# AN INFORMAL INTRODUCTION TO THE MINIMAL MODEL PROGRAM

ALEX KÜRONYA

*This is an expanded version of a talk given in the Fall 2007 Forschungsseminar at the Universität Duisburg-Essen. The purpose of our seminar was to understand the recent seminal work* [2] *of Hacon and McKernan. This note is written for non-specialists by a non-specialist, and hence might contain more or simply different details than experts would think necessary. Naturally, no claims regarding originality are made.*

## 1. Birational equivalence in low dimensions

There is an obvious desire to understand the structure of algebraic varieties, and this is what we will try to do in some sense. During this quest we will restrict ourselves to varieties over the complex numbers (for safety reasons), although a substantial amount of what is said here goes through over other fields as well.

Very early on in one's career as a student in algebraic geometry, we meet two different notions for two algebraic varieties 'being the same': isomorphy, and birational equivalence. In the case of affine varieties, these correspond to isomorphisms between the coordinate rings or the function fields, respectively.

Although for general abstract varieties the isomorphism between the rings of globally defined regular functions is no longer equivalent to the underlying varieties being isomorphic[1], the abovementioned characterization of birational equivalence carries through in the general case: two abstract varieties are birationally equivalent if and only if their function fields are isomorphic. In this sense, birational geometry of varieties is nothing else than characterizing function fields of varieties.

The problem of describing isomorphy classes of varieties is very difficult, and hence one might as well opt for a two-step method: first understand varieties up to birational equivalence, and then work our way from there. It is important to point out that it is not obvious which of the two steps is more difficult in general.

In what follows we will primarily care about birational geometry. Historically one wanted to work with smooth objects only, but as we will see soon, we had to give up this idea at a certain point in order to be able to proceed.

We are grateful to the authors of [1] for providing us with a very early version of their work.

[1]For example if $X$ is an arbitrary projective variety then $\Gamma(X, \mathcal{O}_X) = k$, the base field.

Let us start with the case of curves. Here the picture is particularly simple from the point of view of birational equivalence: two smooth projective curves are birational if and only if they are isomorphic to each other. In short, each irreducible curve is birational to a unique smooth projective curve, thus the investigation of smooth projective curves up to isomorphism is equivalent to the study of all curves up to birational equivalence.

The case of smooth surfaces is a lot more complicated, and the classification of smooth projective surfaces up to birational equivalence is one of the major algebro-geometric achievements of the first half of the 20th century done by the so-called Italian school. Today it is often called the Kodaira–Enriques classification.

For starters, note that each irreducible surface is birational to infinitely many smooth projective surfaces (for example take a smooth point, blow it up, and so on). Our aim here is to find a unique 'simple' element of each birational equivalence class of smooth projective surfaces. The method of the Italian school is simple: if a smooth projective surface contains a smooth rational curve with self-intersection $-1$, then by Castelnuovo's contractibility theorem this curve can be contracted, and the result will again be a smooth projective surface.

More precisely:

**Theorem 1.1** (Castelnuovo, Theorem V.5.7 in [3])**.** *Let $X$ be a smooth projective surface, $E \subseteq X$ a smooth rational curve with $E^2 = -1$. Then there exists a smooth projective surface $Y$ , and a morphism $f : X \to Y$ such that $f$ is the blow-up of a point on $Y$ with $E$ being the exceptional divisor.*

Now if one repeats this process over and over again, then after finitely many steps it will come to a halt as the Picard number drops by one after each contraction. In most cases we obtain a unique minimal model this way. Within this framework minimal models are distiguished by the lack of $(-1)$-curves.

On the other hand, birational maps can also be described explicitly:

**Theorem 1.2** (Zariski, Theorem V.5.5 in [3])**.** *Let $\phi : X \dashrightarrow Y$ be a birational map. Then we can factor $\phi$ into a sequence of blow-ups of points and their inverses.*

## 2. The birational classification of surfaces and Mori's cone theorem

A crucial observation made in the last third of $20^{\text{th}}$ century was the outstanding role the canonical class plays in the classification process. The idea is closely related to the following simple observation.

**Lemma 2.1.** *Let $X$ be a smooth projective surface, $C \subseteq X$ an irreducible curve with $C^2 = -1$. Then $C$ is a smooth rational curve if and only if $K_X \cdot C = -1$.*

*Proof.* The proof follows from the fact that for an integral projective curve $\Gamma$, $\Gamma \simeq \mathbb{P}^1$ is equivalent to $p_a(\Gamma) = 0$ via the (arithmetic) genus formula [3, V. Ex. 1.3 (a)] [2]:

$$p_a(C) = \frac{1}{2}(K_X + C) \cdot C + 1$$

applied to our curve $C \subseteq X$. □

To put it a little bit more generally (same proof applies): an integral curve $C \subseteq X$ is a $(-1)$-curve if and only if $C^2 < 0$ and $K_X \cdot C < 0$.

