# EIGENFUNCTION EXPANSION FOR THE THREE-DIMENSIONAL DIRAC OPERATOR

M. Horváth

Abstract. We consider any selfadjoint Dirac operator with discrete spectrum defined on a three-dimensional domain. We show the local uniform convergence of the corresponding eigenfunction expansion if the function expanded is sufficiently smooth. Then we derive the Riemann's localization principle for these expansions.

## Setting of the problem

Consider the eigenfunctions of the stationary Dirac operator

$$H\Psi = \lambda\Psi, \Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)^T \tag{1}$$

with

$$H = \hbar c_0 \sum_{k=1}^{3} \gamma_4\gamma_k \partial_{x_k} - ie\sum_{k=1}^{3}\gamma_4\gamma_k A_k + e\Phi + \mu c_0^2\gamma_4 \tag{2}$$

where $\hbar$ is the Planck constant, $c_0$ is the velocity of light, $e$ is the electric charge of the particle, $\mu$ is its rest mass, $\Phi$ is the scalar potential, $(A_1, A_2, A_3)$ is the vector potential of the space in which the particle moves and $\gamma_k$ are the $4\times 4$ Dirac matrices satisfying

$$\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma_4\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, k = 1, 2, 3.$$

Here $I$ is the $2\times 2$ identity matrix and

$$\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

are the so-called Pauli matrices. It is known that $\sigma_k^2 = -I$ and if $i \neq j$ then

$$\sigma_i\sigma_j = \begin{cases} \sigma_k & \text{if } (i,j,k) \text{ is a cyclic permutation of } (1,2,3) \\ -\sigma_k & \text{otherwise.} \end{cases}$$

Let $\Omega \subset \mathbb{R}^3$ be an arbitrary domain. Consider a selfadjoint extension of $H$ from $C_0^\infty(\Omega)$ with discrete spectrum. It corresponds to a system $(\Psi_n)$ , complete and orthonormal in $L_2(\Omega, \mathbb{C}^4)$ , of the eigenfunctions of $H$ :

$$H\Psi_n = \lambda_n\Psi_n.$$

---

Research supported by the Hungarian National Science Foundation OTKA T-024144

The *spectral expansion* of a function $f \in L_2(\Omega, \mathbb{C}^4)$ is the series

$$f = \sum_n \Psi_n \int_\Omega \Psi_n^* f \qquad , \qquad \Psi_n^* = \overline{\Psi_n^T}$$

convergent in $L_2$ . Its $\mu$-th *partial sum* is

$$S_\mu(f,x) = \sum_{|\lambda_n|<\mu\hbar c_0} \Psi_n(x) \int_\Omega \Psi_n^* f.$$

This sum may contain infinitely many members; for example in the special case investigated by Evans [2], [3] there is an eigenvalue with infinite multiplicity. However the sum is absolutely convergent; this follows from the square sum estimate, see below in Part 3.
Define the *Bessel-Macdonald kernel* of order $\alpha > 0$ :

$$v_\alpha(r) = \frac{2^{1-\alpha/2}}{\Gamma(\alpha/2)} \frac{K_{\frac{3-\alpha}{2}}(r)}{r^{\frac{3-\alpha}{2}}}$$

where $K_\nu$ denotes the Bessel function of the third kind of order $\nu$ , see Bateman, Erdélyi [1]
The *Liouville classes* $L_p^\alpha(\mathbb{R}^3, \mathbb{C}^4)$ , $\alpha > 0$ , $1 \le p \le \infty$ consist, by definition, of all four-component functions $f$ representable in the form

$$f(x) = \int_{\mathbb{R}^3} g(y) v_\alpha(|x-y|)\, dy \quad , \quad g \in L_p(\mathbb{R}^3, \mathbb{C}^4)$$

and

$$\|f\|_{L_p^\alpha} = \|g\|_{L_p}.$$

It is known (Nikolskii [15])that $v_\alpha(|x|)$ is the inverse Fourier transform of $(1+|\xi|^2)^{-\alpha/2}$. Consequently $L_p^\alpha$ can be interpreted as the set of all functions in $L_p$ , having derivatives in $L_p$ "up to order $\alpha$ ". If $\alpha$ is integer, $L_p^\alpha$ is indeed the usual Soboleff space $W_p^\alpha$, see Nikolskii [15].
We suppose in this paper that the potential functions $A_1$, $A_2$, $A_3$ and $\Phi$ are continuously differentiable in $\Omega$. Under this condition the following statements will be proved about the convergence of the spectral expansion considered:

**Theorem 1.** *Let $f \in L_p^\alpha(\mathbb{R}^3, \mathbb{C}^4)$ where $\alpha > 1$ , $1 < p < \infty$ , $\alpha p > 3$. Suppose that $f$ has compact support in $\Omega$. Then the spectral expansion of $f$ converges locally uniformly in $\Omega$. In other words*

$$S_\mu(f,x) \to f(x) \qquad (\mu \to \infty)$$

*and the convergence is uniform if $x$ runs over a compact subset $K$ of $\Omega$.*

**Theorem 2.** *Let $f \in L_2^\alpha(\mathbb{R}^3, \mathbb{C}^4)$, $\alpha > 1$ and suppose that $f = 0$ on a subdomain $\Omega_0 \subset \Omega$. Then*

$$S_\mu(f,x) \to 0 \qquad (\mu \to \infty)$$

*locally uniformly in $\Omega_0$ .*

**Theorem 3.** *Let $f \in L_p^\alpha(\mathbb{R}^3, \mathbb{C}^4)$, $\alpha > 3/2$, $\alpha p > 3$, $1 < p \leq \infty$ and let $f$ have compact support in $\Omega$. Then the spectral expansion of $f$ converges to $f$ absolutely and locally uniformly.*

*Previous results.* The statements corresponding to Theorems 1-3. in case of selfadjoint extension of the Laplace operator is given by Il'in [11] and for nonselfadjoint Laplace operators in three-dimensional domains by Joó [14] , in N-dimensional domains by Horváth [5] , [6] , [7] , see also [9]. In the one-dimensional case the above theorems follow immediately from so-called equiconvergence theorems. It means that the difference of $S_\mu$ and the corresponding partial sum of the trigonometric Fourier series tends to zero locally uniformly. Such results are obtained in Joó, Komornik [12] , Joó [13] for Schrödinger operators and in Horváth [8] for Dirac operators.

*Some basic ideas and the organization of the paper.* As it is seen from the formulation of Theorems 1-3, we do not assume any boundary condition in order to produce a complete system of orthonormal eigenfunctions. In case of ordinary differential operators the asymptotic behaviour of the eigenfunctions and eigenvalues is known and it makes possible to study the eigenfunction expansions. For partial differential operators such a general theory of the asymptotic properties is not available; however some expressions of eigenfunctions have known asymtotic forms. An important example is the spectral function i.e. the kernel function of the $\mu$-th partial sum operator, see Part 3. It is investigated e.g. in Gårding [4] and Hörmander [10]. In these papers the potential is supposed to belong to $C^\infty$ . Another method is presented by Il'in [11] where smoothness of the potential is not required. His method is based on the estimate of the square sum of eigenfunctions, see Part 2.
The paper consists of five parts. In Part 1. we fix the main notations and give the fundamental solution. It is then used in Part 2. to establish a mean value formula, Lemma 5. It gives a connection between the integral mean of an eigenfunction in a sphere and the value of the eigenfunction in the centre. The analogous formula for the Schrödinger operator is due to Titchmarsh [17] . In Part 3. we provide a square sum estimate of eigenfunctions; it is used everywhere in later parts. The main result of Part 4. is Lemma 12. stating that the main part of the spectral function is that of the trigonometric Fourier series; main part in the sense that the $L_2$-norm of the remainder is much less than that of the main part. Finally Part 5. includes the proof of our Theorems; they are based on the Banach-Steinhaus principle, namely that convergence follows from the convergence in a dense subset if the partial sum operators are uniformly bounded in the corresponding norms.

## Part 1. The fundamental solution of the Dirac operator

Consider the eigenfunctions (1). In the notation of the operator $H$ we shall use the shorter symbolic form

$$H = \sum_{k=1}^{3} B_k \partial_{x_k} + C(x) \tag{3}$$

where

$$B_k = \hbar c_0 \gamma_4 \gamma_k = \hbar c_0 \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

are constant $4\times 4$ matrices and

$$C(x) = -ie\sum_{k=1}^{3}\gamma_4\gamma_k A_k + e\Phi(x)\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} + \mu c_0^2\gamma_4 \tag{4}$$

is a $4\times 4$ matrix depending on $x = (x_1, x_2, x_3)$. Introduce finally the matrix

$$A(y) = \sum_{k=1}^{3} y_k\sigma_k = \begin{pmatrix} -iy_3 & -iy_1 - y_2 \\ -iy_1 + y_2 & iy_3\end{pmatrix}.$$

We shall give the fundamental solution of the unperturbed operator

$$H_0 = \sum_{k=1}^{3} B_k\partial_{x_k}$$

in the space $\mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$ of Schwartz (or tempered) distributions. Namely, let

$$E_0(x) = \frac{-1}{4\pi\hbar c_0}\frac{1}{|x|^3}\begin{pmatrix} 0 & A(x) \\ A(x) & 0\end{pmatrix}.$$

**Lemma 1.**

$$\sum_{k=1}^{3}\partial_{x_k}(E_0)B_k = 0 \quad \textit{for } x\neq 0. \tag{5}$$

*Proof.* Using the properties

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 0 \quad (i\neq j)$$

of the Pauli matrices we calculate

$$\begin{aligned}\sum_{k=1}^{3}\partial_{x_k}(E_0)B_k &= c\sum_{k=1}^{3}\partial_{x_k}\left[|x|^{-3}\begin{pmatrix} 0 & \sum_i\sigma_i x_i \\ \sum_i\sigma_i x_i & 0\end{pmatrix}\right]\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix} \\ &= c\sum_{k=1}^{3}\partial_{x_k}\left[|x|^{-3}\begin{pmatrix} \sum_i\sigma_i\sigma_k x_i & 0 \\ 0 & \sum_i\sigma_i\sigma_k x_i\end{pmatrix}\right] \\ &= c\sum_{i<k}(x_i\partial_{x_k}(|x|^{-3}) - x_k\partial_{x_i}(|x|^{-3}))\begin{pmatrix} \sigma_i\sigma_k & 0 \\ 0 & \sigma_k\sigma_i\end{pmatrix} \\ &+ c\sum_{k=1}^{3}\partial_{x_k}(\frac{x_k}{|x|^3})\begin{pmatrix} -I & 0 \\ 0 & -I\end{pmatrix} = 0\end{aligned}$$

(since $|x|^{-1}$ is harmonic if $x\neq 0$). $\square$

Analogously we get

$$\sum_{k=1}^{3} B_k\partial_{x_k}(E_0) = 0 \quad \text{for } x\neq 0 \tag{5'}$$

**Lemma 2.** *$E_0$ is the fundamental solution of the operator $H_0$ i.e.*

$$\sum_{k=1}^{3} \partial_{x_k}(E_0)B_k = \delta \quad \text{in } \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4) \tag{6}$$

