } i > q, j > 0, \text{ and } m \geq \lambda\cdot j.$$

3. Fix a very ample invertible sheaf $\mathcal{A}$ on $X$. We say that $\mathcal{L}$ is *q-T-ample*, if there exists a positive integer $m_1 = m_1(\mathcal{A},\mathcal{L})$ satisfying

$$\begin{aligned} H^{q+1}\left(X,\mathcal{L}^{\otimes m_1}\otimes\mathcal{A}^{\otimes -(n+1)}\right) &= H^{q+2}\left(X,\mathcal{L}^{\otimes m_1}\otimes\mathcal{A}^{\otimes -(n+2)}\right) \\ \dots &= H^n\left(X,\mathcal{L}^{\otimes m_1}\otimes\mathcal{A}^{\otimes -2n+q}\right) = 0 \end{aligned}$$

**Remark 3.25.** Naive $q$-ampleness is the immediate extension of the Grothendieck–Cartan–Serre vanishing criterion for ampleness. Uniform $q$-ampleness first appeared in [28]; the term $q$-$T$-ampleness was coined by Totaro in [77, Section 7] extending the idea of Castelnuovo–Mumford regularity.

A line bundle is ample if and only if it is naively 0-ample, while all line bundles are $n = \dim X$-ample.

**Remark 3.26.** There are various implications among the three definitions in general. As it was verified in [28, Proposition 1.2] by an argument via resolving coherent sheaves by direct sums of ample line bundles[5] that a uniformly $q$-ample line bundle is necessarily naively $q$-ample. On the other hand, naive $q$-ampleness implies $q$-$T$-ampleness by definition.

The upshot of $q$-$T$-ampleness is to reduce the question of $q$-ampleness to the vanishing of finitely many cohomology groups. In positive characteristic it is not known whether the three definitions coincide.

The main contribution of Totaro in this direction is [77, Theorem 8.1].

**Theorem 3.27** (Totaro)**.** *Over a field of characteristic zero, the three definitions of partial ampleness are equivalent.*

The proof uses on the one hand methods from positive characteristics to generalize a vanishing result of Arapura [7, Theorem 5.4], on the other hand it relies on earlier work of Kawamata on resolutions of the diagonal via Koszul-ample line bundle. This latter leads to a statement of independent interest about the regularity of tensor products of sheaves.

**Theorem 3.28.** (*Totaro, [77, Theorem 4.4]*) *Let $X$ be a connected and reduced projective scheme of dimension $n$, $\mathcal{A}$ a $2n$-Koszul line bundle, $\mathcal{E}$ a vector bundle, $\mathcal{F}$ a coherent sheaf on $X$. Then*

$$\operatorname{reg}(\mathcal{E}\otimes\mathcal{F}) \leq \operatorname{reg}(\mathcal{E}) + \operatorname{reg}(\mathcal{F}) .$$

**Remark 3.29.** In the case of $X = \mathbb{P}^n_{\mathbb{C}}$ the above theorem is a simple application of Koszul complexes (see [61, Proposition 1.8.9] for instance).

[5]Note that the resulting resolution is not guaranteed to be finite; see [61, Example 1.2.21] for a discussion of this idea.

From now on we return to our usual blanket assumption and work over the complex number field. The equivalence of the various definitions brings all sorts of perks.

First, we point out that $q$-ampleness enjoys many formal properties analogous to ampleness. The following statements have been part of the folklore, for precise proofs we refer the reader to [74, Proposition 2.3] and [28, 1.5].

**Lemma 3.30.** *Let $X$ be a projective scheme, $\mathcal{L}$ a line bundle on $X$, $q$ a natural number. Then*

1. *$\mathcal{L}$ is $q$-ample if and only if $\mathcal{L}|_{\mathrm{red}}$ on $X_{\mathrm{red}}$ is $q$-ample.*
2. *$\mathcal{L}$ is $q$-ample precisely if $\mathcal{L}|_{X_i}$ is $q$-ample on $X_i$ for every irreducible component $X_i$ of $X$.*
3. *For a finite morphism $f : Y \to X$, if $\mathcal{L}$ on $X$ is $q$-ample then so is $f^*\mathcal{L}$. Conversely, if $f$ is surjective as well, then the $q$-ampleness of $f^*\mathcal{L}$ implies the $q$-ampleness of $\mathcal{L}$.*

The respective proofs of the ample case go through with minimal modification. Another feature, which goes through as is is the fact that to check (naive) $q$-ampleness we can restrict our attention to line bundles.

**Lemma 3.31.** *Let $X$ be a projective scheme, $\mathcal{L}$ a line bundle, $\mathcal{A}$ an arbitrary ample line bundle on $X$. Then $\mathcal{L}$ is $q$-ample precisely if there exists a natural number $m_0 = m_0(\mathcal{A},\mathcal{L})$ having the property that*

$$H^i\left(X, \mathcal{L}^{\otimes m} \otimes \mathcal{A}^{\otimes -k}\right) = 0$$

*for all $i > q$, $k \geq 0$, and $m \geq m_0 k$.*

*Proof.* Follows immediately by decreasing induction on $q$ from the fact that every coherent sheaf $\mathcal{F}$ on $X$ has a possibly infinite resolution by finite direct sums of non-positive powers of the ample line bundle $\mathcal{A}$ ([61, Example 1.2.21]). □

Ample line bundles are good to work with for many reasons, but the fact that they are open both in families and in the Néron–Severi space contributes considerably. As it turns out, the same properties are valid for $q$-ample line bundles as well.

**Theorem 3.32.** (*Totaro,[77, Theorem 9.1]*) *Let $\pi : X \to B$ be a flat projective morphism of schemes (over $\mathbb{Z}$) with connected fibres, $\mathcal{L}$ a line bundle on $X$, $q$ a natural number. Then the subset of points $b$ of $B$ having the property that $\mathcal{L}|_{X_b}$ is $q$-ample is Zariski open.*

*Sketch of proof.* This is one point where $q$-$T$-ampleness plays a role, since in that formulation one only needs to check vanishing for a finite number of cohomology groups.

Assume that $\mathcal{L}|_{X_b}$ is $q$-$T$-ample for a given point $b \in B$; let $U$ be an affine open neighbourhood on $b \in B$, and $\mathcal{A}$ a line bundle on $\pi^{-1}(U) \subseteq X$, whose restriction to $X_b$ is Koszul-ample. Since Koszul-ampleness is a Zariski-open property, $\mathcal{A}|_{X_{b'}}$ is again Koszul-ample for an open subset of points $b' \in U$; without loss of generality we can assume that this holds on the whole of $U$.

We will use the line bundle $\mathcal{A}|_{X_{b'}}$ to check $q$-$T$-ampleness of $\mathcal{L}_{X_{b'}}$ in an open neighbourhood on $b \in U$. By the $q$-$T$-ampleness of $\mathcal{L}|_{X_b}$ there exists a positive integer $m_0$ satisfying

$$H^{q+1}\left(X_b, \mathcal{L}^{\otimes m_0} \otimes \mathcal{A}^{\otimes -n-1}\right) = \ldots = H^n\left(X_b, \mathcal{L}^{\otimes m_0} \otimes \mathcal{A}^{\otimes -2n+q}\right) = 0 \, .$$

It follows from the semicontinuity theorem that the same vanishing holds true for points in an open neighbourhood of $b$. □

In a different direction, Demailly–Peternell–Schneider proved that uniform $q$-ampleness is open in the Néron–Severi space. To make this precise we need the fact that uniform $q$-ampleness is a numerical property; once this is behind us, we can define $q$-ampleness for numerical equivalence classes of $\mathbb{R}$-divisors.

**Remark 3.33.** Note that a line bundle $\mathcal{L}$ is $q$-ample if and only if $\mathcal{L}^{\otimes m}$ is $q$-ample for some positive integer $m$. Therefore it makes sense to talk about $q$-ampleness of $\mathbb{Q}$-Cartier divisors (one does not customarily use the notion of line bundles with rational or real coefficients); a $\mathbb{Q}$-divisor $D$ is said to be $q$-ample, if it has a multiple $mD$ that is integral and $\mathcal{O}_X(mD)$ is $q$-ample.

**Theorem 3.34.** *Let $D$ and $D'$ be numerically equivalent integral Cartier divisors on an irreducible complex projective variety $X$, $q$ a natural number. Then*

$$D \text{ is } q\text{-ample} \iff D' \text{ is } q\text{-ample}.$$

Demailly–Peternell–Schneider in [28, Proposition 1.4] only prove this claim for smooth projective varieties. The proof in [28] cites the completeness of $\mathrm{Pic}^0(X)$, hence it is far from obvious how to extend it. Here we give a proof that is valid under the given more general hypothesis. Instead of dealing with uniform $q$-ampleness we use the naive formulation.

*Proof.* Let $\mathcal{N}'$ be a numerically trivial line bundle, $\mathcal{L}$ a $q$-ample line bundle on $X$. This means that for a given coherent sheaf $\mathcal{F}$, we have

$$H^i\left(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}\right) = 0 \quad \text{for } i > q \text{ and } m_0 = m_0(\mathcal{L}, \mathcal{F}).$$

We need to prove that

$$H^i\left(X, (\mathcal{L} \otimes \mathcal{N}')^{\otimes m} \otimes \mathcal{F}\right) = 0$$

holds for all $i > q$, and for suitable $m \geq m_1(\mathcal{L}, \mathcal{N}', \mathcal{F})$.