The upshot is that instead of trying to find and contract $(-1)$-curves (which does not generalize to higher dimensions) one should concentrate on curves that intersect the canonical class negatively. Hence from now on we will call a smooth projective variety a *minimal model*, if it is devoid of such curves, that is, if $K_X$ is nef[3]. The two notions of a minimal model coincide very often[4]. With this in our hand it is easy to believe that extremal rays (at least $K_X$-negative ones) play a crucial role in the development of birational geometry. In this direction the basic object is the so-called *Mori cone*, which is also called the closed cone of curves.

**Definition 2.2.** Let $X$ be a smooth projective variety; a 1-*cycle* $C = \sum_{i=1} a_i C_i$ over $X$ is a finite linear combination of proper integral curves with $a_i \in \mathbb{Z}$ (or $\mathbb{Q}$ or $\mathbb{R}$). The 1-cycle $C$ is called *effective* if $a_i \geq 0$ for all $i$'s.

Two one cycles $C, C'$ are called *numerically equivalent* if

$$D \cdot C = D \cdot C'$$

for all Cartier divisors $D$ on $X$. The set of equivalence classes of 1-cycles with real coefficients under numerical equivalence is a real vector space, it is denoted by $N_1(X)$. We denote the numerical equivalence class of a 1-cycle by $[C]$.

*Remark* 2.3. It is a non-trivial fact (nevertheless not too difficult over $\mathbb{C}$) that $\dim_{\mathbb{R}} N_1(X)$ is finite.

---

[2]The adjunction formulas one often finds require the effective divisor $D \subseteq X$ we restrict on to be smooth. There are however considerably more general versions in existence. As described in [4, Proposition 5.73], if $X$ is a projective Cohen–Macaulay scheme of pure dimension $n$ over an arbitrary field, $D$ an effective Cartier divisor on $X$, then

$$\omega_D \simeq \omega_X(D) \otimes \mathcal{O}_D$$

with $\omega$ is the appropriate dualizing sheaf. This in turn has the property that $\omega_X \simeq \mathcal{O}_X(K_X)$ whenever $X$ is a normal projective variety.

[3]A Cartier divisor $D$ is called *nef* if $D \cdot C \geq 0$ for every effective 1-cycle. It is of course enough to test nefness on irreducible curves.

[4]A notable exception is $\mathbb{P}^2$, which is minimal in the classical sense, but $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$ is not nef. The reason for this phenomenon is basically that the morphism contracting the whole projective plane to a point is the contraction of a single $K_X$-negative extremal ray spanned by the class of a line.

**Definition 2.4.** Let $X$ be a smooth projective variety. Then define

$$\begin{aligned} \mathrm{NE}(X) &\stackrel{\mathrm{def}}{=} \left\{\sum a_i[C_i] \,|\, C_i \subseteq X \text{ integral proper curve }, 0 \leq a_i \in \mathbb{Q}\right\} \\ &\subset N_1(X) \\ \overline{\mathrm{NE}}(X) &\stackrel{\mathrm{def}}{=} \text{ the closure of } \mathrm{NE}(X) \text{ in } N_1(X) \ . \end{aligned}$$

The cone $\overline{\mathrm{NE}}(X)$ is called the *Mori cone* of $X$.

The following result of Mori was one of his first major breakthroughs.

**Theorem 2.5** (Mori's cone theorem, Theorems 1.24 and 3.7 in [4])**.** *Let $X$ be a smooth projective variety; then there exist countably many rational curves $C_i$ on $X$ such that $0 < -K_X \cdot C_i \leq \dim X + 1$, and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{KX\geq 0} + \sum_i \mathbb{R}^{\geq 0}[C_i] \ .$$

*Moreover, if $H$ is any ample divisor class, and $\epsilon > 0$ an arbitrary real number, then*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{KX+\epsilon H\geq 0} + \sum \mathbb{R}^{\geq 0}[C_i] \ .$$

*where the latter sum is now finite.*

What links extremal rays to morphisms between varieties contracting certain curves is the following.

**Definition 2.6** (Contraction)**.** Let $X$ be a projective variety, $F \subseteq \overline{\mathrm{NE}}(X)$ an extremal face. A morphism $c_F : X \to Y$ is called the contraction of $F$ if the following conditions are satisfied.