*Proof.* Let $\varphi \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$ ; then

$$\begin{aligned}(\sum_k \partial_{x_k}(E_0)B_k)(\varphi) &= -E_0(\sum_k \partial_{x_k} B_k \varphi) = -\int_{\mathbb{R}^3} E_0 \sum_k B_k \partial_{x_k} \varphi \\ &= -\lim_{\varepsilon \to 0+} \int_{|x| \geq \varepsilon} E_0 \sum_k B_k \partial_{x_k} \varphi \\ &= \lim_{\varepsilon \to 0+} \left( \int_{|x|=\varepsilon} E_0 \sum_k B_k \varphi n_k + \int_{|x| \geq \varepsilon} \sum_k \partial_{x_k}(E_0) B_k \varphi \right)\end{aligned}$$

where $n_k$ is the $k$-th component of the outward normal of the surface $|x| = \varepsilon$ i.e.

$$n_k = \frac{x_k}{|x|} = \frac{x_k}{\varepsilon}.$$

The second summand vanishes by Lemma 1. In the first one write the integral in the form

$$\begin{aligned}\int_{|x|=\varepsilon} E_0 \sum_k B_k \varphi n_k &= \sum_k \int_{|x|=\varepsilon} \frac{-1}{4\pi\varepsilon^3} \begin{pmatrix} 0 & A(x) \\ A(x) & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_k n_k \\ \sigma_k n_k & 0 \end{pmatrix} \varphi(x)\, d\varrho(x) \\ &= \frac{-1}{4\pi\varepsilon^4} \int_{|x|=\varepsilon} \sum_k \begin{pmatrix} A(x)\sigma_k x_k & 0 \\ 0 & A(x)\sigma_k x_k \end{pmatrix} \varphi(x)\, d\varrho(x) \\ &= \frac{-1}{4\pi\varepsilon^4} \int_{|x|=\varepsilon} \begin{pmatrix} A^2(x) & 0 \\ 0 & A^2(x) \end{pmatrix} \varphi(x)\, d\varrho(x)\end{aligned}$$

where $d\varrho$ denotes the surface measure. Since

$$A^2(x) = \left( \sum_k \sigma_k x_k \right)^2 = \sum_{i<k} x_i x_k (\sigma_i \sigma_k + \sigma_k \sigma_i) + \sum_k x_k^2 \sigma_k^2 = -|x|^2 I \tag{7}$$

hence

$$\int_{|x|=\varepsilon} E_0 \sum_k B_k \varphi n_k = \frac{1}{4\pi\varepsilon^2} \int_{|x|=\varepsilon} \varphi(x)\, d\varrho(x) \to \varphi(0) \quad (\varepsilon \to 0+)$$

which proves Lemma 2. □

Analogously we get

$$\sum_k B_k \partial_{x_k} E_0 = \delta. \tag{6'}$$

## PART 2. THE MEAN VALUE FORMULA FOR THE EIGENFUNCTIONS

It is given in Lemma 5. below through some intermediate steps.

**Lemma 3.** *If* $\Psi$ *satisfies* (1) *in a domain* $\Omega \subset \mathbb{R}^3$ *then*

(8)
$$\Psi(x) = \frac{1}{4\pi r^2}\int_{|x-y|=r}\Psi + \int_{|x-y|\le r} E_0(y-x)C(y)\Psi(y)\,dy - \int_{|x-y|\le r}\lambda E_0(y-x)\Psi(y)\,dy$$

*if* $\{y : |x-y| \le r\} \subset \Omega$.

*Proof.* Let $0 < \varepsilon < r$; then

$$\int_{\varepsilon\le|x-y|\le r}\lambda E_0(y-x)\Psi(y)\,dy = \int_{\varepsilon\le|x-y|\le r} E_0(y-x)C(y)\Psi(y)\,dy +$$
$$\int_{\varepsilon\le|x-y|\le r} E_0(y-x)\sum_{k=1}^{3} B_k\partial_{y_k}\Psi(y)\,dy.$$

In the second integral we integrate by parts:

$$\begin{aligned}
&\int_{\varepsilon\le|x-y|\le r} E_0(y-x)\sum_k B_k\partial_{y_k}\Psi(y)\,dy\\
&= \int_{|x-y|=r} E_0(y-x)\sum_k B_k\Psi(y)\frac{y_k-x_k}{r}\,d\varrho(y)\\
&- \int_{|x-y|=\varepsilon} E_0(y-x)\sum_k B_k\Psi(y)\frac{y_k-x_k}{\varepsilon}\,d\varrho(y)\\
&- \int_{\varepsilon\le|x-y|\le r}\sum_k \partial_{y_k}(E_0(y-x))B_k\Psi(y)\,dy.
\end{aligned}$$

The third summand is zero by Lemma 1. The first one has the form

$$\frac{-1}{4\pi r^4}\int_{|x-y|=r}\begin{pmatrix} 0 & A(y-x)\\ A(y-x) & 0\end{pmatrix}\begin{pmatrix} 0 & \sum_k \sigma_k(y_k-x_k)\\ \sum_k \sigma_k(y_k-x_k) & 0\end{pmatrix}\Psi(y)\,d\varrho(y)$$
$$= \frac{-1}{4\pi r^4}\int_{|x-y|=r}\begin{pmatrix} A^2(y-x) & 0\\ 0 & A^2(y-x)\end{pmatrix}\Psi(y)\,d\varrho(y) = \frac{1}{4\pi r^2}\int_{|x-y|=r}\Psi.$$

Analogously,the second summand is

$$\frac{-1}{4\pi\varepsilon^2}\int_{|x-y|=\varepsilon}\Psi \longrightarrow -\Psi(x) \quad (\varepsilon\to 0+).$$

Hence

$$\int_{|x-y|\le r}\lambda E_0(y-x)\Psi(y)\,dy = \lim_{\varepsilon\to 0+}\int_{\varepsilon\le|x-y|\le r}\lambda E_0(y-x)\Psi(y)\,dy$$
$$\int_{|x-y|\le r} E_0(y-x)C(y)\Psi(y)\,dy + \frac{1}{4\pi r^2}\int_{|x-y|=r}\Psi - \Psi(x)$$

as we asserted □

*Remark.* Since $E_0(x)$ has the magnitude order $|x|^{-2}$, the integral involving $C(y)$ in (8) exists only if $C$ belongs locally to $L_q$ with some $q > 3$ . However if we define the integral in the Cauchy principal value sense

(9)
$$\int_{|x-y|\le r} E_0(y-x)C(y)\Psi(y)\,dy = \lim_{\varepsilon\to 0+}\int_{\varepsilon\le|x-y|\le r} E_0(y-x)C(y)\Psi(y)\,dy$$

then the above proof shows that (8) holds if $C$ is locally in $L_1$.

**Lemma 4.** *Denote*

$$S_r(\Psi) = \int_{|x-y|=r} \Psi(y)\, d\varrho(y).$$

*Then*

$$\begin{aligned} \frac{1}{4\pi r^2} S_r(\Psi) &= \Psi(x) + \frac{1}{4\pi\hbar c_0} \sum_{k=1}^{3} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \int_0^r \frac{1}{t^2} S_t(n_k C\Psi)\, dt \\ &\quad - \frac{\lambda}{4\pi\hbar^2 c_0^2} \int_0^r \frac{1}{t^2} \int_0^t (\lambda S_\tau(\Psi) - S_\tau(C\Psi))\, d\tau\, dt \end{aligned} \tag{10}$$

*where $n_k$ is the $k$-th component of the outward normal of the surface $|x-y| = t$.*

*Proof.* For any four-component function $f$,

$$\int_{|x-y|\le r} f = \int_0^r S_t(f)\, dt.$$

Using this we get from (8) that

$$\begin{aligned} \Psi(x) &= \frac{1}{4\pi r^2} S_r(\Psi) - \frac{1}{4\pi\hbar c_0} \int_0^r \frac{1}{t^2} \sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} t S_t(n_k C\Psi)\, dt \\ &\quad + \frac{\lambda}{4\pi\hbar c_0} \int_0^r \frac{1}{t^2} \sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} S_t(n_k \Psi)\, dt. \end{aligned}$$

Integration by parts shows that

$$S_t(n_k\Psi) = \int_{|x-y|=t} n_k \Psi = \int_{|x-y|\le t} \partial_{x_k} \Psi.$$

Hence

$$\begin{aligned} &\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} S_t(n_k\Psi) = \int_{|x-y|\le t} \sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \partial_{x_k} \Psi \\ &= \frac{1}{\hbar c_0} \int_{|x-y|\le t} (\lambda\Psi - C\Psi) = \frac{1}{\hbar c_0} \int_0^t (\lambda S_\tau(\Psi) - S_\tau(C\Psi))\, d\tau \end{aligned}$$

and then

$$\begin{aligned} \Psi(x) &= \frac{1}{4\pi r^2} S_r(\Psi) - \frac{1}{4\pi\hbar c_0} \sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \int_0^r \frac{1}{t^2} S_t(n_k C\Psi)\, dt \\ &\quad + \frac{\lambda^2}{4\pi\hbar^2 c_0^2} \int_0^r \frac{1}{t^2} \int_0^t S_\tau(\Psi)\, d\tau\, dt = \frac{\lambda}{4\pi\hbar^2 c_0^2} \int_0^r \frac{1}{t^2} \int_0^t S_\tau(C\Psi)\, d\tau\, dt \end{aligned}$$

as we asserted. $\square$

**Lemma 5.**

$$\frac{1}{r^2}S_r(\Psi) = \int_{|\theta|=1} \Psi(x+r\theta)\,d\theta = 4\pi\frac{\sin\lambda^* r}{\lambda^* r}\Psi(x) \tag{11}$$
$$+\frac{1}{r}\int_0^r \left[\frac{1}{t}\beta(t) + \int_0^t \frac{1}{\tau^2}\beta(\tau)\,d\tau\right]\cos\lambda^*(r-t)\,dt\,dr$$

*where*

$$\beta(r) = \frac{1}{\hbar c_0}\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} S_r(n_k C\Psi) + \frac{\lambda^*}{\hbar c_0}\int_0^r S_t(C\Psi)\,dt \tag{12}$$

*and*

$$\lambda^* = \frac{\lambda}{\hbar c_0}.$$

*Proof.* Denote

$$F(r) = \int_0^r \frac{1}{t}S_t(\Psi)\,dt.$$

Then

$$\int_0^t S_\tau(\Psi) = \int_0^t \tau\frac{1}{\tau}S_\tau(\Psi)\,d\tau = tF(t) - \int_0^t F.$$

We substitute it into (10) to obtain

$$\frac{1}{r}F'(r) = \frac{1}{r^2}S_r(\Psi) = 4\pi\Psi(x) + \frac{1}{\hbar c_0}\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\int_0^r \frac{1}{t^2}S_t(n_k C\Psi)\,dt$$
$$+\frac{\lambda^*}{\hbar c_0}\int_0^r \frac{1}{t^2}\int_0^t S_\tau(C\Psi)\,d\tau\,dt - \lambda^{*2}\int_0^r \frac{1}{t^2}\left(tF(t) - \int_0^t F\right)dt.$$