To this end, we will study the function

$$f_i^{(m)} : \mathcal{N} \mapsto h^i\left(X, \mathcal{L}^{\otimes m} \otimes \mathcal{N} \otimes \mathcal{F}\right)$$

as a function on the closed points of the scheme $\mathcal{X}$ parametrizing numerically trivial line bundles on $X$, which is a scheme of finite type by the boundedness result Theorem 3.21.

We know that

$$f_i^{(m)}(\mathcal{O}_X) = h^i\left(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}\right) = 0$$

for $i > q$, and $m \geq m_0$. By the semicontinuity theorem $f_i^{(m)}$ has the same value on a dense open subset of $\mathcal{X}$. By applying noetherian induction and semicontinuity on the irreducible components of the complement we will eventually find a value $m_0' = m_0'(\mathcal{L},\mathcal{F})$ such that

$$f_i^{(m)} \equiv 0$$

for all $i > q$ and $m \geq m_0'$. But this implies that

$$H^i\left(X,(\mathcal{L}\otimes\mathcal{N}')^{\otimes m}\otimes\mathcal{F}\right) = H^i\left(X,\mathcal{L}^{\otimes m}\otimes((\mathcal{N}')^{\otimes m})\otimes\mathcal{F}\right) = 0$$

for $m \geq m_0'$, since the required vanishing holds for an arbitrary numerically trivial divisor in place of $(\mathcal{N}')^{\otimes m}$.

□

**Remark 3.35.** As a result, we are in a position to extend the definition of $q$-ampleness elements of $N^1(X)_{\mathbb{Q}}$: if $\alpha$ is a numerical equivalence class of $\mathbb{Q}$-divisors, then we will call it $q$-ample, if it contains a $q$-ample representative.

**Definition 3.36** ($q$-ampleness for $\mathbb{R}$-divisors)**.** An $\mathbb{R}$-divisor $D$ on a complex projective variety is *$q$-ample*, if

$$D = D' + A\ ,$$

where $D'$ is a $q$-ample $\mathbb{Q}$-divisor, $A$ an ample $\mathbb{R}$-divisor.

The result the $q$-ample $\mathbb{R}$-divisors form an open cone in $N^1(X)_{\mathbb{R}}$ was proved in [28]. Here we face the same issue as with Theorem 3.34: in the article [28] only the smooth case is considered, and the proof given there does not seem to generalize to general varieties.

**Definition 3.37.** Given $\alpha \in N^1(X)_{\mathbb{R}}$, we set

$$q(\alpha) \stackrel{\text{def}}{=} \min\{q\in\mathbb{N}\,|\,\alpha \text{ is } q\text{-ample}\}\ .$$

**Theorem 3.38.** *Let $X$ be an irreducible projective variety over the complex numbers. Then the function*

$$q\ :\ N^1(X)_{\mathbb{R}} \longrightarrow \mathbb{N}$$

*is upper-semicontinuous.*

*In particular, for a given $q\in\mathbb{N}$, the set of $q$-ample classes forms an open cone.*

In order to be able to prove this result, we need some auxiliary statements. To this end, Demailly–Peternell–Schneider introduce the concept of height of coherent sheaves with respect to a given ample divisor. Roughly speaking the height of a coherent sheaf tells us, what multiples of the given ample divisor we need to obtain a linear resolution.

**Definition 3.39** (Height)**.** Let $X$ be an irreducible projective variety, $\mathcal{F}$ a coherent sheaf, $\mathcal{A}$ an ample line bundle. Consider the set $\mathcal{R}$ of all resolutions

$$\ldots \to \bigoplus_{1\leq l\leq m_k} \mathcal{A}^{\otimes -d_{k,l}} \to \ldots \to \bigoplus_{1\leq l\leq m_0} \mathcal{A}^{\otimes -d_{0,l}} \to \mathcal{F} \to 0$$

of $\mathcal{F}$ by non-positive powers of $\mathcal{A}$ (that is, $d_{k,l}\geq 0$). Then

$$\mathrm{ht}_{\mathcal{A}}(\mathcal{F}) \stackrel{\text{def}}{=} \min_{\mathcal{R}} \max_{0\leq k\leq 1+\dim X, 1\leq l\leq m_k} d_{k,l}\ .$$

**Remark 3.40.** On could define the height by looking at resolutions without truncating, that is, by
$$\widetilde{\mathrm{ht}}_{\mathcal{A}}(\mathcal{F}) \stackrel{\mathrm{def}}{=} \min_{\mathcal{R}} \max_{0\leq k,\, 1\leq l\leq m_k} d_{k,l}\ .$$

On a general projective variety there might be sheaves that do not possess finite locally free resolutions at all, and it can happen that the height of a sheaf is infinite if we do not truncate resolutions.

A result of Arapura [6, Corollary 3.2] gives effective estimates on the height of a coherent sheaf in terms of its Castelnuovo–Mumford regularity.

**Lemma 3.41.** *Let $X$ be an irreducible projective variety, $\mathcal{A}$ an ample and globally generated line bundle, $\mathcal{F}$ a coherent sheaf on $X$. Given a natural number $k$, there exist vector spaces $V_i$ for $1 \leq i \leq k$ and a resolution*

$$V_k \otimes \mathcal{A}^{\otimes -r_{\mathcal{F}}-kr_X} \to \ldots \to V_1 \otimes \mathcal{A}^{\otimes -r_{\mathcal{F}}-r_X} \to V_0 \otimes \mathcal{A}^{\otimes -r_{\mathcal{F}}} \to \mathcal{F} \to 0\ ,$$

*where*
$$r_{\mathcal{F}} \stackrel{def}{=} \mathrm{reg}_{\mathcal{A}}(\mathcal{F}) \quad \text{and} \quad r_X \stackrel{def}{=} \max\{1, \mathrm{reg}_{\mathcal{A}}(\mathcal{O}_X)\}\ .$$

**Corollary 3.42.** *With notation as above, the height of an $r_{\mathcal{F}}$-regular coherent sheaf $\mathcal{F}$ is*
$$\mathrm{ht}_{\mathcal{A}}(\mathcal{F}) \leq r_{\mathcal{F}} + r_X \cdot \dim X\ .$$

**Proposition 3.43** (Properties of height)**.** *Let $X$ be an irreducible projective variety of dimension $n$, $\mathcal{A}$ an ample line bundle. Then the following hold.*

1. *For coherent sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ we have*
$$\mathrm{ht}_{\mathcal{A}}(\mathcal{F}_1 \otimes \mathcal{F}_2) \leq \mathrm{ht}_{\mathcal{A}}(\mathcal{F}_1) + \mathrm{ht}_{\mathcal{A}}(\mathcal{F}_2)\ .$$

2. *There exists a positive constant $M = M(X, \mathcal{A})$ having the property that*
$$\mathrm{ht}_{\mathcal{A}}(\mathcal{N}) \leq M$$
*for all numerically trivial line bundles $\mathcal{N}$ on $X$.*

*Proof.* The first statement is an immediate consequence of the fact that the tensor product of appropriate resolutions of $\mathcal{F}_1$ and $\mathcal{F}_2$ is a resolution of $\mathcal{F}_1 \otimes \mathcal{F}_2$ of the the required type.

The second claim is a consequence of the fact that numerically trivial divisors on a projective variety are parametrized by a quasi-projective variety. Indeed, it follows by Proposition 3.18 and the noetherian property of the Zariski topology that there exists a constant $M'$ satisfying
$$\mathrm{reg}_{\mathcal{A}}(\mathcal{N}) \leq M'$$
for all numerically trivial line bundles $\mathcal{N}$. By the Corollary of Lemma 3.41,
$$\mathrm{ht}_{\mathcal{A}}(\mathcal{N}) \leq M \stackrel{\mathrm{def}}{=} M' + r_X \cdot \dim X\ ,$$
as required. $\square$

**Lemma 3.44** (Demailly–Peternell–Schneider, [28], Proposition 1.2)**.** *Let $\mathcal{L}$ be a uniformly q-ample line bundle on X with respect to an ample line bundle $\mathcal{A}$ for a given constant $\lambda = \lambda(\mathcal{A},\mathcal{L})$. Given a coherent sheaf $\mathcal{F}$ on X,*

$$H^i\left(X,\mathcal{L}^{\otimes m}\otimes\mathcal{F}\right) = 0$$

*for all $i > q$ and $m \geq \lambda\cdot \mathrm{ht}_{\mathcal{A}}(\mathcal{F})$.*