(a) if $X$ is an irreducible curve, then $c_F(C)$ is a point iff $[C] \in F$;
(b) $(c_F)_*\mathcal{O}_X = \mathcal{O}_Y$.

*Remark* 2.7 (Properties of contractions). There is no guarantee that for a certain extremal face $F$ the contraction morphism $c_F$ exists, and there are examples where it does not. In fact, it is a very important subtask of the minimal model program to find conditions that guarantee the contractability of certain extremal faces. If however $c_F$ exists, then it is uniquely determined by the extremal face $F$.

With this result in our hands, one would like to know how to describe extremal rays on projective varieties. It turns out, that the determination of the Mori cone of a variety is an extremely difficult task, it is only known in a very few cases[5] What one can do though is to give a rough classification, according to their behaviour.

[5]To see the extent of our lack of knowledge here is a simple example: let $C$ be a smooth curve of genus 5. Then one has no idea whatsoever about the extremal rays of $C \times C$.

**Theorem 2.8** (Theorem 1.28 in Kollár–Mori)**.** *Let $X$ be a smooth projective surface, $R \subseteq \overline{\mathrm{NE}}(X)$ a $K_X$-negative extremal ray. Then the corresponding contraction $c_R : X \to Y$ exists, and it falls into exactly one of the following three classes:*

- *(a) $Y$ is a smooth projective surface with $\rho(Y) = \rho(X) - 1$, $c_R$ is the blowup of a (closed) point in $Y$.*
- *(b) $Y$ is a smooth projective curve, $\rho(X) = 2$, and $X$ is a minimal ruled surface over $Y$.*
- *(c) $Y$ is a point, $\rho(X) = 1$, and $-K_X$ is ample.*

*Remark* 2.9. Let $C$ be an irreducible curve such that $[C] \in R$. As we will shortly see, the three cases above correspond to the sign of $C^2$. In case 3 Mori also proved that $Y$ is isomorphic to the projective plane[6].

*Proof of 2.8.* As mentioned earlier, the three cases are distinguished according to the sign of the self-intersection of an irreducible curve generating the extremal ray $R$.

Let us first assume that $(C^2) < 0$. Then the adjunction formula says that

$$2p_a(C) - 2 = (K_X + C) \cdot C \leq -2$$

and so $g(C) = 0$, $K_X \cdot C = (C^2) = -1$, hence $C$ is a $(-1)$-curve, and one can apply Theorem 1.1 to obtain the desired conclusion.

Next we treat the case $(C^2) = 0$. The proof proceeds by showing that for $m \geq 1$ the complete linear system $|mC|$ is base-point free, and provides the required contraction. First let us see what the Riemann–Roch theorem has to offer: as $C$ is effective, $H^2\big(X, mC\big) = 0$ for all $m \gg 1$, and so

$$\begin{aligned} H^0\big(X, mC\big) &= H^0\big(X, mC\big) - H^0\big(X, mC\big) \\ &= \chi(X, mC) \\ &= \frac{1}{2}(mC - K_X) \cdot (mC) - \chi(X, \mathcal{O}_X) \\ &= \frac{(-K_X \cdot C)}{2} m + \chi(X, \mathcal{O}_X) \\ &\geq 2\ . \end{aligned}$$

once $m$ becomes large enough. Since any fixed component is a multiple of $C$, there exists an integer $m_0 > 0$ such that $|m_0 C|$ has no fixed component, and so has a base-point free multiple by a result of Zariski[7]. It can be shown that that $|mC|$ has to be base-point free for $m = 1$.

[6]The statement that $-K_X$ very ample implies $X$ is a projective space had been previously known as Hartshorne's conjecture

[7]Zariski proved that if some multiple of a base-point free complete linear system has only isolated base points, then it will eventually become base-point free.

Let $c_R : X \to Z$ be the Stein factorization of the corresponding morphism. Consider a fibre $\sum_i a_i C_i$ of the morphism $c_R$. Then

$$\sum_i a_i[C_i] = [C] \in R \ ,$$

and since $R$ is an extremal ray, we have $[C_i] \in R$ for every $i$, in particular $C_i^2 = 0$ and $K_X \cdot C_i < 0$. An application of the adjunction formula gives

$$2p_a(C) - 2 = (K_X + C) \cdot C = K_X \cdot C < 0$$

which implies $C_i \simeq \mathbb{P}^1$ and $K_X \cdot C_i = -2$. Hence

$$-2 = K_X \cdot C = K_X \cdot (\sum_i a_i C_i) = -2 \cdot \sum_i a_i C_i \ ,$$

and so $\sum_i a_i C_i$ is a reduced irreducible rational curve. We conclude that $c_R : X \to Y$ is a minimal ruled surface over $Y$.