Differentiating it with respect to $r$ we get

$$\frac{1}{r}F"(r) - \frac{1}{r^2}F'(r) + \lambda^{*2}\left(\frac{1}{r}F(r) - \frac{1}{r^2}\int_0^r F\right)$$
$$= \frac{1}{\hbar c_0}\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\frac{1}{r^2}S_r(n_k C\Psi) + \frac{\lambda^*}{\hbar c_0}\frac{1}{r^2}\int_0^r S_t(C\Psi)\,dt$$
$$= \frac{1}{r^2}\beta(r).$$

Consequently

$$\frac{1}{r}\beta(r) + \int_0^r \frac{1}{t^2}\beta(t)\,dt = F"(r) - \frac{1}{r}F'(r) + \lambda^{*2}\left(F(r) - \frac{1}{r}\int_0^r F\right)$$
$$+\frac{1}{r}F'(r) - \lim_{0+}(\frac{1}{t}F(t)) + \lambda^{*2}\left[\frac{1}{r}\int_0^r F - \lim_{0+}(\frac{1}{t}\int_0^t F)\right].$$

Clearly $\lim_{0+}\frac{1}{t}\int_0^t F = 0$ , $\lim_{0+}\frac{1}{t}F'(t) = 4\pi\Psi(x)$ ; hence

$$\frac{1}{r}\beta(r) + \int_0^r \frac{1}{t^2}\beta(t)\,dt = F"(r) + \lambda^{*2}F(r) - 4\pi\Psi(x).$$

This is a second-order differential equation to $F(r)$ the general solution of which has the form
(13)
$$F(r) = A\cos\lambda^* r + B\sin\lambda^* r + \int_0^r \left[\frac{1}{t}\beta(t) + \int_0^t \frac{1}{\tau^2}\beta(\tau)\,d\tau + 4\pi\Psi(x)\right]\frac{\sin\lambda^*(r-t)}{\lambda^*}\,dt$$

From $\lim_{0+} F = 0$ resp. $\lim_{0+} F' = 0$ it follows $A = 0$ resp. $B = 0$ . Thus
$$\frac{1}{r}S_r(\Psi) = F'(r) = \int_0^r \left[\frac{1}{t}\beta(t) + \int_0^t \frac{1}{\tau^2}\beta(\tau)\,d\tau + 4\pi\Psi(x)\right]\cos\lambda^*(r-t)\,dt.$$

Here the coefficient of $\Psi(x)$ is
$$4\pi\int_0^r \cos\lambda^*(r-t)\,dt = 4\pi\frac{\sin\lambda^* r}{\lambda^*}.$$

Dividing by $r$ we get the statement of Lemma 5. $\square$

**Lemma 6.** *Denote*
$$\hat{\sigma}_j = -\left(\sum_k \sigma_k n_k\right)\sigma_j\left(\sum_k \sigma_k n_k\right)$$

*and*
$$\hat{C} = -ie\sum_{j=1}^{3}\begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0\end{pmatrix}A_j + e\Phi\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} - \mu c_0^2\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}.$$

*Then we have*
$$\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix} n_k C = \hat{C}\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix} n_k.$$

*Proof.* We know $\left(\sum \sigma_k n_k\right)^2 = -I$ . Consequently, multiplying the equation defining $\hat{\sigma}_j$ from the right by $\sum \sigma_k n_k$ implies
$$\left(\sum_k \sigma_k n_k\right)\sigma_j = \hat{\sigma}_j\sum_k \sigma_k n_k;$$

whence
$$\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix} n_k \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0\end{pmatrix} = \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0\end{pmatrix}\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix} n_k,$$
$$\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix} n_k \sum_j \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0\end{pmatrix} A_j = \sum_j \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0\end{pmatrix} A_j \sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix}.$$

On the other hand, $e\Phi\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}$ remains the same, $\mu c_0^2\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}$ changes sign if we put the factor $\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix}$ from its left to its right. This proves Lemma 6. $\square$

Now we give a decomposition of the integral of the remainder term of Lemma 5.

**Lemma 7.** *Suppose that the potentials $A_i$ , $\Phi$ have continuous first-order derivatives. Then*

$$b(r)=\int_0^r\left[\frac{1}{t}\beta(t)+\int_0^t\frac{1}{\tau^2}\beta(\tau)\,d\tau\right]\cos\lambda^*(r-t)\,dt=\frac{b_1(r)+b_2(r)}{\hbar c_0}$$

*where*

$$b_1(r)=\int_0^r\frac{\sin\lambda^*(r-t)}{t}S_t([C+\hat{C}]\Psi)\,dt,$$

$$b_2(r)=\int_0^r\frac{\sin\lambda^*(r-t)}{\lambda^*t}S_t\left(\left[\sum_k\partial_{y_k}\hat{C}\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}-\frac{1}{\hbar c_0}\hat{C}C\right]\Psi\right)\,dt.$$

*Proof.* Integration by parts shows that

$$\int_0^r\int_0^t\frac{1}{\tau^2}\beta(\tau)\,d\tau\cos\lambda^*(r-t)\,dt=\int_0^r\beta(t)\frac{\sin\lambda^*(r-t)}{\lambda^*t^2}\,dt.$$

Consequently
(14)

$$b(r)=\int_0^r\beta(t)\left[\frac{\cos\lambda^*(r-t)}{t}+\frac{\sin\lambda^*(r-t)}{\lambda^*t^2}\right]\,dt=\int_0^r\beta(t)\frac{d}{dt}\left[\frac{-\sin\lambda^*(r-t)}{\lambda^*t}\right]\,dt.$$

For the second term of $\beta(t)$ in (12) we have
(15)

$$\int_0^r\frac{d}{dt}\left[\frac{-\sin\lambda^*(r-t)}{\lambda^*t}\right]\frac{\lambda^*}{\hbar c_0}\int_0^t S_\tau(C\Psi)\,d\tau\,dt=\frac{1}{\hbar c_0}\int_0^r\frac{\sin\lambda^*(r-t)}{t}S_t(C\Psi)\,dt\,.$$

On the other hand by Lemma 6.

$$\begin{aligned}&\frac{d}{dt}\sum_k\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}S_t(n_kC\Psi)=\frac{d}{dt}S_t\left(\hat{C}\sum_k n_k\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}\Psi\right)\\&=\frac{d}{dt}\int_{|x-y|\le t}\sum_k\partial_{y_k}\left[\hat{C}(y)\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}\Psi(y)\right]\,dy=S_t\left(\sum_k\partial_{y_k}\left[\hat{C}\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}\Psi\right]\right)\\&=S_t\left(\sum_k\partial_{y_k}\hat{C}\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}\Psi\right)+S_t\left(\hat{C}\sum_k\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}\partial_{y_k}\Psi\right)\\&=S_t\left(\sum_k\partial_{y_k}\hat{C}\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}\Psi\right)+S_t\left(\hat{C}\left[\lambda^*\Psi-\frac{C}{\hbar c_0}\Psi\right]\right).\end{aligned}$$

Substituting the first part of (12) into (14) we get

$$\begin{aligned}&\frac{1}{\hbar c_0}\int_0^r\sum_k\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}S_t(n_kC\Psi)\frac{d}{dt}\frac{-\sin\lambda^*(r-t)}{\lambda^*t}\,dt\\&=\frac{1}{\hbar c_0}\int_0^r\frac{d}{dt}\left[\sum_k\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}S_t(n_kC\Psi)\right]\frac{\sin\lambda^*(r-t)}{\lambda^*t}\,dt\\&=\frac{1}{\hbar c_0}\int_0^r S_t\left(\sum_k\partial_{y_k}\hat{C}\begin{pmatrix}0&\sigma_k\\\sigma_k&0\end{pmatrix}\Psi\right)\frac{\sin\lambda^*(r-t)}{\lambda^*t}\,dt\\&+\frac{1}{\hbar c_0}\int_0^r S_t\left(\hat{C}\left[\lambda^*\Psi-\frac{C}{\hbar c_0}\Psi\right]\right)\frac{\sin\lambda^*(r-t)}{\lambda^*t}\,dt.\end{aligned}$$

With (15) this proves Lemma 7. □

## Part 3. The square sum of eigenfunctions

**Definition.** The system $(\Psi_n)$ of eigenfunctions (1) of the Dirac operator is called Bessel system in $L_2(\Omega,\mathbb{C}^4)$ if there exists a constant $c_1$ satisfying

$$\sum_n \Big|\int_\Omega \Psi_n^* f\Big|^2 \le c_1 \int_\Omega |f|^2 \quad \left(f\in L_2(\Omega,\mathbb{C}^4)\right) .$$

**Lemma 8.** *Let* $f\in L_2(\Omega,\mathbb{C}^4)$, $\quad x\in\Omega$, $\quad 0<r<dist(x,\partial\Omega)$, $\quad |\alpha_n|\le\alpha$ *and suppose that* $(\Psi_n)$ *is a Bessel system in* $L_2(\Omega,\mathbb{C}^4)$. *Then*

$$\sum_n \Big|\int_{|x-y|\le r} \Psi_n^*(y)f(y)\sin(\alpha_n|x-y|)\,dy\Big|^2 \le c_1\int_{|x-y|\le r}|f|^2\,dy\,e^{2r\alpha} .$$

*Analogous estimate holds with* cos *instead of* sin.

*Proof.* We expand the sine function into Taylor series:

$$a_n = \int_{|x-y|\le r} \Psi_n^*(y)f(y)\sin(\alpha_n|x-y|)\,dy = \sum_k a_{n,k},$$
$$a_{n,k} = (-1)^k\frac{\alpha_n^{2k+1}}{(2k+1)!}\int_{|x-y|\le r}\Psi_n^*(y)f(y)|x-y|^{2k+1}\,dy.$$

We know from the Bessel property that

$$\sum_n |a_{n,k}|^2 \le \frac{\alpha^{4k+2}}{[(2k+1)!]^2}c_1\int_{|x-y|\le r}|f(y)|^2|x-y|^{4k+2}\,dy \le \left[\frac{(r\alpha)^{2k+1}}{(2k+1)!}\right]^2 c_1\|f\|_{L_2}^2.$$

This implies that

$$\|(a_n)\|_{l_2} \le \sum_k \|(a_{n,k})\|_{l_2} \le \sqrt{c_1}\|f\|_{L_2}e^{r\alpha}$$

and we are done. $\square$

By the same method we can check that

**Lemma 8'.** *Under the conditions of Lemma 8. we have*

$$\sum_n \Big|\int_{|x-y|\le r}\Psi_n^*(y)f(y)\frac{\sin\alpha_n|x-y|}{\alpha_n|x-y|}\,dy\Big|^2 \le c_1\left(\frac{\sinh\alpha r}{\alpha r}\right)^2\int_{|x-y|\le r}|f|^2.$$

**Lemma 9.** *Square sum of eigenfunctions*
*If* $(\Psi_n)$ *is a Bessel system in* $L_2(\Omega,\mathbb{C}^4)$ , $K\subset\Omega$ *is compact and* $\mu\ge 1$ *then*

$$\sum_{||\lambda_n^*|-\mu|\le 1}|\Psi_n(x)|^2 \le c\mu^2 \quad (x\in K) \tag{16}$$

*where the constant* $c=c(K,c_1)$ *is independent of* $x$ *and* $\mu$ .