*Proof of Theorem 3.38.* [6] Fix an integral ample divisor $A$, and integral Cartier divisors $B_1,\dots,B_\rho$ whose numerical equivalence classes form a basis of the rational Néron–Severi space. Let $D$ be an integral uniformly $q$-ample divisor (for a constant $\lambda = \lambda(D,A)$), $D'$ a $\mathbb{Q}$-Cartier divisor, and write

$$D' \equiv D+\sum_{i=1}^{\rho}\lambda_i B_i$$

for rational numbers $\lambda_i$. Assume that $D$ is Let $k$ be a positive integer clearing all denominators, then

$$kD' = kD+\sum_{i=1}^{\rho}k\lambda_i B_i+N$$

for a numerically trivial (integral) divisor $N$. We want to show that

$$H^i\left(X,mkD'-pA\right) = 0$$

whenever $m\geq\lambda(D',A)\cdot p$ for a suitable positive constant $\lambda$. By Lemma 3.44 applied with

$$\mathcal{L} = \mathcal{O}_X(D)\ ,\ \mathcal{A} = \mathcal{O}_X(A)\ ,\ \text{and}\ \mathcal{F} = \mathcal{O}_X(\sum_{i=1}^{\rho}mk\lambda_i B_i+mN-pA)\ ,$$

this will happen whenever

$$m\geq\lambda(D,A)\cdot\mathrm{ht}_A(\sum_{i=1}^{\rho}mk\lambda_i B_i+mN-pA)\ .$$

Observe that

$$\begin{aligned}\mathrm{ht}_A(\sum_{i=1}^{\rho}mk\lambda_i B_i+mN-pA) &= \sum_{i=1}^{\rho}\mathrm{ht}_A(mk\lambda_i B_i)+\mathrm{ht}_A(mN)+\mathrm{ht}_A(-pA)\\ &\leq \sum_{i=1}^{\rho}mk|\lambda_i|\cdot\max\{\mathrm{ht}_A(B_i),\mathrm{ht}_A(-B_i)\}+M+p\ ,\end{aligned}$$

where $M$ is the constant from Proposition 3.43; note that $\mathrm{ht}_A(-pA)=p$ for $p\geq 0$.

Therefore, if the $\lambda_i$'s are small enough so that

$$\lambda\cdot\sum_{i=1}^{\rho}k|\lambda_i|\cdot\max\{\mathrm{ht}_A(B_i),\mathrm{ht}_A(-B_i)\} < \frac{1}{2}\ ,$$

then it suffices to require

$$m\geq 2\lambda(M+p)\ ,$$

and $D'$ will be $q$-ample. This shows the upper-semicontinuity of uniform $q$-ampleness. □

[6] The following modification of the proof of [28, Proposition 1.4] was suggested by Burt Totaro.

**Remark 3.45.** If $D_1$ is a $q$-ample and $D'$ is an $r$-ample divisor, then their sum $D+D'$ can only be guaranteed to be $q+r$-ample; this bound is sharp. As a consequence, the cone of $q$-ample $\mathbb{R}$-divisor classes is not necessarily convex. We denote this cone by $\mathrm{Amp}^q(X)$.

It is an interesting question how to characterize the cone of $q$-ample divisors for a given integer $q$. If $q=0$, then the Cartan–Serre–Grothendieck theorem implies that the $\mathrm{Amp}^0(X)$ equals the ample cone. Totaro describes the $(n-1)$ ample cone with the help of duality theory.

**Theorem 3.46** (Totaro, [77], Theorem 10.1)**.** *For an irreducible projective variety $X$,*

$$\mathrm{Amp}^{n-1}(X) \;=\; N^1(X)_{\mathbb{R}} \setminus (-\overline{\mathrm{Eff}(X)})\ .$$

**Remark 3.47.** Ottem [74] applies the theory of $q$-ample line bundles to study varieties of higher codimension with considerable success. Among the many results he obtains, there is a Lefschetz-type theorem for ample subschemes.

Totaro links partial positivity to the vanishing of higher asymptotic cohomology. Generalizing Theorem 2.21, he asks the following question.

**Question 3.48** (Totaro)**.** Let $D$ be an $\mathbb{R}$-divisor class on a complex projective variety, $0 \le q \le n$ an integer. Assume that $\widehat{h}^i(X,D') = 0$ for all $i > q$ and all $D' \in N^1(X)_{\mathbb{R}}$ in a neighbourhood of $D$. Is is true that $D$ is $q$-ample?

### 3.D Positivity of restrictions and vanishing of higher cohomology

Vanishing theorems classically apply to ample or big and nef line bundles. However, with the recent shift of attention towards big line bundles, it became very important to know what kind of vanishing properties big line bundles or even certain non-big ones retain.

An asymptotic statement along these lines appears in Proposition 2.17. Matsumura in [65, Theorem 1.6] gave a partial generalization of the Kawamata–Viehweg vanishing theorem using restricted base loci.

We will study the relationship between ampleness of restrictions of line bundles to general complete intersections and the vanishing properties of higher cohomology groups. Our inspiration comes from [77], and some unpublished ideas related to [23]. It turns out that this approach eventually leads to a generalization of Fujita's vanishing theorem to all big line bundles.

Employing techniques from [60] and [23], we can prove strong vanishing theorems for non-ample — oftentimes not even pseudo-effective — divisors. Our proofs follow the same fundamental principle: positivity of restrictions of line bundles results in partial vanishing of higher cohomology groups.

**Theorem 3.49** (Küronya, [59], Theorem 1.2)**.** *Let $X$ be a complex projective scheme, $L$ a Cartier divisor, $A_1,\dots,A_q$ very ample Cartier divisors on $X$ such that $L|_{E_1\cap\cdots\cap E_q}$ is ample for general $E_j \in |A_j|$. Then for any coherent sheaf $\mathcal{F}$ on $X$ there exists an integer $m(L,A_1,\dots,A_q,\mathcal{F})$ such that*

$$H^i(X,\mathcal{F}\otimes\mathcal{O}_X(mL+N+\sum_{j=1}^{q} k_jA_j)) \;=\; 0$$

*for all $i > q$, $m \geq m(L, A_1, \ldots, A_q, \mathcal{F})$, $k_j \geq 0$, and all nef divisors $N$.*

**Remark 3.50.** By setting $k_1 = \cdots = k_q = 0$ in the Theorem we obtain that $L$ is $q$-ample. This way we recover a slightly weaker version of [28, Theorem 3.4].

Once we know what to aim at, the proof is an induction on codimension using Fujita's vanishing theorem. The basic principle is in fact straightforwards: a assume $L$ is a Cartier divisor, $A$ a very ample Cartier divisor on $X$, for which $L|_E$ is ample for a general element $E \in |A|$. Then the cohomology long exact sequence associated to

$$0 \to \mathcal{O}_X(mL + (k-1)A) \to \mathcal{O}_X(mL + kA) \to \mathcal{O}_E(mL + kA) \to 0$$

shows that $H^i(X, \mathcal{O}_X(mL + (k-1)A)) \simeq H^i(X, \mathcal{O}_X(mL + kA))$ for all $i \geq 1$ and $k \geq 0$ by Fujita vanishing on $E$ whenever $m$ is sufficiently large. The common isomorphism class of all the groups $H^i(X, \mathcal{O}_X(mL + kA))$ for $k \geq 1$ is 0 according to Serre vanishing for the (very) ample divisor $A$.

Here is the complete proof.

*Proof.* For every $1 \leq j \leq q$, pick a general element $E_j \in |A_j|$ $(1 \leq j \leq q)$, and let $N$ be an arbitrary nef divisor on $X$. Consider the set of standard exact sequences

$$\begin{aligned} 0 \quad \to \quad & \mathcal{F} \otimes \mathcal{O}_{Y_j}(mL + N + \sum_{l=1}^{q} k_l A_l) \to \mathcal{F} \otimes \mathcal{O}_{Y_j}(mL + N + \sum_{l=1}^{q} k_l A_l + A_{j+1}) \\ \to \quad & \mathcal{F} \otimes \mathcal{O}_{Y_{j+1}}(mL + N + \sum_{l=1}^{q} k_l A_l + A_{j+1}) \to 0 \end{aligned} \tag{8}$$

for all $0 \leq j \leq q-1$, all $m$, and all $k_1, \ldots, k_q \geq 0$. Here $Y_j \stackrel{\text{def}}{=} E_1 \cap \cdots \cap E_j$ for all $1 \leq j \leq q$, for the sake of completeness set $Y_0 \stackrel{\text{def}}{=} X$. Take a look at the sequences with $j = q-1$.

Fujita's vanishing theorem on $Y_q = E_1 \cap \cdots \cap E_q$ applied to the ample divisor $L|_{Y_q}$ gives that

$$H^i(Y_q, \mathcal{F} \otimes \mathcal{O}_{Y_q}(mL + N + \sum_{l=1}^{q} k_l A_l)) = 0$$

for all $i \geq 1$, $m \geq m(\mathcal{F}, L, A_1, \ldots, A_q, Y_q)$, all $k_1, \ldots, k_q \geq 0$, and all nef divisors $N$ on $X$.