Lastly, consider the case when $C^2 > 0$. One can show that the classes of curves with positive self-intersection lie in the interior of $\overline{\mathrm{NE}}(X)$. However, $[C]$ generates an extremal ray as well, so it has to be the case that $N^1(X) \simeq \mathbb{R}^1$. As we have assumed that $K_X \cdot C > 0$, the anticanonical divisor is positive on $\overline{\mathrm{NE}}(X) - 0$, and hence ample by a form of the Nakai–Moishezon ampleness criterion. □

As a consequence, we can describe minimal models of smooth surfaces.

**Corollary 2.10.** *Let $X$ be a smooth projective surface. Then one can find a sequence of contraction morphisms $X \to X_1 \to \cdots \to X_n = Y$ such that exactly one of the following happens.*

*(a) $-K_Y$ is nef;*
*(b) $Y$ is a minimal ruled surface over a curve;*
*(c) $Y \simeq \mathbb{P}^2$.*

*Remark* 2.11. If $Y$ is a minimal model (i.e. $-K_Y$ is nef), then $Y$ is uniquely determined by $X$.

Let us summarize what has just happened.

*Algorithm* 2.12 (The Mori program for surfaces).

(a) Start with a smooth projective surface $X$.
(b) If $K_X$ is nef then stop. If $K_X$ is not nef then the Cone Theorem 2.5 provides us with a $K_X$-negative extremal ray $R$.
(c) Theorem 2.8 implies that the contraction morphism $c_R : X \to Y$ exists.
   (a) If $\dim Y = \dim X$ then replace $X$ with $Y$ and go back to (a).
   (b) If $\dim Y < \dim X$ then Theorem 2.8 gives detailed information about the structure of $X$.

This is the framework one would want to generalize to higher dimensions to obtain minimal models. In this last formulation everything makes sense on a normal $\mathbb{Q}$-factorial quasi-projective variety, but there the contractions that might occur often produce singular varieties, which causes various parts of the machine to break down.

## 3. Extremal rays on higher-dimensional varieties

After having determined the various types of extremal rays that can occur on a smooth projective surface, it is natural to ask what happens in higher dimensions. Mori has classified the various possibilities in the case of a smooth threefold. The classification runs much in parallel with the one we have seen on surfaces, there is however one very basic difference: it can and indeed does happen that the result of a contraction of a $K_X$-negative extremal ray is a *singular* variety, and hence we cannot apply our previous methods directly.

One obvious way around the difficulty would be to allow some classes of singular varieties, however this means walking on really thin ice, as many of the methods we have used break down if the singularities become too nasty. The question which class of singularities we should allow has no clear answer. The approach we will take is roughly to start with smooth varieties, and include everything we find along the way.

Whatever happens, we will want

(a) to have a canonical class $K_X$ in some sense;
(b) and to be able to intersect curves with $K_X$.

Therefore it is absolutely necessary that $K_X$ be a $\mathbb{Q}$-Cartier divisor under all circumstances. As we will see in later talks, $K_X$ exists on all normal varieties as a Weil divisor class. In particular, this minimal working requirement can be maintained while all varieties in sight are $\mathbb{Q}$-factorial normal varieties (actually, $\mathbb{Q}$-Gorenstein normal would be enough, but that is a lot more difficult to guarantee) and we will always assume this. However, in order for the minimal model program to work (for example to have the Cone Theorem 2.5), one needs to impose further conditions on the varieties in question.

Let us have a look at what happens to contractions in higher dimensions.

**Proposition 3.1.** *Let $X$ be a $\mathbb{Q}$-factorial normal projective variety, $f : X \to Y$ the contraction corresponding to an extremal ray $R \subseteq \overline{NE}(X)$. Then we have the following options.*

*(a) (Fibre type contraction)* $\dim X > \dim Y$*;*
*(b) (Divisorial contraction) $f$ is birational and the exceptional locus* $\mathrm{Ex}(f)$ *is a prime divisor;*
*(c) (Small contraction) $f$ is birational and* $\operatorname{codim}_X \mathrm{Ex}(f) \geq 2$.

*Remark* 3.2. It is instructive to check the correspondence with Theorem 2.8. In the case of fibre type contractions the general fibre $F$ of the morphism $f$ is a variety with $-K_F = -K_X|_F$ ample (such varieties are called Fano varieties). In some sense $f$ allows us to reduce the study of $X$ to that of the lower-dimensional variety $Y$ and $F$. In dimension two, this happens when an irreducible generator of the contracted extremal ray has nonnegative self-intersection number.