*Proof.* Choose a number $0 < R < \min(1/2\mathrm{dist}(K, \partial\Omega), \pi/4)$. Suppose first that $\mu$ is large enough, $\mu > \mu_0$ with some $\mu_0$ to be precised later. Introduce the function

$$d(r, \mu) = \begin{cases} \mu\frac{\sin(\mu r)}{r} & R \le r \le 2R \\ 0 & \text{elsewhere.} \end{cases}$$

Count the coefficients

$$\begin{aligned} d_n^* &= \int_\Omega d(|x-y|, \mu)\Psi_n(y)\,dy = \int_R^{2R} \mu\frac{\sin \mu r}{r} r^2 \int_{|\theta|=1} \Psi_n(x + r\theta)\,d\theta\,dr \\ &= \int_R^{2R} 4\pi\frac{\mu}{\lambda_n^*} \sin \mu r \sin \lambda_n^* r\,dr\Psi_n(x) \\ &+ \int_R^{2R} \mu \sin \mu r \int_0^r \left[\frac{1}{t}\beta(t) + \int_0^t \frac{1}{\tau^2}\beta(\tau)\,d\tau\right] \cos \lambda_n^*(r-t)\,dt\,dr \\ &= A_n\Psi_n(x) + B_n. \end{aligned}$$

The Bessel property implies that

$$\sum_n |A_n\Psi_n(x)+B_n|^2 \le c_1 \int_\Omega [d(|x-y|, \mu)]^2\,dy = c_1 \int_R^{2R} 4\pi r^2\mu^2\frac{\sin^2 \mu r}{r^2}\,dr \le c_1 4\pi\mu^2 R.$$

Here the coefficient $A_n$ has the estimate in case $\lambda_n^* > 0$ ( the case $\lambda_n^* < 0$ is similar)

$$\begin{aligned} A_n &= 4\pi\frac{\mu}{\lambda_n^*}\int_R^{2R} \sin \mu r \sin \lambda_n^* r\,dr = c\int_R^{2R} (\cos(\mu - \lambda_n^*)r - \cos(\mu + \lambda_n^*)r)\ dr \\ &= c\frac{\sin(\mu - \lambda_n^*)2R - \sin(\mu - \lambda_n^*)R}{\mu - \lambda_n^*} + \underline{\underline{O}}(\mu^{-1}) \\ &= c\frac{\sin(\mu - \lambda_n^*)R}{\mu - \lambda_n^*}(2\cos(\mu - \lambda_n^*)R - 1) + \underline{\underline{O}}(\mu^{-1}) \ge cR \end{aligned}$$

with $c$ independent also of $R$ if $\mu > \mu_0$ is large enough. Next we show that

$$\sum_{||\lambda_n^*|-\mu|\le 1} |B_n|^2 \le c(R)\mu^2; \tag{17}$$

it will imply the statement of Lemma 9., since then

$$\begin{aligned} cR^2 \sum_{||\lambda_n^*|-\mu|\le 1} |\Psi_n(x)|^2 &\le \sum_{||\lambda_n^*|-\mu|\le 1} |A_n\Psi_n(x)|^2 \\ &\le 2\sum_{||\lambda_n^*|-\mu|\le 1} (|B_n|^2 + |A_n\Psi_n(x) + B_n|^2) \le c(R)\mu^2. \end{aligned}$$

Now check (17). In $B_n$ we use the decomposition of the inner integral (see (12)) given in Lemma 7. Suppose again that $\lambda_n^* = \mu + \varepsilon_n > 0$. Then the expression in the inner integral of $B_n$ corresponding to $b_1(r)$ has the form

$$\begin{aligned} &\sin \lambda_n^* r \cdot \int_{|x-y|\le r} (\cos \mu|x-y| \cos \varepsilon_n|x-y| - \sin \mu|x-y| \sin \varepsilon_n|x-y|) \\ &\frac{C(y) + \hat{C}(y)}{|x-y|}\Psi_n(y)\,dy \\ &- \cos \lambda_n^* r \cdot \int_{|x-y|\le r} (\sin \mu|x-y| \cos \varepsilon_n|x-y| + \cos \mu|x-y| \sin \varepsilon_n|x-y|) \\ &\frac{C(y) + \hat{C}(y)}{|x-y|}\Psi_n(y)\,dy. \end{aligned}$$

Lemma 8. states that the square sum of these quantities is bounded by

$$c(R)\int_{|x-y|\le r}\frac{|C(y)+\hat{C}(y)|^2}{|x-y|^2}\,dy \le c(R)\|C+\hat{C}\|_{L_\infty}^2 r.$$

The same estimate is true for the inner integral of $B_n$ corresponding to $b_2(r)$, only $C+\hat{C}$ is substituted by another bounded matrix function and the quantity is divided by $\lambda_n^*$ , a number not close to zero. The above considerations imply that the square sum of the inner integrals of $B_n$ is $\underline{\underline{O}}(r)$; hence

$$\sum_n |B_n(x)|^2 = \sum_n |\int_R^{2R}\mu\sin\mu r\cdot\int_0^r \cdots\,dt\,dr|^2 \le \mu^2 R\int_R^{2R}\sum_n|\int_0^r\cdots dt|^2\,dr \le c\mu^2$$

which proves Lemma 9. if $\mu \ge \mu_0$.
Now consider the case $\mu \le \mu_0$. We define in this case

$$d(r,\mu) = \begin{cases} \frac{1}{r} & R_1 \le r \le 2R_1 \\ 0 & \text{elsewhere} \end{cases}$$

and restart the above proof. Now

$$A_n = 4\pi\frac{\cos\lambda_n^* R_1 - \cos 2\lambda_n^* R_1}{{\lambda_n^*}^2},\; B_n = \int_{R_1}^{2R_1}(b_1(r)+b_2(r))\,dr.$$

If $0 < R_1$ is small enough then the numerator of $A_n$ has the magnitude order ${\lambda_n^*}^2 R_1^2$ , hence

$$|A_n| \ge cR_1^2$$

with a universal constant $c > 0$. The square sum of $b_1(r)$ can be estimated by Lemma 8. (with $\alpha_n = \lambda_n^*$ ). On the other hand

$$b_2(r) = \frac{\sin\lambda_n^* r}{\lambda_n^*}\int_0^r\frac{\cos\lambda_n^* t}{t}S_t(\tilde{C}\Psi_n)\,dt - \cos\lambda_n^* r\cdot\int_0^r\frac{\sin\lambda_n^* t}{\lambda_n^* t}S_t(\tilde{C}\Psi_n)\,dt$$

where $\tilde{C}$ is the simple notation of the matrix given in the brackets of the definition of $b_2(r)$. Here in the first summand we use $|\frac{\sin\lambda_n^* r}{\lambda_n^*}| < r$ and Lemma 8, in the second summand Lemma 9. to show

$$\sum_n |b_1(r)+b_2(r)|^2 \le c$$

which in turn implies $\sum_n |B_n|^2 \le c$. Remark that if summands $\lambda_n^* = 0$ occur, we use the corresponding limit relations resulting from $\lambda_n^* \to \infty$ and the same estimates hold true. Lemma 9. is proved. $\square$

## Part 4. The spectral function

We call *spectral function* the kernel of the partial sum operator ; it has the form

$$\Theta(x,y,\mu)=\sum_{|\lambda_n^*|<\mu}\Psi_n(x)\Psi_n^*(y). \tag{18}$$

This is a $4\times 4$-matrix; the sum (if infinite) is convergent pointwise and for fixed $x$ in $L_2$-norm in $y$ (this can also be derived from Lemma 9.). Consequently

$$S_\mu(f,x)=\int_\Omega \Theta(x,y,\mu)f(y)\,dy. \tag{19}$$

We shall see below that $\Theta(x,y,\mu)$ "approximately" equals to

$$\Theta_0(x,y,\mu)=V(|x-y|,\mu)\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}$$

where

$$V(r,\mu)=\begin{cases}\frac{1}{2\pi^2}\frac{\mu}{r^2}(-\cos\mu r+\frac{\sin\mu r}{\mu r}) & (0<r<R)\\ 0 & (r\ge R)\end{cases}.$$

Some preliminary results are needed.

**Lemma 10.** *Let $\tilde C$ be an arbitrary $4\times 4$ matrix function, continuous in $\Omega$ and let $F$ be continuous in $(0,R]$. Then*

$$\begin{aligned}&\sum_n\Big|\int_{|x-y|\le R}\frac{\tilde C(y)}{|x-y|}\Psi_n(y)\int_{|x-y|}^R F(r)\sin\lambda_n^*(r-|x-y|)\,dr\,dy\Big|^2\\ &\le c\|\tilde C\|_\infty^2\int_0^R rF^2(r)\,dr\end{aligned} \tag{20}$$

*whenever the right hand side is finite.*

*Proof.* Denote $a_n$ the quantity in the absolute value in the left of (20). We can substitute $\lambda_n^*$ by $|\lambda_n^*|$ in $a_n$. For every $n$ choose the natural number $k\ge 1$ satisfying $|\lambda_n^*|=k+\varepsilon_n$ , $-1\le\varepsilon_n<0$. Then the inner integral of $a_n$ can be transformed as follows

$$\begin{aligned}&\int_{|x-y|}^R F(r)\left[\sin k(r-|x-y|)\cos\varepsilon_n(r-|x-y|)+\cos k(r-|x-y|)\sin\varepsilon_n(r-|x-y|)\right]dr\\ &=\sum_{d=0}^\infty(-1)^d\Big[\frac{\varepsilon_n^{2d}}{(2d)!}\int_{|x-y|}^R F(r)\sin k(r-|x-y|)\cdot(r-|x-y|)^{2d}\,dr\\ &+\frac{\varepsilon_n^{2d+1}}{(2d+1)!}\int_{|x-y|}^R F(r)\cos k(r-|x-y|)\cdot(r-|x-y|)^{2d+1}\,dr\Big]\\ &=\sum_{d=0}^\infty(-1)^d\left[\frac{\varepsilon_n^{2d}}{(2d)!}\alpha_1(k,d,|x-y|)+\frac{\varepsilon_n^{2d+1}}{(2d+1)!}\alpha_2(k,d,|x-y|)\right].\end{aligned}$$