This implies that first and the last group in the exact sequence

$$\begin{aligned} & H^{i-1}(Y_q, \mathcal{F} \otimes \mathcal{O}_{Y_q}(mL + N + \sum_{l=1}^{q} k_l A_l + A_q)) \to H^i(Y_{q-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{q-1}}(mL + N + \sum_{l=1}^{q} k_l A_l)) \\ & \to H^i(Y_{q-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{q-1}}(mL + N + \sum_{l=1}^{q} k_l A_l + A_q)) \to H^i(Y_q, \mathcal{F} \otimes \mathcal{O}_{Y_q}(mL + N + \sum_{l=1}^{q} k_l A_l + A_q)) \end{aligned}$$

vanishes for $i \geq 2$, $m \geq m(\mathcal{F}, L, A_1, \ldots, A_q, Y_q)$, $k_1, \ldots, k_q \geq 0$, and all nef divisors $N$. Consequently,

$$H^i(Y_{q-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{q-1}}(mL + N + \sum_{l=1}^{q} k_l A_l)) = H^i(Y_{q-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{q-1}}(mL + N + \sum_{l=1}^{q} k_l A_l + kA_q))$$

for all $i \geq 2$, $m \geq m(\mathcal{F}, L, A_1, \ldots, A_q, Y_q)$, $N$ nef, $k \geq 0$, and $k_1, \ldots, k_q \geq 0$. Then

$$H^i(Y_{q-1}, \mathcal{F} \otimes \mathcal{O}_{Y_{q-1}}(mL + N + \sum_{l=1}^{q} k_l A_l + kA_q)) = 0$$

follows for all $k \geq 0$ from Serre vanishing applied to the ample divisor $A_q|_{Y_{q-1}}$. By the semicontinuity theorem and the general choice of the $E_j$'s we can drop the dependence on $Y_q$.

Working backwards along the cohomology sequences associated to (8), we obtain by descending induction on $j$ that

$$H^i(Y_j, \mathcal{F} \otimes \mathcal{O}_{Y_j}(mL + N + \sum_{l=1}^{q} k_l A_l)) = 0$$

for $i > q - j$, $m \gg 0$, and all $k_1, \ldots, k_q \geq 0$. This gives the required result when $j = 0$. □

**Corollary 3.51.** *With notation as above, if $L|_{E_1 \cap \cdots \cap E_q}$ is ample for general $E_j \in |A_j|$, then $L$ is $q$-ample.*

The above vanishing result translates naturally into asymptotic cohomology. There we have the The advantage is that it is enough to check positivity along a flag after pulling back to a proper birational model.

**Corollary 3.52.** *Let $X$ be an irreducible projective variety, $L$ a Cartier divisor on $X$. Assume that there exists a proper birational morphism $\pi : Y \to X$, a natural number $q$, and very ample divisors $A_1, \ldots, A_q$ on $Y$ such that $\pi^* L|_{E_1 \cap \cdots \cap E_q}$ is ample for general elements $E_i \in |A_i|$, for all $1 \leq i \leq q$. Then*

$$\widehat{h}^i(X, L) = 0 \quad \textit{for } i > q.$$

*Proof.* As a consequence of Theorem 3.49, one has $H^i(Y, \pi^* \mathcal{O}_X(mL)) = 0$ for all $i > q$ and $m \gg 0$. This gives $\widehat{h}^i(Y, \pi^* L) = 0$ for all $i > q$. By the birational invariance of asymptotic cohomology [60, Corollary 2.10]

$$\widehat{h}^i(X, L) = \widehat{h}^i(Y, \pi^* L) = 0 \ \text{ for all } i > q.$$

□

These ideas lead to relations between invariants expressing partial positivity, and the inner structure of various cones of divisors in the Néron–Severi space. We do not pursue this circle of ideas here; for details we refer the reader to [59, Section 1].

We move on to another geometric formulation of Theorem 3.49 in terms of base loci. Following [32, Remark 1.3], one defines the augmented base locus of a $\mathbb{Q}$-Cartier divisor $L$ via

$$\mathbf{B}_+(L) \stackrel{\text{def}}{=} \bigcap_A \mathbf{B}(L - A) \ ,$$

where $A$ runs through all ample $\mathbb{Q}$-Cartier divisors. As remarked in [59], this notion extends to projective schemes without modification.

**Remark 3.53.** Let $X$ be a projective scheme over the complex numbers. Arguing as in the proof of [32, Proposition 1.5] we can see that for a given $\mathbb{Q}$-divisor $L$ one can always find an $\varepsilon > 0$ such that

$$\mathbf{B}_+(L) = \mathbf{B}(L - A)$$

for any ample $\mathbb{Q}$-divisor with $\|A\| < \varepsilon$ (with respect to an arbitrary norm on the Néron–Severi space). This implies that

$$\dim \mathbf{B}_+(L|_Y) \leq \dim \mathbf{B}_+(L)$$

for any closed subscheme $Y$ in $X$.

In addition, if $A$ is very ample and $E \in |A|$ is a general element, then

$$\dim \mathbf{B}_+(L|_E) < \dim \mathbf{B}_+(L) \ .$$

**Corollary 3.54.** *Let $X$ be a complex projective scheme, $L$ a Cartier divisor, $\mathcal{F}$ a coherent sheaf on $X$. Then there exists a positive integer $m_0(L,\mathcal{F})$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mL+D)) = 0$$

*for all $i > \dim \mathbf{B}_+(L)$, $m \geq m_0(L,\mathcal{F})$, and all nef divisors $D$ on $X$.*

*Proof.* If $\dim \mathbf{B}_+(L) = \dim X$, then the claim holds for dimension reasons. Assume that $\dim \mathbf{B}_+(L) < \dim X$, and fix a very ample divisor $A$ on $X$. By Remark 3.53 coupled with the fact that divisors with an empty augmented base locus are ample, $L$ satisfies the conditions of Theorem 3.49 with $q \geq \dim \mathbf{B}_+(L)$ with respect to $A$. Then Theorem 3.49 gives the required vanishing. □

**Remark 3.55.** Fujita's original proof also generalizes to give Corollary 3.54. For this, one needs to verify statements about base loci on schemes that are very similar in nature to Lemma 3.30. The other tools one needs to use are a birational version of Theorem 3.57, and the following result about resolution of sheaves.

**Proposition 3.56.** *Let $X$ be an irreducible projective variety, $L$ a Cartier divisor on $X$. Then $\mathbf{B}_+(L)$ is the smallest subset $V$ of $X$ with the property that for all coherent sheaves $\mathcal{F}$ on $X$ there exists a possibly infinite sequence of sheaves of the form*

$$\cdots \to \bigoplus_{i=1}^{r_i} \mathcal{O}_X(-m_iL) \to \cdots \to \bigoplus_{i=1}^{r_1} \mathcal{O}_X(-m_1L) \to \mathcal{F} \to 0 \ ,$$

*which is exact off $V$.*

*Proof.* If $L$ is a non-big divisor, then $\mathbf{B}_+(L) = X$, and the statement is obviously true. Hence we can assume without loss of generality that $L$ is big.

First we prove the following claim: let $\mathcal{F}$ be an arbitrary coherent sheaf on $X$; then there exist positive integers $r$,$m$, and a map of sheaves

$$\bigoplus_{i=1}^{r} \mathcal{O}_X(-mL) \xrightarrow{\phi} \mathcal{F} \ , \tag{9}$$

which is surjective away from $\mathbf{B}_+(L)$.

Fix an arbitrary ample divisor $A$ on $X$. The sheaves $\mathcal{F} \otimes \mathcal{O}_X(m'A)$ are globally generated for $m'$ sufficiently large. According to [32, Proposition 1.5] $\mathbf{B}_+(L) = \mathbf{B}(L - \varepsilon A)$ for any rational $\varepsilon > 0$ small enough. Pick such an $\varepsilon$, set $L' \stackrel{\text{def}}{=} L - \varepsilon A$ and let $m \gg 0$ be a positive integer such that $m' \stackrel{\text{def}}{=} m\varepsilon$ is an integer, and

$$\mathrm{Bs}(mL') = \mathbf{B}(mL') = \mathbf{B}_+(L) \ .$$

By picking $m$ large enough, we can in addition assume that $\mathcal{F}\otimes\mathcal{O}_X(m'A)$ is globally generated. As a consequence,

$$\mathcal{F}\otimes\mathcal{O}_X(m'A)\otimes\mathcal{O}_X(mL')\simeq\mathcal{F}\otimes\mathcal{O}_X(m'A+mL')$$

is globally generated away from $\mathrm{Bs}(mL')=\mathbf{B}_+(L)$. On the other hand

$$mL'+m'A\,=\,m(L-\varepsilon A)+(m\varepsilon)A\,=\,mL\,,$$

hence we have found $m\gg 0$ such that $\mathcal{F}\otimes\mathcal{O}_X(mL)$ is globally generated away from $\mathbf{B}_+(L)$. Thanks to the map

$$H^0\left(X,\mathcal{F}\otimes\mathcal{O}_X(m'A)\right)\otimes H^0\left(X,\mathcal{O}_X(mL')\right)\longrightarrow H^0\left(X,\mathcal{F}\otimes\mathcal{O}_X(mL)\right)$$

one can find a finite set of sections giving rise to a map of sheaves

$$\bigoplus_{i=1}^{r}\mathcal{O}_X\longrightarrow\mathcal{F}\otimes\mathcal{O}_X(mL)$$

surjective away from $\mathbf{B}_+(L)$. Tensoring by $\mathcal{O}_X(-mL)$ gives the map in (9).

Next we will prove that $\mathbf{B}_+(L)$ satisfies that property described in the Proposition. Let $\mathcal{G}$ be the kernel of the map $\bigoplus_{i=1}^{r_1}\mathcal{O}_X(-m_1L)\xrightarrow{\phi_1}\mathcal{F}$ coming from (9). Applying (9) to $\mathcal{G}$ we obtain a map

$$\bigoplus_{i=1}^{r_2}\mathcal{O}_X(-mL)\xrightarrow{\phi_2}\mathcal{G}$$

surjective off $\mathbf{B}_+(L)$, hence a two-term sequence

$$\bigoplus_{i=1}^{r_2}\mathcal{O}_X(-m_2L)\rightarrow\bigoplus_{i=1}^{r_1}\mathcal{O}_X(-m_1L)\rightarrow\mathcal{F}$$

exact away from the closed subset $\mathbf{B}_+(L)$. Continuing in this fashion we arrive at a possibly infinite sequence of the required type.