Divisorial contractions correspond to the blowup of a point in the surface case. In fact, they are a generalization thereof: any blowup of a smooth variety along a smooth center is a divisorial contraction[8]. In order for this to be of use to us, it would be important that $Y$ be $\mathbb{Q}$-factorial (or at the very least that $K_Y$ is $\mathbb{Q}$-Cartier). But this luckily holds in all relevant situations. In addition, it is also true that $\rho(Y) = \rho(X) - 1$.

Small contractions did not arise on surfaces for dimension reasons, and they do not occur on smooth threefolds either. In general however they do, and cause major problems. Here is why: let $f : X \to Y$ be a small contraction, assume $K_X$ is $\mathbb{Q}$-Cartier. We claim that no multiple of $K_Y$ can be Cartier.

Indeed, suppose that both $mK_X$ and $mK_Y$ are Cartier. Then $mK_X$ and $f^*(mK_Y)$ are two Cartier divisors that are linearly equivalent away from the codimension at least two subset $\mathrm{Ex}(f)$. It follows that the two are linearly equivalent on X. But this is impossible, as $mK_X \cdot R < 0$ and $f^*(mK_Y) \cdot R = 0$.

Thus small contractions inevitably lead us out of the class of varieties that we can hope to control. The way out of this seemingly hopeless situation is a new operation called a flip. The rough idea is that instead of contracting the locus of the extremal ray, we cut it out, and glue in something else (this process corresponds to the topological notion of surgery) hoping all the time that we end up with a more amenable variety.

**Definition 3.3** (Flip)**.** Let $f : X \to Y$ be a proper birational morphism with $\mathrm{codim}_X \mathrm{Ex}(f) \leq 2$, assume furthermore that $-K_X$ is $\mathbb{Q}$-Cartier and $f$-ample. A variety $X^+$ together with a proper birational morphism $f^+ : X^+ \to Y$ is called *a flip of* $X$ if $\mathrm{codim}_X^+ \mathrm{Ex}(f^+) \geq 2$, $K_X^+$ is $\mathbb{Q}$-Cartier and $f^+$-ample.

Very often it is the birational map $X \dashrightarrow X^+$ which is referred to as a flip. At this point it is not at all clear if a flip exists, or if it is unique, or that it helps us at all, but we'll sort this out for the most part.

Now comes a brief detour about the singularities we will encounter. Currently the class of singularities with which the minimal model program works is the so-called terminal singularities, this is what we will mostly use. They come from the following situation: let $f : X \to Y$ be a birational morphism of smooth varieties, $s \in H^0\big(X, mK_Y\big)$. Then $f^*s$ as a section of $mK_X$ will wanish along the exceptional divisor of $f$. This motivates the following definition.

**Definition 3.4** (Terminal singularities)**.** A normal variety $Y$ has terminal singularities if $K_Y$ is $\mathbb{Q}$-Cartier, and for every resolution of singularities

[8]The converse is not true though, there exist divisorial contractions that are not blowups.

$f : X \to Y$ one has

$$f_* \mathcal{O}_X(mK_X - E) = \mathcal{O}_Y(mK_Y)$$

where $E \subseteq X$ is the reduced exceptional divisor.

**Definition 3.5** (Minimal model)**.** A proper normal variety $X$ is called a *minimal model* if it has terminal singularities, and $K_X$ is nef.

After this rather long preparation we are in a position to state the minimal model program.

*Algorithm* 3.6 (Minimal Model Program).

(a) Start with a $\mathbb{Q}$-factorial normal projective variety $X$ with terminal singularities.
(b) If $K_X$ is nef then stop. If $K_X$ is not nef then the Cone Theorem guarantees us the existence of a $K_X$-negative extremal ray $R$.
(c) Let $c_R : X \to Y$ be the contraction of $R$.
    (a) If $c_R$ is a fibre type contraction, then stop. We have reduced the study of $X$ to that of the fibres of $c_R$ and a smaller-dimensional variety $Y$.
    (b) If $c_R$ is a divisorial contraction, then we will prove later that $Y$ is also $\mathbb{Q}$-factorial with terminal singularities, so replace $X$ by $Y$, and return to (b).
    (c) If $c_R$ is a small contraction, then we will again prove that the flip $X^+$ of $c_R$ is $\mathbb{Q}$-factorial with terminal singularities. Hence we can replace $X$ by $X^+$, and return to (b).

It is far from clear why the above procedure should stop, let alone why the required flips exist. In any case, once the algorithms stops, it produces either a minimal model or a Fano fibre space.