The corresponding decomposition of $a_n$ has the form

$$a_n = \sum_{d=0}^{\infty} (a_{n,1}^{(d)} + a_{n,2}^{(d)})$$

where

$$a_{n,1}^{(d)} = (-1)^d \frac{\varepsilon_n^{2d}}{(2d)!} \int_{|x-y|\le R} \frac{\tilde{C}(y)}{|x-y|} \alpha_1(k,d,|x-y|)\Psi_n(y)\,dy,$$
$$a_{n,2}^{(d)} = (-1)^d \frac{\varepsilon_n^{2d+1}}{(2d+1)!} \int_{|x-y|\le R} \frac{\tilde{C}(y)}{|x-y|} \alpha_2(k,d,|x-y|)\Psi_n(y)\,dy.$$

Since the functions $\sin kr$ form an orthogonal basis in $L_2(0,\pi)$,

$$\sum_{k=1}^{\infty} |\alpha_1(k,d,t)|^2 \le c \int_t^R F^2(r)(r-t)^{4d}\,dr \le cR^{4d} \int_t^R F^2(r)\,dr$$

and similarly

$$\sum_{k=1}^{\infty} |\alpha_2|^2 \le cR^{4d+2} \int_t^R F^2(r)\,dr.$$

Using this and the basis property of the $\Psi_n$ we get for fixed $d$

$$\begin{aligned}
\sum_n |a_{n,1}^{(d)}|^2 &\le c\frac{1}{[(2d)!]^2} \sum_{k=1}^{\infty} \sum_{k-1\le|\lambda_n^*|<k} \Big| \int_{|x-y|\le R} \frac{\tilde{C}(y)}{|x-y|} \alpha_1(k,d,|x-y|)\Psi_n(y)\,dy \Big|^2 \\
&\le \frac{c}{[(2d)!]^2} \sum_{k=1}^{\infty} \int_{|x-y|\le R} \frac{|\tilde{C}(y)|^2}{|x-y|^2} |\alpha_1|^2\,dy \\
&= \frac{c}{[(2d)!]^2} \int_0^R \frac{1}{t^2} S_t(|\tilde{C}|^2) \sum_{k=1}^{\infty} |\alpha_1(k,d,t)|^2\,dt \\
&\le \frac{c}{[(2d)!]^2} \|\tilde{C}\|_\infty^2 \int_0^R \int_t^R F^2(r)\,dr \cdot R^{4d}\,dt \\
&\le c\|\tilde{C}\|_\infty^2 \frac{R^{4d}}{[(2d)!]^2} \int_0^R rF^2(r)\,dr
\end{aligned}$$

and similarly

$$\sum_n |a_{n,2}^{(d)}|^2 \le c\|\tilde{C}\|_\infty^2 \frac{R^{4d+2}}{[(2d+1)!]^2} \int_0^R rF^2(r)\,dr.$$

These estimates imply that

$$\begin{aligned}
\|a_n\|_{l_2} &\le \sum_d \|a_{n,1}^{(d)}\|_{l_2} + \sum_d \|a_{n,2}^{(d)}\|_{l_2} \le c\|\tilde{C}\|_\infty \sqrt{\int_0^R rF^2(r)\,dr} \cdot \sum_d \left[\frac{R^{2d}}{(2d)!} + \frac{R^{2d+1}}{(2d+1)!}\right] \\
&\le ce^R \|\tilde{C}\|_\infty \sqrt{\int_0^R rF^2(r)\,dr}.
\end{aligned}$$

□

**Lemma 11.** *Let $F$ be arbitrary (real-valued, measurable) function; then*

$$\sum_n \Big| \int_0^R F(r)b(r)\,dr\Big|^2 \le c\int_0^R rF^2(r)\,dr$$

*whenever the right hand side is finite. The constant $c$ may depend on $R$, $K$, $C$ but not on $F$ and $x$. The function $b(r)$ is defined in Lemma 7.*

*Proof.* $\int_0^R Fb_1$ and $\lambda_n^* \int_0^R Fb_2$ both have the form

$$\int_0^R \frac{1}{t} S_t(\tilde{C}\Psi_n) \int_t^R F(r)\sin\lambda_n^*(r-t)\,dt$$

hence Lemma 10. implies that

$$\sum_n \Big| \int_0^R Fb_1\Big|^2 + \sum_{|\lambda_n^*|\ge 1} \Big| \int_0^R Fb_2\Big|^2 \le c\int_0^R rF^2(r)\,dr.$$

On the other hand

$$\begin{aligned}\int_0^R Fb_2 &= \int_0^R F(r)\frac{\sin\lambda_n^* r}{\lambda_n^*}\int_0^r \frac{1}{t}S_t(\tilde{C}\Psi_n)\cos\lambda_n^* t\,dt\,dr\\ &- \int_0^R F(r)\cos\lambda_n^* r\int_0^r S_t(\tilde{C}\Psi_n)\frac{\sin\lambda_n^* t}{\lambda_n^* t}\,dt\,dr\\ &= I_1 + I_2.\end{aligned}$$

Here the square sum of the $I_1$ resp. $I_2$ can be estimated by Lemma 8. resp. Lemma 8' as follows:

$$\begin{aligned}&\sum_{|\lambda_n^*|<1} |I_1|^2\\ &\le c\int_0^R rF^2(r)\,dr\int_0^R r\sum_{|\lambda_n^*|<1}\Big|\int_{|x-y|\le r}\frac{\tilde{C}(y)}{|x-y|}\Psi_n(y)\cos\lambda_n^*|x-y|\,dy\Big|^2\,dr,\\ &\le c\int_0^R rF^2(r)\,dr,\\ &\sum_{|\lambda_n^*|<1} |I_2|^2 \le c\int_0^R rF^2(r)\,dr\int_0^R r\sum_{|\lambda_n^*|<1}\Big|\int_{|x-y|\le r}\frac{\tilde{C}(y)}{|x-y|}\Psi_n(y)\cos\lambda_n^*|x-y|\,dy\Big|^2\,dr\\ &\le c\int_0^R rF^2(r)\,dr.\end{aligned}$$

The proof is complete. $\square$

**Lemma 12.** *Estimates for the spectral function*
*a./ Denote*

$$\hat{\Theta} = \Theta_0 - \Theta$$

*the remainder term of the spectral function. Then*

$$\|\hat{\Theta}\|_{L_2(\Omega)} \le \begin{cases} c\mu\ln\mu & (\mu \ge \mu_0) \\ c & (\mu \le \mu_0) \end{cases} \tag{21}$$

*b./ Let $1 \le q < 3/2$ and let $D \subset \Omega$ be a measurable bounded subset. Then*

$$\|\Theta\|_{L_q(D)} \le \begin{cases} c\mu\ln\mu & (\mu \ge \mu_0) \\ c & (\mu \le \mu_0) \end{cases} \tag{22}$$

*c./ Let $\Omega_0 \subset \Omega$ be a subdomain and $K \subset \Omega$ be compact. Then*

$$\|\Theta\|_{L_2(\Omega\setminus\Omega_0)} \le \begin{cases} c\mu\ln\mu & (\mu \ge \mu_0) \\ c & (\mu \le \mu_0) \end{cases} \qquad (x \in K). \tag{23}$$

*The norms are taken in the second variable $y$ .*

*Proof.* a./ Count the coefficients of $\Theta_0$:

$$\begin{aligned} v_n &= \int_\Omega \Psi_n^*(z)V(|x-z|,\mu)\,dz = \int_0^R V(r,\mu)r^2\int_{|\theta|=1}\Psi_n^*(x+r\theta)\,d\theta\,dr \\ &= 4\pi\int_0^R rV(r,\mu)\frac{\sin\lambda_n^* r}{\lambda_n^*}\,dr\cdot\Psi_n^*(x) + \left[\int_0^R rV(r,\mu)b(r)\,dr\right]^* \\ &= \hat{A}_n\Psi_n^*(x) + \hat{B}_n^*. \end{aligned}$$

Then

$$V(|x-y|,\mu)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \sum_n \Psi_n(y)v_n = \sum_n \Psi_n(y)\Psi_n^*(x)\hat{A}_n + \sum_n \Psi_n(y)\hat{B}_n^*;$$

hence the remainder $\hat{\Theta}$ has the decomposition

$$\hat{\Theta}^* = \hat{\Theta}_1 + \hat{\Theta}_2$$

where

$$\hat{\Theta}_1(x,y,\mu) = \sum_n \Psi_n(y)\Psi_n^*(x)(\hat{A}_n - \delta_{|\lambda_n^*|<\mu}),$$

$$\hat{\Theta}_2(x,y,\mu) = \sum \Psi_n(y)\hat{B}_n^*.$$

It is known (see e.g. Joó [14]) that

$$|\hat{A}_n - \delta_{|\lambda_n^*|<\mu}| \le \frac{1}{2}\delta_{|\lambda_n^*|=\mu} + c\frac{\mu}{1+|\mu-|\lambda_n^*||}\frac{1}{1+|\lambda_n^*|}.$$

It makes easy the norm estimate of $\hat{\Theta}_1$:

$$\begin{aligned} \|\hat{\Theta}_1\|^2 &= \sum_n |\hat{A}_n - \delta_{|\lambda_n^*|<\mu}|^2|\Psi_n(x)|^2 \\ &\le c\sum_{|\lambda_n^*|=\mu}|\Psi_n(x)|^2 + c\mu^2\sum_n\frac{\mu^2}{1+|\mu-|\lambda_n^*||^2}\frac{|\Psi_n(x)|^2}{1+{\lambda_n^*}^2} \\ &\le c(1+\mu^2) + c\mu^2\sum_{k=1}^\infty\frac{1}{1+|\mu-k|^2}\frac{1}{1+k^2}\sum_{k-1\le|\lambda_n^*|<k}|\Psi_n(x)|^2 \\ &\le c(1+\mu^2) + c\mu^2\sum_k\frac{1}{1+|\mu-k|^2} \le c(1+\mu^2). \end{aligned}$$

On the other hand Lemma 11. states that

$$\|\hat{\Theta}_2\|^2 = \sum_n |\hat{B}_n|^2 \le c\int_0^R r^3 V^2(r,\mu)\,dr = c\mu^2\int_0^R \left(-\cos\mu r + \frac{\sin\mu r}{\mu r}\right)\frac{1}{r} \le c\mu^2\ln(2+\mu)$$

which proves a./
b./ Since $D$ is bounded,

$$\|\hat{\Theta}\|_{L_q(D)} \le c(q)\|\hat{\Theta}\|_{L_2(D)}$$

so we have to check the norm estimate of b./ only for $\Theta_0$ :

$$\|\Theta_0\|_{L_q}^q = c\mu^q\int_0^R \left|-\cos\mu r + \frac{\sin\mu r}{\mu r}\right|^q r^{2-2q}\,dr \le c\mu^q\int_0^R r^{2-2q}\,dr \le c\mu^q$$

since $q < 3/2$ . This proves b./
c./ Again it remained to check the norm estimate c./ for $\Theta_0$ . But if $x \in K \subset \Omega_0$ , $y \in \Omega\setminus\Omega_0$ and $0 < R < dist(K, \Omega\setminus\Omega_0)$ , then by definition $\Theta_0(x,y,\mu) = 0$. Lemma 12. is proved. □