Last, if $x\in\mathbf{B}_+(L)$ then for all $\varepsilon\in\mathbb{Q}^{\geq 0}$ and all $m\geq 1$ such that $m\varepsilon\in\mathbb{Z}$, all global sections of $\mathcal{O}_X(m(L-\varepsilon A))$ vanish at $x$. By taking $\mathcal{F}\overset{\text{def}}{=}\mathcal{O}_X(-A)$,

$$\mathcal{F}\otimes\mathcal{O}_X(mL)\,=\,\mathcal{O}_X(mL-A)$$

will then have all global sections vanishing at $x\in X$. Therefore $\mathbf{B}_+(L)$ is indeed the smallest subset of $X$ with the required property. $\square$

Last, we turn to our generalization of the Kawamata–Viehweg vanishing theorem.

**Theorem 3.57** (Küronya, [59], Theorem 3.1)**.** *Let $X$ be a smooth projective variety, $L$ a divisor, $A$ a very ample divisor on $X$. If $L|_{E_1\cap\cdots\cap E_k}$ is big and nef for a general choice of $E_1,\ldots,E_k\in|A|$, then*

$$H^i\left(X,\mathcal{O}_X(K_X+L)\right)\,=\,0\quad\textit{for } i>k.$$

The proof relies on the same cohomological machinery that leads to Theorem 2.21. We look at an acyclic resolution

$$0 \longrightarrow \mathcal{O}_X(L) \longrightarrow \mathcal{O}_X(L+rA) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{E_i}(L+rA) \longrightarrow \tag{10}$$
$$\longrightarrow \bigoplus_{1\le i_1<i_2\le r} \mathcal{O}_{E_{i_1}\cap E_{i_2}}(L+rA) \longrightarrow \ldots \longrightarrow \bigoplus_{1\le i_1<i_2<\cdots<i_n\le r} \mathcal{O}_{E_{i_1}\cap\cdots\cap E_{i_n}}(L+rA) \longrightarrow 0\,.$$

analogous to $K^\bullet_{m,p}$, which, to simplify the subsequent accounting process, we chop up into short exact sequences

$$0 \to \mathcal{O}_X(L) \to \mathcal{O}_X(L+rA) \to \mathcal{C}_1 \to 0$$
$$0 \to \mathcal{C}_1 \to \bigoplus_{i=1}^{r} \mathcal{F}\otimes\mathcal{O}_{E_i}(L+rA) \to \mathcal{C}_2 \to 0$$
$$\vdots$$
$$0 \to \mathcal{C}_{n-1} \to \bigoplus_{1\le i_1<i_2<\cdots<i_{n-1}\le r} \mathcal{O}_{E_{i_1}\cap\cdots\cap E_{i_{n-1}}}(L+rA)$$
$$\to \bigoplus_{1\le i_1<i_2<\cdots<i_n\le r} s\mathcal{O}_{E_{i_1}\cap\cdots\cap E_{i_n}}(L+rA) \to 0\,.$$

*Proof.* We will prove the statement by induction on the codimension of the complete intersections we restrict to; the case $q=0$ is the Kawamata–Viehweg vanishing theorem. Let $E_1,\ldots,E_q \in |A|$ be elements such that the intersection of any combination of them is smooth of the expected dimension, and irreducible when it has positive dimension. As the $E_i$'s are assumed to be general, this can clearly be done via the base-point free Bertini theorem, which works for $\dim X \ge 2$. In the remaining cases (when $\dim X \le 1$) the statement of the proposition is immediate.

Consider the exact sequence (10) with $D = K_X + L + mA$ and $r = q$. First we show that it suffices to verify

$$H^i\left(X, \mathcal{C}_1^{(m)}\right) = 0 \quad \text{for all } m \ge 0 \text{ and } i > q-1,$$

where the upper index of $\mathcal{C}$ is used to emphasize the explicit dependence on $m$. Grant this for the moment, and see how this helps up to prove the statement of the proposition.

Take the following part of the long exact sequence associated to the first piece above

$$H^{i-1}\left(X, \mathcal{C}_1^{(m)}\right) \to H^i(X, \mathcal{O}_X(K_X+L+mA))$$
$$\to H^i(X, \mathcal{O}_X(K_X+L+(m+q)A)) \to H^i\left(X, \mathcal{C}_1^{(m)}\right)\,.$$

By assumption the cohomology groups on the two sides vanish for all $m$ whenever $i > q$, hence

$$H^i(X, \mathcal{O}_X(K_X+L+mA)) \simeq H^i(X, \mathcal{O}_X(K_X+L+(m+q)A)) \quad \text{for all } m \ge 0 \text{ and } i > q.$$

These groups are zero however for $m$ sufficiently large by Serre vanishing, hence

$$H^i(X, \mathcal{O}_X(K_X+L)) = 0 \quad \text{for all } i > q,$$

as we wanted.

As for the vanishing of the cohomology groups $H^i\left(X, \mathcal{C}_1^{(m)}\right)$ for $m \geq 0$ and $i > q-1$, it is quickly checked inductively. Observe that for all $1 \leq j \leq q$ we have

$$K_X + L + (m+q)A|_{E_1\cap\cdots\cap E_j} = K_{E_1\cap\cdots\cap E_j} + (L + (m+(q-j))A)|_{E_1\cap\cdots\cap E_j}$$

by adjunction, and $(L + (m+(q-j))A)|_{E_1\cap\cdots\cap E_j}$ becomes ample when restricted to the intersection with $E_{j+1}\cap\cdots\cap E_q$. Induction on $j$ gives

$$H^i\left(E_1\cap\cdots\cap E_j, K_X + L + (m+q)A|_{E_1\cap\cdots\cap E_j}\right) = 0 \quad \text{for all } m \geq 0 \text{ and } i > q-j\ .$$

By chasing through the appropriate long exact sequences, we arrive at

$$H^i\left(X, \mathcal{C}_j^{(m)}\right) = 0 \quad \text{for all } m \geq 0 \text{ and } i > q-j.$$

For $j = 1$ this is the required vanishing. $\square$

## 4 Negativity of curves on algebraic surfaces

### 4.A Summary

Moving on to a slightly different topic, we study curves on smooth projective surfaces with negative self-intersection. Irreducible curves with negative self-intersection have great significance in the theory of algebraic surfaces. They are to a large extent responsible for generating the Mori cone of a surface, and as such, play a central role in both classical and modern aspects of the field.

The basic example of this phenomenon is the exceptional divisor $E$ of the blow-up of a smooth point; such curves satisfy $(E^2) = -1$.

Staying with the same setup, let $X$ be a smooth projective surface, $f : Y = \mathrm{Bl}_p X \to X$ the blow-up of a point $p$ with exceptional divisor $E \subseteq Y$. Given a curve $C \subset X$ with multiplicity $d$ at $p \in X$, we have

$$(C^2) = ((f^*C)^2) = ((\widetilde{C} + dE)^2) = (\widetilde{C}^2) + d^2\ ,$$

that is, the self-intersection of the strict transform $\widetilde{C}$ of $C$ goes down by $d^2$. Consequently, curves with multiplicity at the point $p$ that is high compared to their self-intersection give rise to negative curves on the blow-up.

As an example, negativity of curves on blow-ups of $\mathbb{P}^2$ is closely related to the geometry of plane curves, which is a long-studied and notoriously difficult area of geometry.

Hirzebruch surfaces illustrate that there exist curves on surfaces with arbitrarily negative self-intersection.

Although it is not totally obvious, nevertheless one can construct surfaces with infinitely many negative curves on them. In the case of $-1$-curves[7] the classical

[7] By a $-1$ curve we mean an irreducible curve $C$ with $(C^2) = -1$; in birational geometry it is customary to add the condition $(K_X \cdot C) = -1$. We do not require this.

example is the projective plane blown up at nine general point. Another non-trivial instance can be obtained as follows: let $E$ be an elliptic curves without complex multiplication, then $E \times E$ is an abelian surface with Picard number 3. Blowing up a point on $E \times E$ results in a surface with infinitely many curves of negative self-intersection: the proper transforms of copies of $E$ through the blown-up point will all have this property [51, Exercise 4.16].

Not unexpectedly, this phenomenon is not restricted to self-intersection $-1$.

**Theorem 4.1** (Küronya et al., [10], Theorem B)**.** *4.14 For every natural number $m > 0$ there exists a smooth projective surface $X$ containing infinitely many smooth irreducible curves of self-intersection $-m$. If $m \geq 2$, then even the genus of these curves may be prescribed.*

Going further down this road, the next we question that comes to mind is whether there exists a surface (always smooth and projective) $X$, and a sequence of irreducible curves $C_n$ on $X$ such that

$$(C_n^2) \longrightarrow -\infty \quad \text{as } n \to \infty.$$

**Remark 4.2.** In positive characteristic it has been long known that such surfaces exist. The following example is taken from [41, Exercise V.1.10], but it had certainly been known to André Weil.