Before proceeding any further, we need to say a few words about pairs, which are a widely used technical tool in the theory of minimal models. A *pair* (or *logarithmic pair* or simply a *log pair*) is defined to be an ordered pair $(X, \Delta)$, with $X$ being an arbitrary variety, and $\Delta$ an $\mathbb{R}$-Weil divisor on $X$. Very often $X$ is required to be normal, $\Delta$ to be effective, and $K_X+\Delta$ to be $\mathbb{Q}$-Cartier. While most of what we have said so far goes through without serious modification upon replacing $K_X$ by $K_X + \Delta$ (the so-called *log-canonical divisor*), the presence of $\Delta$ provides an extra amount of flexibility and power which one cannot do without. In what follows we will often do without pairs. From a certain point on however, one cannot get around them.

## 4. Finite Generation of the Canonical Ring and the Existence of Flips

As far as the minimal model program goes, the two main problems are the existence of flips (meaning that we don't know if the flip of a small contraction of a $K_X$-negative extremal ray exists or not), and the termination of flips (which asks if there can be an infinite sequence of flips). In this generality, these are only known up to dimension four, and even there the proofs are quite hard and technical.

Every variety comes with a handful of objects naturally attached to it. Provided $K_X$ is $\mathbb{Q}$-Cartier, one of these is the canonical ring

$$R(X, K_X) \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} H^0\big(X, im_0K_X\big) \ ,$$

where $m_0K_X$ is a Cartier divisor class.

*Remark* 4.1. Note that as written $R(X, K_X)$ implicitly depends on the choice of $m_0$. In fact, different choices for $m_0$ give possibly different graded rings. However, given that we are really interested only in the problem whether $R(X, K_X)$ is finitely generated (and in $\operatorname{Proj} R(X, K_X)$ if so), we can for the most part ignore the dependence on $m_0$ (see Proposition 4.6).

The following results provide a strong connection between finite generation of the canonical ring and the existence of flips.

**Proposition 4.2** (Lemma 6.2 in [4]). *Let $Y$ be a normal algebraic variety, $D$ a Weil divisor on $Y$. Then the following are equivalent.*

- *(a) $R(Y, D) \stackrel{def}{=} \bigoplus_{m=0}^{\infty} \mathcal{O}_Y(mD)$ is a sheaf of finitely generated $\mathcal{O}_Y$-algebras;*
- *(b) There exists a projective birational morphism $g : Z \to Y$ with $Z$ normal, $\operatorname{codim}_Z \operatorname{Ex}(g) \geq 2$, and $g_*^{-1}D$ $\mathbb{Q}$-Cartier and $g$-ample over $Y$.*

*When these equivalent conditions are satisfied, then $g : Z \to Y$ is unique.*

*Proof.* Let us first assume (b). In this case we observe that

$$\mathcal{O}_Y(mD) \ = \ g_*\mathcal{O}_Z(m \cdot g_*^{-1}D) \ .$$

Indeed, we have an injection $g_*\mathcal{O}_Z(m \cdot g_*^{-1}D) \hookrightarrow \mathcal{O}_Y(mD)$ which we now prove be an isomorphism. Let $U \subseteq Y$ an open set, and $s \in H^0\big(U, \mathcal{O}_Y(mD)\big)$ an arbitrary section. Then $s$ lifts to $H^0\big(g^{-1}(U) - \operatorname{Ex}(g), \mathcal{O}_Z(m \cdot g_*^{-1}D)\big)$. Now the codimension of the exceptional set of $g$ is at least two, hence the lift of $s$ extends to a section $g^*s \in H^0\big(g^{-1}(U), \mathcal{O}_Z(m \cdot g_*^{-1}D)\big)$.

As the divisor $g_*^{-1}D$ is $g$-ample, the section rings

$$\bigoplus_{m=0}^{\infty} \mathcal{O}_Y(mD) \ = \ \bigoplus_{m=0}^{\infty} g_*\mathcal{O}_Z(m \cdot g_*^{-1}D)$$

are finitely generated. Also, since $Z = \operatorname{Proj}_Y \bigoplus_{m=0}^{\infty} \mathcal{O}_Y(mD)$, $Z$ is unique.

For the other direction, after possibly replacing $D$ by a high enough multiple, we can assume that $\mathcal{O}_Y(D)$ generated $R(Y, D)$ as an $\mathcal{O}_Y$-algebra. Setting

$$Z \stackrel{\text{def}}{=} \operatorname{Proj}_Y R(Y, D) \ ,$$

observe that $g_*\mathcal{O}_Z(m) = \mathcal{O}_Y(mD)$, $\mathcal{O}_Z(1)$ is very ample, and $g_*\mathcal{O}_Z(m) = \mathcal{O}_Y(mB)$.