## Part 5. The proof of Theorem 1, 2, and 3.

First we show the statement of Theorem 1. for sufficiently smooth functions:

**Lemma 13.** *If $f \in C_0^\infty(\Omega)$ then*

$$S_\mu(f,x) \to f(x) \quad (\mu\to\infty)$$

*locally uniformly in $\Omega$ .*

*Proof.*

$$\begin{aligned}
\int_\Omega \Psi_n^* f &= \frac{1}{\lambda_n^*}\int_\Omega \left[\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix}\partial_{y_k}\Psi_n + \frac{1}{\hbar c_0}C\Psi_n\right]^* f \\
&= \frac{1}{\lambda_n^*}\int_\Omega \Psi_n^*\left[\sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix}\partial_{y_k} f + \frac{1}{\hbar c_0}Cf\right] \\
&= \frac{1}{\lambda_n^*}\int_\Omega \Psi_n^* f_1.
\end{aligned}$$

Iterating this process once more we obtain

$$\int_\Omega \Psi_n^* f = \frac{1}{{\lambda_n^*}^2}\int_\Omega \Psi_n^* f_2;$$

since $C$ is continuously differentiable, $f_2$ is continuous with compact support in $\Omega$. Consequently

$$\begin{aligned}
|f(x) - S_\mu(f,x)| &\le \sum_{|\lambda_n^*|\ge\mu}\frac{1}{{\lambda_n^*}^2}\left|\int_\Omega \Psi_n^* f_2\right|\left|\Psi_n(x)\right| \\
&\le \sqrt{\sum_n \left|\int_\Omega \Psi_n^* f_2\right|^2}\sqrt{\sum_{|\lambda_n^*|\ge\mu}\frac{1}{{\lambda_n^*}^4}|\Psi_n(x)|^2} \\
&\le \|f_2\|_{L_2}\sqrt{\sum_{k=[\mu]}^\infty \sum_{k-1<|\lambda_n^*|\le k}\frac{|\Psi_n(x)|^2}{k^4}} \\
&\le c\|f_2\|_{L_\infty(\mathrm{supp}\ f)}\sqrt{\sum_k \frac{k^2}{k^4}} \le c\|f_2\|_{L_\infty(\mathrm{supp}\ f)}\sqrt{\frac{1}{\mu}}
\end{aligned}$$

i.e. $S_\mu f - f = \underline{\underline{O}}(\mu^{-1/2})$ tends uniformly to zero. $\square$

We shall need also the next two statements. Define the polynomial

$$w_\alpha(r) = a_0 + a_2 r^2$$

where the constants $a_0$ , $a_2$ are chosen to satisfy

$$v_\alpha(R) = w_\alpha(R), \quad v'_\alpha(R) = w'_\alpha(R)$$

Finally let

$$v^R_\alpha(r) - w^R_\alpha(r) = \begin{cases} v_\alpha(r) - w_\alpha(r) & (r \le R) \\ 0 & (r > R). \end{cases}$$

**Lemma 14.** *Nikolskii* [15]
*Let $0 < \alpha < 2$ and $f \in L^\alpha_p(\mathbb{R}^3, \mathbb{C}^4)$, $suppf \subset \Omega$ be compact and let $0 < R < dist(suppf, \partial\Omega)$. Then there exists a function $g \in L_p(\mathbb{R}^3, \mathbb{C}^4)$ with compact support in $\Omega$ satisfying*

$$f(x) = \int_\Omega (v^R_\alpha(|x-y|) - w^R_\alpha(|x-y|))g(y)\,dy.$$

*The norms $\|g\|_{L_p}$ and $\|f\|_{L^\alpha_p}$ are equivalent.*

The following imbedding theorem is needed

**Lemma 15.** *Triebel* [18]
*Let $1 < p < q < \infty$, $\alpha > 0$ and $\beta = \alpha - 3(\frac{1}{p} - \frac{1}{q}) \ge 0$. then*

$$L^\alpha_p(\mathbb{R}^3) \subset L^\beta_q(\mathbb{R}^3)$$

*and this imbedding is continuous.*

*Proof of Theorem 1.*
*Step 1.* We can suppose that $1 < \alpha < 2$ and $p > 3$ . Indeed, diminish first $\alpha$ to ensure $3 < \alpha p < 4$ . Then define $\beta$ and $p'$ by the formula

$$\beta = 1 + \gamma = \alpha - 3\left(\frac{1}{p} - \frac{1}{p'}\right).$$

This means that

$$\frac{1}{p'} = \frac{1}{p} - \frac{\alpha - 1 - \gamma}{3} = \frac{1}{3} + \frac{3 - \alpha p + \gamma p}{3p} < \frac{1}{3}$$

if $0 < \gamma < \alpha - 1$ is sufficiently small. On the other hand it follows from $\alpha p < 4$ that $\frac{1}{p'}$ is positive. Hence Lemma 15. applies: $L^\alpha_p \subset L^\beta_{p'}$ and it is enough to prove Theorem 1. for $f \in L^\beta_{p'}$. From now on we denote $1 < \beta < 2$ by $\alpha$ and $3 < p' < \infty$ by $p$.
*Step 2.* Take the function $g$ corresponding to $f$ by Lemma 14. Then

$$\begin{aligned} S_\mu(f,x)^* &= \sum_{|\lambda^*_n| < \mu} \int_\Omega f^*(y)\Psi_n(y)\,dy \cdot \Psi^*_n(x) \\ &= \sum_{|\lambda^*_n| < \mu} \int_\Omega \int_\Omega g^*(z)(v^R_\alpha - w^R_\alpha)(|y-z|)\,dz \Psi_n(y)\,dy \cdot \Psi^*_n(x) \\ &= \int_\Omega g^*(z) \sum_{|\lambda^*_n| < \mu} \int_\Omega (v^R_\alpha - w^R_\alpha)(|y-z|)\Psi_n(y)\Psi^*_n(x)\,dy\,dz. \end{aligned}$$

This implies that

$$|S_\mu(f,x)| \le \|g\|_{L_p} \cdot \| \sum_{|\lambda_n^*|<\mu} \int_\Omega (v_\alpha^R - w_\alpha^R)(|y-z|)\Psi_n(y)\Psi_n^*(x)\,dy\|_{L_q(\mathrm{supp}\ g)}$$

where

$$\frac{1}{p}+\frac{1}{q}=1 \quad (\Rightarrow 1<q<\frac{3}{2}).$$

Suppose that for $x\in K$

$$\| \sum_{|\lambda_n^*<\mu} \int_\Omega (v_\alpha^R - w_\alpha^R)(|y-z|)\Psi_n(y)\Psi_n^*(x)\,dy\|_{L_q(\mathrm{supp}\ g)} \le c; \tag{24}$$

with $c=c(K,q,R)$ independent of $\mu$ and $f$ . Then we have

$$|S_\mu(f,x)| \le c\|g\|_{L_p} \le c\|f\|_{L_p^\alpha}. \tag{25}$$

Given an $\varepsilon>0$ there exists a function $f_0\in C_0^\infty$ such that

$$\|f-f_0\|_{L_p^\alpha}<\varepsilon; \quad \|f-f_0\|_{L_\infty}<\varepsilon$$

(see Triebel [18]).But we know from Lemma 13. that

$$\|S_\mu f_0 - f_0\|_{L_\infty(K)} < \varepsilon$$

for $\mu$ sufficiently large. Then (25) implies that

$$|S_\mu f(x)-f(x)| \le |S_\mu(f-f_0,x)| + |f(x)-f_0(x)| + |S_\mu(f_0,x)-f_0(x)| < (c+2)\varepsilon.$$

Thus we have seen that the statement of Theorem 1. is a simple corollary of (24). In what follows we prove (24).

*Step 3.* We evaluate the integral in (24). For the sake of simplicity denote

$$\varphi(r) = r(v_\alpha^R(r) - w_\alpha^R(r)).$$

Then

$$\begin{aligned}
&\int_\Omega (v_\alpha^R - w_\alpha^R)(|y-z|)\Psi_n(y)\,dy \\
&= \int_0^R (v_\alpha^R(r) - w_\alpha^R(r))r^2 \int_{|\theta|=1} \Psi_n(x+r\theta)\,d\theta\,dr \\
&= \int_0^R \varphi(r) 4\pi \frac{\sin\lambda_n^* r}{\lambda_n^*}\,dr \cdot \Psi_n(z) + \int_0^R \varphi(r) b(r)\,dr \\
&= C_n\Psi_n(z) + D_n(z).
\end{aligned}$$

Hence we have to show

$$\| \sum_{|\lambda_n^*|<\mu} (C_n\Psi_n(z) + D_n)\,\Psi_n^*(x)\|_{L_q(\mathrm{supp}\ g)} \le c. \tag{24'}$$

*Step 4.* We prove that

$$C_n = c(1+\lambda_n^{*\,2})^{-\alpha/2} + \delta_n, \quad \delta_n = \underline{\underline{O}}(\frac{1}{1+\lambda_n^{*\,2}}).$$

Indeed, integration by parts shows

$$\int_0^R w_\alpha(r) r \cdot 4\pi \frac{\sin \lambda_n^* r}{\lambda_n^*}\, dr = \underline{\underline{O}}(\frac{1}{1+\lambda_n^{*\,2}}).$$

In order to estimate the other part of $C_n$ we recall the following expansion of the Bessel function $K_\nu$ , valid for noninteger $\nu$ :

$$\begin{aligned} K_\nu(r) &= \frac{\pi}{2\sin\nu\pi}[I_{-\nu}(r) - I_\nu(r)] \\ &= \frac{\pi}{2\sin\nu\pi}\Big[\sum_{m=0}^{\infty} \frac{\left(\frac{r}{2}\right)^{2m-\nu}}{m!\Gamma(m-\nu+1)} \\ &\quad - \sum_{m=0}^{\infty} \frac{\left(\frac{r}{2}\right)^{2m+\nu}}{m!\Gamma(m+\nu+1)}\Big]. \end{aligned}$$

It shows that

$$v_\alpha(r) = c_1 r^{\alpha-3} + c_2 + c_3 r^{\alpha-1} + c_4 r^2 + \dots$$

whence we get

$$\begin{aligned} 4\pi \int_0^R r v_\alpha(r) \frac{\sin\lambda_n^* r}{\lambda_n^*}\, dr &= 4\pi c_1 \int_0^R \frac{\sin \lambda_n^* r}{\lambda_n^* r^{2-\alpha}}\, dr + \underline{\underline{O}}(\frac{1}{1+\lambda_n^{*\,2}}) \\ &= \frac{c}{(1+\lambda_n^{*\,2})^{\alpha/2}} + \underline{\underline{O}}(\frac{1}{1+\lambda_n^{*\,2}}). \end{aligned}$$