Let $C$ be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p$, and consider the surface $X = C \times C$. Then the diagonal $\Delta \subset C \times C$ has self-intersection

$$(\Delta^2) = 2 - 2g < 0 .$$

Let $F : C \to C$ denote the geometric Frobenius endomorphism taking the coordinates of a point of $C$ to their $p^{\text{th}}$ power. Then

$$(((F \times \mathrm{id}_C)^m(\Delta))^2) = p^m \cdot (2 - 2g) \longrightarrow -\infty .$$

The key tool of the above argument was the application of a non-invertible endomorphism to a single curve with negative self-intersection. Interestingly enough, a similar argument proves to be unsuccessful over the complex numbers.

**Theorem 4.3** (Küronya et al., [10], Proposition 2.1)**.** *4.15 Let $X$ be a smooth complex projective surface with a non-invertible endomorphism. Then $X$ has bounded negativity, i.e. there exists a non-negative real number $b(X)$ having the property that*

$$(C^2) \geq -b(X)$$

*for all reduced irreducible curves $C \subseteq X$.*

Independently of the above result, it has been a long-standing folklore conjecture, which, according to eyewitness accounts goes back at least to Enriques, that on a complex surface the negativity of curves is bounded below.

**Conjecture 4.4** (Bounded Negativity Conjecture)**.** *Given a smooth projective surface $X$ over the complex numbers, there exists a non-negative real number $b(X)$ such that*

$$(C^2) \geq -b(X)$$

*for all reduced irreducible curves $C \subseteq X$.*

**Remark 4.5.** The Segre–Gimigliano–Harbourne–Hirschowitz conjecture implies that on blow-ups of $\mathbb{P}^2$ along general finite sets of points, all curves $C$ have self-intersection $\geq -1$.

Such bounds have an immediate effect on various invariants of the surface. We give an example that points out a connection to Seshadri constants.

**Theorem 4.6** (Küronya et al., [12])**.** *4.24 Let $X$ be a smooth projective surface with the property that $(C^2) \geq -b(\widetilde{X})$ for all irreducible curves $C \subseteq \widetilde{X}$ for some non-negative constant $b(\widetilde{X})$, where $\pi : \tilde{X} \to X$ is the blow-up of a point $x$ on $X$. Then the Seshadri constant $\varepsilon(X,x)$ satisfies the lower bound*

$$\varepsilon(X,x) \geq \frac{1}{\sqrt{b(\widetilde{X})+1}} .$$

Beside a large number of examples, further credibility is given to the bounded negativity conjecture by results of Bogomolov [14] and Lu–Miyaoka [64], who obtained strong lower bounds on the self-intersections on curves via vector bundle techniques. Here we present a variant from [12].

**Theorem 4.7** (Lu–Miyaoka, Küronya et al., [64, 12])**.** *4.21 Given a smooth projective surface $X$ over the complex numbers with $\kappa(X) \geq 0$, one has*

$$(C^2) \geq c_1^2(X) - 3c_2(X) + 2 - 2g(C)$$

*for every reduced irreducible curve $C$ of geometric genus $g(C)$.*

**Remark 4.8.** In [64] also prove that under certain topological restrictions on $X$ one can obtain a lower bound on $(C^2)$, which is independent of the genus, thus proving the bounded negativity conjecture in this case. The proof given by the authors is however rather unclear.

### 4.B Negative curves and lowers bounds on self-intersection numbers

Parallel to their importance, determining negative curves on a given surface is an extremely difficult task in general. A strikingly simple open question along these lines is the following.

**Question 4.9.** Let $C$ be a general smooth projective curve of genus $g(C) \geq 5$, $X = C \times C$, $\Delta \subset X$ the diagonal. Is there an irreducible curve $\Gamma \neq \Delta$ on $X$ with negative self-intersection?

One of the obstacles while dealing with negative curves is the almost total loss of control after blowing up. Here is an example: thanks to the work of Kovács we have a good description of the cone of curves of K3 surfaces; in particular, of the negative curves on them.

**Theorem 4.10** (Kovács,[54]). *Let $X$ be a smooth projective K3 surface with $\rho(X) \geq 3$. Then either $X$ contains no curves with negative self-intersection, or the cone of curves is the closure of the cone generated by all smooth rational curves on $X$ with self-intersection $-2$.*

However, one has pretty much no idea about negative curves on blow-ups of K3's, we are not even able to say whether their number in a given case is finite or infinite. An interesting example worked out by Kollár [51, Exercise 4.16] shows that even in the case of a surface with no negative curves, a blow-up can result infinitely many of those. We quickly reproduce this example. Note the arithmetic nature of the surface, we will see later when considering Hilbert modular surfaces that spaces arising from non-trivial number-theoretic situations can lead to very interesting geometry.

**Example 4.11** (Blow-up of an abelian surface). Let $E$ be an elliptic curve without complex multiplication. As it is well-known (see [61, Section 1.5.B] for example), the surface $X = E \times E$ has Picard number three, and the Néron–Severi space is generated by the classes

$$E_{10} = E \times \{0\} \ , \ E_{01} \ = \ \{0\} \times E \ , \ E_{11} \ = \ \Delta \ .$$

Let $(m,n)$ be pair of coprime integers, and set $E_{mn}$ to be the image of the morphism

$$\begin{array}{ccc} E & \longrightarrow & X = E \times E \\ x & \mapsto & (mx,nx) \ . \end{array}$$

Observe that $(E_{mn}^2) = 0$, and all of these curves go through the point $(0,0) \in X$. The abelian surface $X$ contains no curves of negative self-intersection. On the other hand, when blowing up the point $(0,0) \in X$, the resulting surface $\widetilde{X}$ contains infinitely many negative one curves. Beside the exceptional divisor, one has

$$(\widetilde{E}_{mn}^2) \ = \ -1$$

for the strict transforms of all the curves $E_{mn}$.

As it turns out, the classes of the $\widetilde{E}_{mn}$'s and the exceptional divisor generate (up to closure) the Mori cone of $\widetilde{X}$.

In the other direction, one can exert some control over the negative curves on the blow-up under very restrictive conditions on the surface.

**Proposition 4.12.** *Let $X$ be a smooth projective surface over the complex numbers such that*

$$\overline{NE}(X) \ = \ \sum_{i=1}^{r} \mathbb{R}^+[C_i] \ ,$$

*where $C_1, \ldots, C_r$ are irreducible extremal curves. Let furthermore be $p \in X$ be a (closed) point, $d_i \overset{def}{=} \mathrm{mult}_P C_i$, and $\pi : Y \to X$ be the blow-up of the point $p$ in $X$ with exceptional divisor $E$.*

*Assume that for every $1 \leq i, j \leq r$ one has*

$$C_i \cdot C_j \leq d_i d_j \ .$$

*Then*

$$\overline{NE}(Y) = \mathbb{R}^+[E] + \sum_{i=1}^{r} \mathbb{R}^+[\tilde{C}_i] \ ,$$

*where $\tilde{C}_i$ denotes the proper transform of the curve $C_i$.*

*Proof.* First of all, we observe that all the irreducible curves $\tilde{C}_1, \ldots, \tilde{C}_r, E$ are extremal, hence it will be enough to show that irreducible curves $\Gamma \subseteq Y$ satisfy

$$\Gamma \subseteq \mathbb{R}^+[E] + \sum_{i=1}^{r} \mathbb{R}^+[\tilde{C}_i] \ .$$

Write $d \overset{\text{def}}{=} \Gamma \cdot E$, and note that $\tilde{C}_i \cdot E = d_i$ for $1 \leq i \leq r$. As the statement is immediate for the curves $\tilde{C}_1, \ldots, \tilde{C}_r, E$, we will assume that $\Gamma$ is not equal to any of them. In particular, $\pi_* \Gamma$ is an irreducible curve on $X$ and as such can be written in the form

$$\pi_*[\Gamma] = \sum_{i=1}^{r} a_i [C_i] \ ,$$

with all $a_i$'s being nonnegative.

Then

$$\pi^*(\pi_*[\Gamma]) = \sum_{i=1}^{r} a_i (\pi^*[C_i]) = \sum_{i=1}^{r} a_i [\tilde{C}_i] + \left( \sum_{i=1}^{r} a_i d_i \right) [E] \ .$$

On the other hand,

$$\pi^*(\pi_*[\Gamma]) = [\Gamma] + d[E] \ ,$$

since $\widetilde{\pi_* \Gamma} = \Gamma$, and therefore

$$[\Gamma] = \sum_{i=1}^{r} a_i [\tilde{C}_i] + \left( \sum_{i=1}^{r} a_i d_i - d \right) [E] \ .$$

We are left with proving that

$$\sum_{i=1}^{r} a_i d_i \geq d \ .$$

To this end, let us compute the intersection numbers of $\Gamma$ against the curves $\tilde{C}_j$. Given that $\Gamma$ is not equal to any of them, we have

$$\begin{aligned} 0 \leq \Gamma \cdot \tilde{C}_j &= \sum_{i=1}^{r} a_i (\tilde{C}_i \cdot \tilde{C}_j - d_i d_j) + \left( \sum_{i=1}^{r} a_i d_i - d \right) d_j \\ &= \sum_{i=1}^{r} a_i (C_i \cdot C_j) - d d_j \ , \end{aligned}$$

more concisely

$$\sum_{i=1}^{r} a_i(C_i \cdot C_j) - dd_j \geq 0 \ .$$

After multiplying this inequality by the nonnegative $a_j$ and summing over $j$, we arrive at

$$\sum_{j=1}^{r}\sum_{i=1}^{r} a_i a_j (C_i \cdot C_j) \geq d \cdot \sum_{j=1}^{r} a_j d_j \ .$$

The conditions $C_i \cdot C_j \leq d_i d_j$ then imply that

$$\left(\sum_{i=1}^{r} a_i d_i\right)^2 = \sum_{i=1}^{r}\sum_{j=1}^{r} a_i a_j d_i d_j \geq d \cdot \sum_{j=1}^{r} a_j d_j \ .$$

Since $\sum_{i=1}^{r} a_i d_j$ is nonnegative, we can conclude that

$$\sum_{i=1}^{r} a_i d_i \geq d$$

as we wanted. □

The conditions of the Proposition are satisfied for example when all extremal rays of the Mori cone of $X$ go through the point $p$ that we blow up. This happens on ruled surfaces, in particular on all Hirzebruch surfaces.