We will now show that $g$ is small. Suppose on the contrary that $g : Z \to Y$ has an exceptional divisor $E$. Then the inclusion $\mathcal{O}_Z \subset \mathcal{O}_Z(E)$ is proper, and we obtain an injection

$$\tau : \mathcal{O}_Y(mB) = f_*\mathcal{O}_Z(m) \subsetneq f_*(\mathcal{O}_Z(m)(E))$$

for all $m$ large enough. But this is a contradiction as $\mathcal{O}_Y(mB)$ is a reflexive sheaf, and $\tau$ is an isomorphism away from the codimension two subset $g(\mathrm{Ex}(g))$. □

**Corollary 4.3.** *Let $f : X \to Y$ be the small contraction of a $K_X$-negative extremal ray with $X$ being a normal $\mathbb{Q}$-factorial projective variety with terminal singularities only.*

*Then a flip $f^+ : X^+ \to Y$ exists if and only if*

$$\bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(mK_X)$$

*is finitely generated as an $\mathcal{O}_Y$-algebra. In this case a flip of $f$ is isomorphic to*

$$\mathrm{Proj}_Y \bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(mK_X) \ .$$

*Proof.* Let us first assume then a flip $f^+ : X^+ \to Y$ of $f$ exists. Then

$$\bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(mK_X) = \bigoplus_{m=0}^{\infty} \mathcal{O}_X(mK_Y) = \bigoplus_{m=0}^{\infty} f_*^+\mathcal{O}_{X^+}(mK_{X^+}) \ .$$

This last $\mathcal{O}_Y$-algebra is however finitely generated Proposition 4.2 as $K_{X^+}$ is $f^+$-ample.

For the other implication assume that $\bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(mK_X)$ is a finitely generated $\mathcal{O}_Y$-algebra. Then again the equivalence in Proposition 4.2 implies that the flip of $f$ exists and is equal to $\mathrm{Proj}_Y \bigoplus_{m=0}^{\infty} f_*\mathcal{O}_X(mK_X)$. □

*Remark* 4.4. Note that Corollary 4.3 implies immediately that the flip of $f$ is unique. In addition, we obtain that the existence of flips is a local question on the base ($Y$ in our case). Therefore not only can we assume without loss of generality that $Y$ is affine, but we can shrink $Y$ to our heart's content if it seems necessary. As we will see in the talks about the actual work of Hacon–McKernan, this is often done and typically without further notice.

Assuming now that $f : X \to Y$ is a flipping contraction with $Y = \mathrm{Spec}\, S$ affine, the question of the existence of flips gets simplified to some extent. By [3, Proposition 8.5]

$$f_*\mathcal{O}_X(mK_X) \simeq H^0\big(\widetilde{X, mK_X}\big) \ ,$$

and so the existence of the flip of $f$ is quickly seen to be equivalent to the finite generation of the canonical ring $R(X, K_X)$.

**Definition 4.5.** Let $R = \oplus_{i=0}^{\infty} R_i$ be a graded ring, $d$ an arbitrary natural number. The $d^{th}$ *truncation of* $R$ is defined as

$$R^{(d)} \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} R_{id} \ .$$

**Proposition 4.6.** *Let $R$ be a graded ring as above. Then $R$ is finitely generated if and only if there exists $d_0 \geq 1$ for which $R^{(d_0)}$ is finitely generated. In this case $R^{(d)}$ is finitely generated for all $d \geq 1$.*

*Proof.* Observe that $R^{(d)}$ is the ring of invariants of $R$ under the finite group action of $\mu_d$, hence itself finitely generated once $R$ is.

For the other implication, let $r \in R$ be a homogeneous element of $R$. Then $r$ satisfies a monic equation

$$X^{d_0} - r^{d_0} = 0 \ ,$$

with $r^{d_0} \in R^{(d_0)}$, which implies that the ring $R$ is integral over $R^{(d_0)} \subseteq R$. Now if $R^{(d_0)}$ is finitely generated then so is $R$ by the finiteness of integral closure. $\square$

*Remark* 4.7. The previous result is a small but very useful observation. Whenever we deal with the issue of finite generation, it provides us with the possibility of passing to truncations at our leisure to get into a possibly more favourable situation. In the existing literature this happens quite often (and most of the time without warning).

The area around the existence of flips in higher-dimensions has been dominated by ideas of Shokurov in recent years. One of his crucial insights is the following reduction step.