Indeed, if $|\lambda_n^*|$ is not large then

$$\Big| \int_0^R \frac{\sin\lambda_n^* r}{\lambda_n^* r^{2-\alpha}}\, dr \Big| \le \int_0^R \frac{r}{r^{2-\alpha}}\, dr = \underline{\underline{O}}(1)$$

and if $|\lambda_n^*|$ is large then

$$\begin{aligned} \int_0^R \frac{\sin \lambda_n^* r}{\lambda_n^* r^{2-\alpha}}\, dr &= \int_0^{|\lambda_n^*|R} \frac{\sin t}{t^{2-\alpha}}\, dt \frac{1}{\lambda_n^{*\,\alpha}} = \left(\int_0^\infty - \int_{|\lambda_n^*|R}^\infty\right) \frac{\sin t}{t^{2-\alpha}}\, dt \frac{1}{\lambda_n^{*\,\alpha}} \\ &= \frac{c}{\lambda_n^{*\,\alpha}} + \underline{\underline{O}}(\frac{1}{\lambda_n^{*\,2}}) \\ &= \frac{c}{(1+\lambda_n^{*\,2})^{\alpha/2}} + \underline{\underline{O}}(\frac{1}{1+\lambda_n^{*\,2}}). \end{aligned}$$

*Step 5.* We show that

$$\| \sum_{|\lambda_n^*|<\mu} C_n \Psi_n(z) \Psi_n^*(x) \|_{L_q(\mathrm{supp}\ g)} \le c.$$

Following the decomposition given in Step 4. we take first the remainder of $C_n$

$$\begin{aligned}
&\| \sum_{|\lambda_n^*|<\mu} \delta_n \Psi_n(z)\Psi_n^*(x)\|^2_{L_q(\text{supp } g)} \\
&\le c\| \sum_{|\lambda_n^*|<\mu} \delta_n \Psi_n(z)\Psi_n^*(x)\|^2_{L_2(\text{supp } g)} \le c \sum_{|\lambda_n^*|<\mu} \delta_n^2 |\Psi_n(x)|^2 \\
&\le c \sum_{k=1}^{[\mu]+1} \frac{1}{1+k^4} \sum_{k-1\le|\lambda_n^*|<k} |\Psi_n(x)|^2 \le c \sum_k \frac{1}{k^2} \le c.
\end{aligned}$$

The main term of $C_n$ gives the sum

$$\sum_{|\lambda_n^*|<\mu} \frac{\Psi_n(z)\Psi_n^*(x)}{(1+{\lambda_n^*}^2)^{\alpha/2}} = \int_0^\mu \frac{d_\mu \Theta(z,x,t)}{(1+t^2)^{\alpha/2}} = \frac{\Theta(z,x,\mu)}{(1+\mu^2)^{\alpha/2}} + \alpha/2 \int_0^\mu \Theta(z,x,t) \frac{2t}{(1+t^2)^{\alpha/2+1}}\, dt.$$

Then by (22) applied with $D = \text{supp } g$

$$\| \sum_{|\lambda_n^*|<\mu} \frac{\Psi_n(z)\Psi_n^*(x)}{(1+{\lambda_n^*}^2)^{\alpha/2}} \|_{L_q(D)} \le c \frac{(1+\mu)\ln(2+\mu)}{(1+\mu^2)^{\alpha/2}} + c \int_0^\mu (1+t)\ln(2+t) \frac{t}{(1+t^2)^{\alpha/2+1}}\, dt \le c$$

and we are ready.
*Step 6.* We finally show that

$$\| \sum_{|\lambda_n^*|<\mu} D_n(z)\Psi_n^*(x)\|_{L_2(\text{supp } g)} \le c; \tag{26}$$

then the same will be of course valid for the $L_q$-norm. Recall that

$$D_n(z) = \int_0^R \varphi(r) b(r)\, dr$$

and, by the definition of $w_\alpha$

$$\varphi(R) = \varphi'(R) = 0.$$

From the expansion of the Bessel function $K_\nu$ mentioned in Step 4. we derive the estimates

$$\varphi(r) \le c r^{\alpha-2}, \quad \varphi'(r) \le c r^{\alpha-3}, \quad \varphi''(r) \le c r^{\alpha-4}.$$

Using this we consider first the summands of (26) with small eigenvalue. From Lemma 9. and 11. we get

$$\begin{aligned}
| \sum_{|\lambda_n^*|<1} D_n(z)\Psi_n^*(x)| &\le \sqrt{\sum_{|\lambda_n^*|<1} |D_n(z)|^2} \sqrt{| \sum_{|\lambda_n^*|<1} |\Psi_n(x)|^2} \\
&\le c\sqrt{\int_0^R r\varphi^2(r)\, dr} \le c\sqrt{\int_0^R r^{2\alpha-3}\, dr} \le c.
\end{aligned}$$

Now consider the decomposition of $D_n$ corresponding to Lemma 7. The first part of it is

$$D_{n,1}(z) = \int_0^R \varphi(r) b_1(r)\, dr = \int_0^R \frac{1}{t} S_t((C+\hat{C})\Psi_n) \int_t^R \varphi(r) \sin \lambda_n^*(r-t)\, dr\, dt.$$

The inner integral has the form

$$\begin{aligned}\int_t^R \varphi(r) \sin \lambda_n^*(r-t)\, dr &= \frac{\varphi(t)}{\lambda_n^*} + \int_t^R \varphi'(r) \frac{\cos \lambda_n^*(r-t)}{\lambda_n^*}\, dr \\ &= \frac{\varphi(t)}{\lambda_n^*} - \int_t^R \varphi"(r) \frac{\sin \lambda_n^*(r-t)}{{\lambda_n^*}^2}\, dr \\ &= \frac{\varphi(t)}{\lambda_n^*} + \delta_n; \\ \delta_n &= \underline{\underline{O}}\left(\frac{1}{{\lambda_n^*}^2 t^{3-\alpha}}\right)\end{aligned} \tag{27}$$

The remainder term of (27) put into $D_{n,1}$ can be easily estimated:

$$\begin{aligned}&\| \sum_{1\le|\lambda_n^*|<\mu} \int_0^R \frac{1}{t} S_t((C+\hat{C})\Psi_n)\delta_n(t)\, dt \cdot \Psi_n^*(x)\|_{L_2} \\ &= \| \int_{|w|\le R} \frac{1}{|w|}(C+\hat{C})(z+w) \sum_{1\le|\lambda_n^*|<\mu} \Psi_n(z+w)\Psi_n^*(x)\delta_n(|w|)\, dw\|_{L_2} \\ &\le c \int_{|w|\le R} \frac{1}{|w|} \| \sum_n \Psi_n(z+w)\Psi_n^*(x)\delta_n(|w|)\|_{L_2}\, dw \\ &\le c \int_{|w|\le R} \frac{1}{|w|} \sqrt{\sum_n |\Psi_n(x)|^2 \delta_n^2(|w|)}\, dw \\ &\le c \int_{|w|\le R} |w|^{\alpha-4} \sqrt{\sum_n \frac{|\Psi_n(x)|^2}{{\lambda_n^*}^4}}\, dw \le c \int_0^R t^2 t^{\alpha-4}\, dt \le c.\end{aligned}$$

The part of $D_{n,1}$ corresponding to the main term of (27) can be expressed by (1) as follows:

$$\begin{aligned}I(z) &= \frac{1}{\lambda_n^*} \int_0^R \frac{1}{t} S_t((C+\hat{C})\Psi_n)\varphi(t)\, dt \\ &= \frac{1}{{\lambda_n^*}^2} \int_{|w|\le R} \frac{\varphi(|w|)}{|w|}(C+\hat{C})(z+w) \Big[ \frac{C(z+w)}{\hbar c_0} \Psi_n(z+w) \\ &+ \sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \partial_{w_k} \Psi_n(z+w) \Big]\, dw \\ &= \frac{1}{{\lambda_n^*}^2} \int_{|w|\le R} \Big[ \frac{\varphi(|w|)}{|w|} \Big( \frac{(C^2+\hat{C}C)(z+w)}{\hbar c_0} \\ &- \sum_k \partial_{w_k}(C+\hat{C})(z+w) \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \Big) \\ &- (C+\hat{C})(z+w) \sum_k \partial_{w_k} \frac{\varphi(|w|)}{|w|} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \Big] \Psi_n(z+w)\, dw.\end{aligned} \tag{28}$$

Consider in (26) the terms represented by $I(z)$ :

$$
\begin{aligned}
&\| \sum_{1\le|\lambda_n^*|<\mu} I(z)\Psi_n^*(x)\|_{L_2} \\
&\le c\int_{|w|\le R}\left(\frac{|\varphi(|w|)|}{|w|}+\sum_k\left|\partial_{w_k}\frac{\varphi(|w|)}{|w|}\right|\right)\cdot\|\sum_{1\le|\lambda_n^*|<\mu}\frac{\Psi_n(z+w)\Psi_n^*(x)}{{\lambda_n^*}^2}\|_{L_2}\,dw \\
&\le c\int_{|w|\le R}|w|^{\alpha-4}\sqrt{\sum_{1\le|\lambda_n^*|<\mu}\frac{|\Psi_n(x)|^2}{{\lambda_n^*}^4}}\|_{L_2}\,dw \\
&\le c\int_0^R t^{\alpha-2}\,dt\le c.
\end{aligned}
$$

Thus we proved (26) with $D_{n,1}$ . The other part of $D_n$ is

$$
D_{n,2}(z)=\int_0^R\varphi(r)b_2(r)\,dr=\int_0^R\frac{1}{t}S_t(\tilde{C}\Psi_n)\delta_n(t)\,dt \tag{29}
$$

where

$$
\delta_n(t)=\frac{1}{\lambda_n^*}\int_t^R\varphi(r)\sin\lambda_n^*(r-t)\,dr=\underline{\underline{O}}\left(\frac{1}{{\lambda_n^*}^2t^{2-\alpha}}\right); \tag{30}
$$

it can be checked just like (27) . Hence

$$
\|\sum_{1\le|\lambda_n^*|<\mu}D_{n,2}(z)\Psi_n^*(x)\|_{L_2}\le c
$$

follows by the above showed way. Thus the proof of (26) became complete; since it was the last step in proving Theorem 1, we are ready. □

*Proof of Theorem 2..* Observe first that we can suppose $1<\alpha<2$ . Consider the function $g$ corresponding to $f$ by Lemma 14. Then $g$ must be zero a.e. in the domain $\Omega_{0,R}=\{y\in\Omega_0 : dist(y,\partial\Omega_0)>R\}$ . Let $K\subset\Omega_{0,R}$ be compact; we have to show that $S_\mu f\to f\ (=0)$ uniformly on K. We follow the Steps 2-6. of the proof of Theorem 1. By Steps 2. and 3. we have only to show that

$$
\|\sum_{|\lambda_n^*|<\mu}(C_n\Psi_n(z)+D_n(z))\Psi_n^*(x)\|_{L_2(\text{supp } g)}\le c. \tag{31}
$$

For the summands with $D_n$ Step 6. applies directly. Step 5. can also be repeated with the only difference that we use (23) instead of (22) (and in Lemma 12. we consider $\Omega_{0,R}$ instead of $\Omega_0$) . So the proof of (31) is ready. □

*Proof of Theorem 3..*
*Step 1.* We can suppose $p=2$ . Indeed, if $p>2$ then we can simply diminish $p$ to 2 since by the compactness of $suppf$, $f\in L_p^\alpha\Rightarrow f\in L_2^\alpha$ . If $1<p<2$ , we apply the imbedding theorem (Lemma 15.), namely $L_p^\alpha\subset L_2^\beta$ , where $\beta=\alpha-3\left(\frac{1}{p}-\frac{1}{2}\right)>\frac{3}{2}$ , since $\alpha-\frac{3}{p}>0$ . From now on we suppose $f\in L_2^\alpha,\quad \alpha>\frac{3}{2}$ .