Next, we move on to the issue of having infinitely many negative curves on a surface, and the question of a lower bound on their self-intersection number. Beside Example 4.11, the canonical example of a surface with infinitely many negative curves is the blow-up of $\mathbb{P}^2$ at nine points. Here again we follow the exposition of Kollár [51, Example 4.15.3].

**Example 4.13.** Let $C_1, C_2 \subset \mathbb{P}^2$ be two smooth cubics with nine intersection points $P_1, \ldots, P_9$; assume moreover that every element of the pencil generated by the $C_i$'s is irreducible. Set

$$X \stackrel{\text{def}}{=} \mathrm{Bl}_{P_1,\ldots,P_9}\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2 \ .$$

The pencil given by $\widetilde{C}_1$ and $C_2$ becomes base-point free on $X$, therefore it gives rise to a morphism

$$f \ : \ X \longrightarrow \mathbb{P}^1 \ ,$$

the fibres of which correspond to irreducible plane cubics.

Let $\Sigma \subseteq X$ be the finite set of singular points of the fibres of $f$. The nine exceptional divisors $E_1, \ldots, E_9$ of $\pi$ (with $E_i$ sitting over $P_i$) then become sections of the morphism $f$. Let us fix the section $E_9$, then we can consider $X - \Sigma$ as a family of algebraic groups over $\mathbb{P}^1$.

For a general choice of $C_1$ and $C_2$, the sections $E_1, \ldots, E_8$ generate a free abelian subgroup of rank 8 of $\mathrm{Aut}(X \setminus \Sigma)$. Since $\Sigma$ is a finite set of points, $\mathrm{Aut}(X \setminus \Sigma) \subseteq \mathrm{Aut}(X)$.

Given a vector $m = (m_1, \ldots, m_8) \in \mathbb{Z}^8)$, let $\sigma_m \in \mathrm{Aut}(X)$ be the automorphism of $X$ corresponding to $\sum_{i=1}^{8} m_i E_i$. Note that $\sigma_m(E_9)$ is a smooth rational curve of self-intersection $-1$ on $X$.

In both examples listed so far, the infinitely many negative curves all have self-intersection $-1$. We show how one can obtain examples with highly negative curves.

**Theorem 4.14** (Küronya et al., [10])**.** *For every natural number $m > 0$ there exists a smooth projective surface $X$ containing infinitely many smooth irreducible curves of self-intersection $-m$. If $m \geq 2$, then even the genus of these curves may be prescribed.*

*Proof.* Let $f : X \to B$ be a smooth complex projective minimal elliptic surface with a section, fibered over a smooth base curve $B$ of genus $g(B)$. Then $X$ can have no multiple fibers, so that by Kodaira's well-known result (cf. [8, V,Corollary 12.3]), $K_X$ is a sum of a specific choice of $2g(B)-2+\chi(\mathcal{O}_X)$ fibers of the elliptic fibration. Let $C$ be any section of the elliptic fibration $f$. By adjunction, $C^2 = -\chi(\mathcal{O}_X)$.

Take $X$ to be rational and $f$ to have infinitely many sections; for example, blow up the base points of a general pencil of plane cubics. Then $\chi(\mathcal{O}_X) = 1$, so that $C^2 = -1$ for any section $C$.

Pick any $g \geq 0$ and any $m \geq 2$. Then, as is well-known [44], there is a smooth projective curve $C$ of genus $g$ and a finite morphism $h : C \to B$ of degree $m$ that is not ramified over points of $B$ over which the fibers of $f$ are singular. Let $Y = X \times_B C$ be the fiber product. Then the projection $p : Y \to C$ makes $Y$ into a minimal elliptic surface, and each section of $f$ induces a section of $p$. By the property of the ramification of $h$, the surface $Y$ is smooth and each singular fiber of $f$ pulls back to $m$ isomorphic singular fibers of $p$. Since $e(Y)$ is the sum of the Euler characteristics of the singular fibers of $p$ (cf. e.g. [8, III, Proposition 11.4]), we obtain from Noether's formula that $\chi(\mathcal{O}_Y) = e(Y)/12 = me(X)/12 = m\chi(\mathcal{O}_X) = m$. Therefore, for any section $D$ of $p$, we have $D^2 = -m$; i.e., $Y$ has infinitely many smooth irreducible curves of genus $g$ and self-intersection $-m$. $\square$

Now that the question of having infinitely many negative curves of given self-intersection has been settled, one can ask the very natural question whether one can tend to infinity with the negativity of the curves. As it was shown above, it is easy to obtain such sequences of curves in positive characteristic via taking a single negative curve and applying a non-trivial endomorphism to it. Interestingly enough, this process fails spectacularly over the complex numbers.

**Theorem 4.15.** *Let $X$ a smooth projective complex surface admitting a surjective endomorphism that is not an isomorphism. Then $X$ has bounded negativity, i.e., there is a bound $b(X)$ such that*

$$(C^2) \geq -b(X)$$

*for every reduced irreducible curve $C \subset X$.*

*Proof.* It is a result of Fujimoto and Nakayama ([36] and [70]) that a surface $X$ satisfying our hypothesis is of one of the following types:

(1) $X$ is a toric surface;
(2) $X$ is a $\mathbb{P}^1$-bundle;
(3) $X$ is an abelian surface or a hyperelliptic surface;

(4) $X$ is an elliptic surface with Kodaira dimension $\kappa(X) = 1$ and topological Euler number $e(X) = 0$.

In cases (1) and (2) the assertion is clear as $X$ then carries only finitely many negative curves. In case (3) bounded negativity follows from the adjunction formula (cf. [12, Prop. 3.3.2]). Finally, bounded negativity for elliptic surfaces with $e(X) = 0$ is the topic of Proposition 4.16. □

**Proposition 4.16.** *Let $X$ be a smooth projective complex elliptic surface with $e(X) = 0$. Then there are no negative curves on $X$.*

*Proof.* Let $\pi : X \to B$ be an elliptic fibration with $B$ a smooth curve, and $F$ the class of a fiber of $\pi$. By the properties of $e(X)$ on a fibered surface (cf. [8, III, Proposition 11.4 and Remark 11.5]), the only singular fibers of $X$ are possibly multiple fibers, and the reduced fibers are always smooth elliptic curves. In particular, $X$ must be minimal and its fibers do not contain negative curves.

Aiming at a contradiction, suppose that $C \subset X$ is a negative curve. Then, by the above, the intersection number $n \overset{\text{def}}{=} C \cdot F$ is positive. This means that $\pi$ restricts to a map $C \to B$ of degree $n$. Taking an embedded resolution $f : \widetilde{X} \to X$ of $C$, we get a smooth curve $\widetilde{C} = f^*C - \Gamma$, where the divisor $\Gamma$ is supported on the exceptional locus of $f$. The Hurwitz formula, applied to the induced covering $\widetilde{C} \to B$, yields

$$2g(\widetilde{C}) - 2 = n \cdot (2g(B) - 2) + \deg R \ , \tag{11}$$

where $R$ is the ramification divisor.