We consider a somewhat special situation, which eventually will turn out to imply the general case under certain (strong) hypothesis[9] From now will not the able to avoid the use of pairs.

Let $f : (X, \Delta) \to Z$ be a flipping contraction where $Z$ is affine, $\Delta = S + B$, $S$ is a prime divisor, and $\lfloor B \rfloor = 0$. Assume in addition that $-S$ and $-(K_X + \Delta)$ are $f$-ample, and $\rho(X/Z) = 1$. Instead of dealing with the canonical ring of $X$ directly, we will try to replace it by some kind of a restriction to $S$. In what follows, let $m_0$ be an integer so that $m_0(K_X + \Delta)$ is a Cartier divisor.

**Definition 4.8.** With notation as above, we define the *restricted algebra* of $R(X, K_X + \Delta)$ as

$$R_S(X, \Delta) \stackrel{\text{def}}{=}$$
$$\bigoplus_{m=0}^{\infty} \operatorname{im}\big(\text{res} : H^0\big(X, mm_0(K_X + S + B)\big) \to H^0\big(S, mm_0(K_S + B|_S)\big)\big) \ ,$$

[9]This is the part where induction on dimension plays a crucial role. The abovementioned hypothesis is namely the log minimal model program in $\dim X - 1$.

where 'res' denotes the restriction of sections to $S$.

*Remark* 4.9. It follows from an extension of the adjunction formula specially crafted for such situations (see [5, ???]) that

$$(K_X + B + S)\,|_S \;=\; K_S + B|_S \;.$$

**Proposition 4.10.** *With notation as above, $R(X, K_X + \Delta)$ is finitely generated if and only if $R_S$ is.*

*Proof.* As $S \subseteq X$ is a prime divisor, we can find a rational function $\phi \in \mathbb{C}(X)$ which has a zero of order one along $S$. Set $D \stackrel{\text{def}}{=} S - \operatorname{div}\phi$, then on the one hand $D \sim_{\text{lin}} S$, on the other hand $D$ has neither a pole nor a zero alongside $S$.

We have assumed that the relative Picard number $\rho(X/Z) = 1$, therefore modulo pullbacks from Cartier divisors on $Z$ there exists a rational number $r$ for which

$$D \sim_{f,\mathbb{Q}} r(K_X + \Delta)\;.$$

It follows that $R(X, D)$ and $R$ have a common truncation, in particular, one is finitely generated if and only if the other one is.

Since $\phi \in H^0\big(X, D\big)$, it will be enough to verify that $\phi$ generates

$$K \stackrel{\text{def}}{=} \bigoplus_{m \geq 0} \ker(H^0\big(X, mD\big) \to H^0\big(S, mD|_S\big))\;.$$

Observe that we have

$$\operatorname{div}\phi_m + mD - S \geq 0$$

whenever $\phi_m \in H^0\big(X, mD\big) \cap K$. Setting $\phi_m = \phi\phi'_m$ for a suitable $\phi'_m \in k(X)$, we obtain

$$\begin{aligned} \operatorname{div}\phi'_m + (m-1)D &= \operatorname{div}\phi_m - \operatorname{div}\,\phi + (m-1)D \\ &= \operatorname{div}\phi_m + mD - S \\ &\geq 0\;. \end{aligned}$$

This means that $\phi'_m \in H^0\big(X, (m-1)D\big)$, and therefore we are done. $\square$

Another crucial idea is to relate the various section rings we have seen to function algebras associated to certain sequences of b-divisors. The importance of such a connection lies in the fact that there is a criterion for the finite generation of such b-divisorial algebras which has a scope wide enough to be practical.

The result goes as follows

**Theorem 4.11.** *A saturated semi-ample adjoint b-divisorial algebra if finitely generated.*

The essential content of the paper [2] is the fact the under certain hypotheses the restricted algebras $R_S$ is isomorphic to a saturated semi-ample adjoint b-divisorial algebra, and as such, is finitely generated. It is important to point out, that the proof of this result in [2] relies on induction on the dimension of $X$, and uses the log minimal model program in dimension $\dim X - 1$.

## References

[1] Alessio Corti, Anne-Sophie Kaloghiros, Vladimir Lazić: Introduction to the minimal model program and the existence of flips, preprint.
[2] Christopher D. Hacon, James McKernan: On the existence of flips, preprint, ariXiv:math.AG/0507597.
[3] Robin Hartshorne: Algebraic geometry. Springer 1977.
[4] János Kollár, Shigefumi Mori: Birational geometry of algebraic varieties.
[5] János Kollár et al.: Flips and abundance