*Step 2.* Let $g$ be the function corresponding to $f$ by Lemma 14. Then the calculations of Steps 2. and 3. of Theorem 1. imply

$$\begin{aligned}\sum_n \Big| \int_\Omega \Psi_n^* f \Big| |\Psi_n(x)| &= \sum_n \Big| \int_\Omega g^*(z) \int_\Omega (v_\alpha^R - w_\alpha^R)(|y-z|)\Psi_n(y)\,dy\,dz \Big| |\Psi_n(x)| \\ &= \sum_n \Big| \int_\Omega g^*(z)(C_n \Psi_n(z) + D_n(z))\,dz \Big| |\Psi_n(x)|.\end{aligned}$$

We have to show that this sum is finite.
*Step 3.* Denote

$$g_n = \int_\Omega g^*(z)\Psi_n(z)\,dz.$$

We have seen in Step 4. of Theorem 1. that

$$C_n = \underline{\underline{O}}\left(\frac{1}{(1+{\lambda_n^*}^2)^{\alpha/2}}\right).$$

Consequently

$$\begin{aligned}&\sum_n \Big| \int_\Omega g^*(z) C_n \Psi_n(z)\,dz \Big| |\Psi_n(x)| = \sum_n |g_n||C_n||\Psi_n(x)| \\ &\le c\|g\|_{L_2} \sqrt{\sum_n \frac{|\Psi_n(x)|^2}{(1+{\lambda_n^*}^2)^\alpha}} \le c\|g\|_{L_2} \sqrt{\sum_{k=1}^\infty k^{-2\alpha} \sum_{k-1 \le |\lambda_n^*| < k} |\Psi_n(x)|^2} \\ &\le c\|g\|_{L_2} \sqrt{\sum_k k^{2-2\alpha}} \le c\|g\|_{L_2}\end{aligned}$$

since $\alpha > \frac{3}{2}$ .
*Step 4.* It remained to prove that

$$\sum_n \Big| \int_\Omega g^*(z) D_n(z)\,dz \Big| |\Psi_n(x)| \le c. \tag{32}$$

As we have seen in the proof of Theorem 1, Step 6,

$$\sum_{|\lambda_n^*|<1} |D_n(z)||\Psi_n(x)| \le c$$

hence

$$\begin{aligned}\sum_{|\lambda_n^*|<1} \Big| \int_\Omega g^*(z) D_n(z)\,dz \Big| |\Psi_n(x)| &\le \int_\Omega |g(z)| \sum_{|\lambda_n^*|<1} |D_n(z)||\Psi_n(x)|\,dz \\ &\le c \int_\Omega |g(z)|\,dz \le c\|g\|_{L_2};\end{aligned}$$

so in (32) we have to sum up only for $n$ with $|\lambda_n^*| \geq 1$ . With the notations of the proof of Theorem 1. consider first $D_{n,1}$ and the remainder term of (27) in it:

$$
\begin{aligned}
&\sum_n \Big| \int_\Omega g^*(z) \int_{|w|\leq R} \frac{1}{|w|}(C+\hat{C})(z+w)\Psi_n(z+w)\delta_n(|w|)\, dw\, dz \Big| |\Psi_n(x)| \\
=&\sum_n \Big| \int_\Omega \frac{\delta_n(|w|)}{|w|} \underbrace{\int_\Omega g^*(z)(C+\hat{C})(z+w)\Psi_n(z+w)\, dz}_{g_n(w)}\, dw \Big| |\Psi_n(x)| \\
\leq &\sum_n \int_\Omega \frac{|\delta_n(|w|)|}{|w|} |g_n(w)|\, dw |\Psi_n(x)| \\
\leq &c \int_\Omega |w|^{\alpha-4} \sqrt{\sum_n |g_n(w)|^2} \sqrt{\sum_n \frac{|\Psi_n(x)|^2}{{\lambda_n^*}^4}}\, dw \\
\leq &c\|g\|_{L_2} \int_\Omega |w|^{\alpha-4} dw \leq c\|g\|_{L_2} \int_0^R t^{\alpha-2}\, dt \leq c\|g\|_{L_2}.
\end{aligned}
$$

Concerning the main part of (27) we apply (28) . Define

$$
\begin{aligned}
g_n(w) = &\int_\Omega g^*(z) \Big[ \frac{\varphi(|w|)}{|w|} \Big( \frac{(C^2+\hat{C}C)(z+w)}{\hbar c_0} - \sum_k \partial_{w_k}(C+\hat{C})(z+w) \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \Big) \\
&- (C+\hat{C})(z+w) \sum \partial_{w_k} \frac{\varphi(|w|)}{|w|} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \Big] \Psi_n(z+w)\, dz;
\end{aligned}
$$

then

$$
\sum_n |g_n(w)|^2 \leq c\|g\|_{L_2} \left( \frac{|\varphi(|w|)|}{|w|} + \sum_k \Big| \partial_{w_k} \frac{\varphi(|w|)}{|w|} \Big| \right)^2 \leq c\|g\|_{L_2} |w|^{2\alpha-8}
$$

and hence

$$
\begin{aligned}
&\sum_{|\lambda_n^*|\geq 1} \Big| \int_\Omega g^*(z) I(z)\, dz \Big| |\Psi_n(x)| \leq \int_{|w|\leq R} \sum_n |g_n(w)| \frac{|\Psi_n(x)|}{{\lambda_n^*}^2}\, dw \\
\leq &\int_{|w|\leq R} \sqrt{\sum_n |g_n(w)|^2} \sqrt{\sum_n \frac{|\Psi_n(x)|^2}{{\lambda_n^*}^4}}\, dw \leq c\|g\|_{L_2} \int_{|w|\leq R} |w|^{\alpha-4} dw \leq c\|g\|_{L_2}.
\end{aligned}
$$

This proves (32) with $D_{n,1}$ instead of $D_n$ . The analogous estimate for $D_{n,2}$ goes by using the formulas (29), (30) as follows:

$$
\begin{aligned}
&\sum_{|\lambda_n^*|>1} \Big| \int_\Omega g^*(z) D_{n,2}(z)\, dz \Big| |\Psi_n(x)| \\
= &\sum_n \Big| \int_{|w|\leq R} \frac{\delta_n(|w|)}{|w|} \underbrace{\int_\Omega g^*(z)\tilde{C}(z+w)\Psi_n(z+w)\, dz}_{g_n(w)}\, dw \Big| |\Psi_n(x)| \\
\leq &\, c \int_{|w|\leq R} |w|^{\alpha-3} \sqrt{\sum_n |g_n(w)|^2} \sqrt{\sum_n \frac{|\Psi_n(x)|^2}{{\lambda_n^*}^4}}\, dw \leq c\|g\|_{L_2} \int_0^R t^{\alpha-1}\, dt \leq c\|g\|_{L_2}.
\end{aligned}
$$

□

*Concluding remarks.* Discussing the conditions given in Theorems 1-3. we mention first that $\alpha p > 3$ is necessary to ensure continuity of the expanded function, see Triebel [18]. In case $\alpha < 1$ Il'in [11] constructed a function with divergent expansion by the eigenfunctions of the Laplace operator. Since in case of zero potentials the eigenfunctions of the Dirac operator are eigenfunctions of the Laplace operator as well (see Horváth [9]), this shows that $\alpha \geq 1$ is necessary. In the present paper the case $\alpha = 1$ is left open; it is neither clear whether we can drop the assumption on the differentiability of the potential.

## References


1. H. Bateman and A. Erdélyi, *Higher transcendental functions, volume 2*, Mc Graw-Hill, New York, 1953.
2. W. D. Evans, *Eigenfunction expansions associated with the Dirac relativistic equations*, Quart. J. Math. Oxford **17** (1966), 211–233.
3. ———, *Eigenfunction expansions associated with the Dirac relativistic equations (II)*, Quart. J. Math. Oxford **18** (1967), 239–262.
4. L. Gårding, *On the asymptotic properties of the spectral function belonging to a self-adjoint semi-bounded extension of an elliptic differential operator*, Kungl. Fisiogr. Sällsk. Lund Förh. **24:21** (1954), 1–18.
5. M. Horváth, *Sur le développement spectral de l'opérateur de Schrödinger*, Comptes Rendus Acad. Sci. Paris, Série I. **311** (1990), 499–502.
6. ———, *Local uniform convergence of the eigenfunction expansion associated with the Laplace operator I.*, Acta Math. Hung. **64** (1994), 1–25.
7. ———, *Local uniform convergence of the eigenfunction expansion associated with the Laplace operator II.*, Acta Math. Hung. **64** (1994), 101–138.
8. ———, *Eigenfunction expansions for one-dimensional Dirac operators*, Acta Sci. Math. (Szeged) **61** (1995), 225–240.
9. ———, *Local uniform convergence of the Riesz means of Laplace and Dirac expansions*, Annales de la Faculté des Sciences de Toulouse (to appear).
10. L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121(3–4)** (1968), 193–218.
11. V. A. Il'in, *Localization and convergence problems for the Fourier series by a fundamental system of functions of the Laplace operator (in Russian)*, Uspechi Mat. Nauk **23:2** (1968), 61–120.
12. I. Joó and V. Komornik, *On the equiconvergence of expansions by Riesz bases formed by eigenfunctions of the Schrödinger operator*, Acta Sci. Math. (Szeged) **46** (1983), 37–375.
13. I. Joó, *On the divergence of eigenfunction expansions*, Annales Univ. Sci. Budap. Sectio Math. **32** (1989), 3–36.
14. ———, *On the convergence of eigenfunction expansions*, Acta Math. Hung. **60(1–2)** (1992), 125–156.
15. S. M. Nikolskii, *Approximation of functions with several variables and imbedding theorems (in Russian)*, Naouka, Moscow, 1977.
16. E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, vol. 2, Clarendon Press, Oxford, 1958.
17. ———, *On the completeness problem for the eigenfunction formulae of relativistic quantum mechanics*, Proc. Royal Soc. A. **262** (1961), 489–502.
18. H. Triebel, *Interpolation theory. Function spaces. Differential operators*, VEB Verlag, Berlin, 1978.



Technical University of Budapest, Faculty of Natural Sciences, Mathematical Institute, Department of Analysis, H-1111 Budapest Stoczek u.2.

*E-mail address*: `horvath@math.bme.hu`