Let $m_1F_1, \ldots, m_kF_k$ denote the multiple fibers of $\pi$. The assumption $e(X) = 0$ implies via Noether's formula that $K_X \equiv (2g(B) - 2)F + \sum(m_i - 1)F_i$. Hence

$$\begin{aligned}
(K_X \cdot C) &= n(2g(B) - 2) + \sum(m_i - 1)(F_i \cdot C) \\
&= n(2g(B) - 2) + \sum(m_i - 1)(f^*F_i \cdot f^*C) \\
&= n(2g(B) - 2) + \sum(m_i - 1)(f^*F_i \cdot \widetilde{C}) \\
&\leq n(2g(B) - 2) + \deg R.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2g(\widetilde{C}) - 2 = (\widetilde{C}^2) + (K_{\widetilde{X}} \cdot \widetilde{C}) &= ((f^*C - \Gamma)^2) + ((f^*K_X + K_{\widetilde{X}/X}) \cdot (f^*C - \Gamma)) \\
&= (C^2) + (\Gamma^2) + (K_X \cdot C) - (K_{\widetilde{X}/X} \cdot \Gamma) \ .
\end{aligned}$$

Consequently, using the Hurwitz formula (11), we obtain

$$(C^2) \geq (K_{\widetilde{X}/X} \cdot \Gamma) - (\Gamma^2) \ .$$

Then Lemma 4.17 yields the contradiction $(C^2) \geq 0$. □

**Lemma 4.17.** *Let $f : Z \to X$ be a birational morphism of smooth projective surfaces, and let $C \subset X$ be any curve, with proper transform $\widetilde{C} = f^*C - \Gamma_{Z/X}$ on $Z$. Then*

$$(K_{Z/X} \cdot \Gamma_{Z/X}) - (\Gamma_{Z/X}^2) \geq 0 \ .$$

*Proof.* As $f$ is a finite composition of blow-ups, this can be seen by an elementary inductive argument. For the convenience of the reader we briefly indicate it. Suppose that $f$ consists of $k$ successive blow-ups. For $k = 1$ the assertion is clear, since then $K_{Z/X}$ is the exceptional divisor $E$, and $\Gamma$ is the divisor $mE$, where $m$ is the multiplicity of $C$ at the blown-up point. For $k > 1$ we may decompose $f$ into two maps

$$Z \xrightarrow{g} Y \xrightarrow{h} X .$$

One has proper transforms

$$C' = h^*C - \Gamma_{Y/X} \quad \text{and} \quad \widetilde{C} = g^*C' - \Gamma_{Z/Y} = f^*C - \Gamma_{Z/X} .$$

The equalities

$$\begin{aligned} K_{Z/X} &= K_{Z/Y} + g^*K_{Y/X} \\ \Gamma_{Z/X} &= \Gamma_{Z/Y} + g^*\Gamma_{Y/X} \end{aligned}$$

then imply

$$\begin{aligned} (K_{Z/X} \cdot \Gamma_{Z/X}) - (\Gamma^2_{Z/X}) &= ((K_{Z/Y} + g^*K_{Y/X}) \cdot (\Gamma_{Z/Y} + g^*\Gamma_{Y/X})) \\ &- ((\Gamma_{Z/Y} + g^*\Gamma_{Y/X})^2) \\ &= ((K_{Z/Y} \cdot \Gamma_{Z/Y}) - (\Gamma^2_{Z/Y})) + (K_{Y/X} \cdot \Gamma_{Y/X}) - (\Gamma^2_{Y/X}) , \end{aligned}$$

and the assertion follows by induction. □

**Remark 4.18.** A direct conceptual proof of Theorem 4.15 which does not build on the classification of surfaces with non-trivial endomorphisms would be very interesting.

Additional argument for the truth of the bounded negativity conjecture in certain cases comes from ideas concentrating on the lack of positivity in the cotangent bundle. The technical basis for such a rationale is the Bogomolov–Sommese vanishing theorem [34, Theorem 6.9].

**Theorem 4.19** (Bogomolov–Sommese)**.** *Let $X$ be a smooth complex projective surface, $\mathcal{L}$ a line bundle, $C$ a normal crossing divisors on $X$. Then*

$$H^0\left(X, \Omega^a_X(\log C) \otimes \mathcal{L}^{-1}\right) = 0 \quad \text{for all } a < \kappa(X, \mathcal{L}).$$

*In particular, under the given circumstances $\Omega^1_X(\log C)$ contains no big line-subbundles.*

Building on the non-existence of big line bundles inside the log-cotangent bundle, Miyaoka in [67] established the inequality

$$c_1(\Omega^1_X)^2 \leq 3c_2(\Omega^1_X) ,$$

which was in turn generalized by Sakai [76, Theorem 7.6] to the log category. The proof in [76] is somewhat sketchy, a detailed version can be found in the Appendix of [12].

**Theorem 4.20** (Logarithmic Miyaoka–Yau inequality)**.** *Let $X$ be a smooth complex projective surface, $C$ a semi-stable curve on $X$ such that $K_X+C$ is big. Then*

$$c_1(\Omega^1_X(\log C))^2 \leq 3c_2(\Omega^1_X(\log C)),$$

*in other words,*

$$(K_X+C)^2 \leq 3(e(X)-e(C)).$$

The actual lower bound on the self-intersection is contained in the following result.

**Theorem 4.21.** *Let $X$ be a smooth projective surface with $\kappa(X)\geq 0$. Then for every reduced, irreducible curve $C\subset X$ of geometric genus $g(C)$ we have*

$$C^2 \geq c_1^2(X)-3c_2(X)+2-2g(C). \tag{12}$$

*Proof.* The idea is to reduce the statement to a smooth curve and use Theorem 4.20.

We blow up $f:\tilde{X}=X_N\to X_{N-1}\to\ldots\to X_0=X$ resolving step-by-step the singularities of $C$. The proper transform of $C$ in $\tilde{X}$ is then a smooth irreducible curve $\tilde{C}$. Applying Lemma 4.22 recursively to every step in the resolution $f$ we see that it is enough to prove inequality (12) for $C$ smooth.

This follows easily from the Logarithmic Miyaoka-Yau inequality 4.20. Note that our assumption $\kappa(X)\geq 0$ implies that $K_{\tilde{X}}+\tilde{C}$ is $\mathbb{Q}$–effective. Hence

$$\begin{aligned} c_1^2(X)+2C\cdot(K_X+C)-C^2 &= c_1^2(\Omega^1_X(\log C)) \\ &\leq 3c_2(\Omega^1_X(\log C)) \\ &= 3c_2(X)-6+6g(C)\;. \end{aligned}$$

By adjunction $C\cdot(K_X+C)=2g(C)-2$ and rearranging terms we arrive at (12). □

**Lemma 4.22.** *Let $X$ be a smooth projective surface, $C\subset X$ a reduced, irreducible curve of geometric genus $g(C)$, $P\in C$ a point with* $\mathrm{mult}_P C\geq 2$*. Let $\sigma:\tilde{X}\to X$ be the blow-up of $X$ at $P$ with the exceptional divisor $E$. Let $\tilde{C}=\sigma^*(C)-mE$ be the proper transform of $C$. Then the inequality*

$$\tilde{C}^2 \geq c_1^2(\tilde{X})-3c_2(\tilde{X})+2-2g(\tilde{C})$$

*implies*

$$C^2 \geq c_1^2(X)-3c_2(X)+2-2g(C).$$

*Proof.* We have

$$C^2=\tilde{C}^2+m^2,\;\; c_1^2(X)=c_1^2(\tilde{X})+1,\;\; c_2(X)=c_2(\tilde{X})-1 \;\text{ and }\; g(C)=g(\tilde{C}).$$

Hence

$$\begin{aligned} C^2 &= m^2+\tilde{C}^2 \\ &\geq m^2+c_1^2(\tilde{X})-3c_2(\tilde{X})+2-2g(\tilde{C}) \\ &= m^2+c_1^2(X)-1-3c_2(X)-3+2-2g(C) \\ &\geq c_1^2(X)-3c_2(X)+2-2g(C). \end{aligned}$$

□

To illustrate the significance of Conjecture 4.4, we present an application concerning an open question of Demailly [25, Question 6.9] about global Seshadri constants.

**Question 4.23.** Let $X$ be a smooth projective surface. Is the global Seshadri constant

$$\varepsilon(X) \stackrel{\text{def}}{=} \inf\{\, \varepsilon(L) \mid L \in \operatorname{Pic}(X) \text{ ample} \,\}$$

positive?

As of today this is unknown, worse, it is unknown whether for every fixed $x \in X$ the quantity

$$\varepsilon(X,x) = \inf\{\, \varepsilon(L,x) \mid L \in \operatorname{Pic}(X) \text{ ample} \,\}$$

is always positive. The latter however would follow from the Bounded Negativity Conjecture.

**Theorem 4.24** (Küronya et al., [12])**.** *Let $X$ be a smooth projective surface with the property that $(C^2) \geq -b(\widetilde{X})$ for all irreducible curves $C \subseteq \widetilde{X}$ for some non-negative constant $b(\widetilde{X})$, where $\pi : \tilde{X} \to X$ is the blow-up of a point $x$ on $X$. Then the Seshadri constant $\varepsilon(X,x)$ satisfies the lower bound*

$$\varepsilon(X,x) \geq \frac{1}{\sqrt{b(\widetilde{X})+1}} \ .$$

*Proof of the proposition.* Let $C \subset X$ be an irreducible curve of multiplicity $m$ at $x$, and let $\tilde{C} \subset Y$ be its proper transform on the blow-up $Y$ of $X$ in $x$. Then

$$C^2 - m^2 = (f^*C - mE)^2 = \tilde{C}^2 \geq -b(Y) \ .$$

Consider first the case where $m \leq \sqrt{b(Y)}$. Then

$$\frac{L \cdot C}{m} \geq \frac{L \cdot C}{\sqrt{b(Y)}} \geq \frac{1}{\sqrt{b(Y)}}$$

In the alternative case, where $m > \sqrt{b(Y)}$, we have

$$C^2 \geq m^2 - b(Y) > 0$$

and hence, using the Index Theorem, we get

$$\frac{L \cdot C}{m} \geq \frac{\sqrt{L^2}\sqrt{C^2}}{m} \geq \sqrt{1 - \frac{b(Y)}{m^2}} \geq \sqrt{1 - \frac{b(Y)}{b(Y)+1}} = \frac{1}{\sqrt{b(Y)+1}}.$$

□

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