INVERSE SCATTERING WITH FIXED ENERGY AND AN INVERSE EIGENVALUE PROBLEM ON THE HALF-LINE

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Abstract. Recently A. G. Ramm [9] has shown that a subset of phase shifts δ_l, $l = 0, 1, \dots$ determines the potential if the indices of the known shifts satisfy the Müntz condition $\sum_{l\neq 0, l\in L} \frac{1}{l} = \infty$. We prove the necessity of this condition in some classes of potentials. The problem is reduced to an inverse eigenvalue problem for the half-line Schrödinger operators.

1. INTRODUCTION

Consider the inverse scattering problem for the operator

$$\varphi''(r) - \frac{\lambda^2 - 1/4}{r^2}\varphi(r) - q(r)\varphi(r) + k^2\varphi(r) = 0 \quad r \geq 0 \tag{1.1}$$

with fixed energy $k^2 = 1$. Suppose that the potential q satisfies $rq(r) \in L_1(0, \infty)$. It is known [6] that for $\Re\lambda \geq 0$ there exists a solution of (1.1), unique up to a constant multiple, satisfying the boundary conditions

$$\varphi(r, \lambda) = \mathbf{O}(r^{\lambda+1/2}), \qquad r \to 0+, \tag{1.2}$$

$$\varphi(r, \lambda) = A(\lambda) \sin(r - \pi/2 \cdot (\lambda - 1/2) + \delta(\lambda)) + \mathbf{o}(1), \qquad r \to \infty. \tag{1.3}$$

The values $\delta(\lambda) \in \mathbf{C}$ are called phase shifts; they are defined by (1.3) only $\mod \pi$. In quantum mechanics most relevant are the shifts

$$\delta_l = \delta(l + 1/2) \qquad l = 0, 1, 2, \dots. \tag{1.4}$$

Concerning the recovery of the potential $q(r)$ by a set of phase shifts $\delta(\lambda_n)$ we mention the following recent result of Ramm:

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Theorem 1.1. [9] *Suppose that* $q(r) = 0$ *for* $r > a$ *and* $rq(r) \in L_2(0, a)$. *Let* $L \subset \mathbf{N}$ *and suppose that the Müntz condition*

$$\sum_{l \neq 0, l \in L} \frac{1}{l} = \infty \tag{1.5}$$

is valid. Then the data δ_l, $l \in L$ *uniquely determine the potential* $q(r)$.

We show that here L_2 can be substituted by L_1 and that the condition (1.5) is "almost" necessary:

Theorem 1.2. *Let* $q(r) = 0$ *for* $r > a$ *and* $rq(r) \in L_1(0, a)$. *Consider arbitrary different complex numbers* λ_n *with* $\Re\lambda_n \geq \gamma > 0$. *If*

$$\sum \frac{\Re\lambda_n}{|\lambda_n|^2} = \infty \tag{1.6}$$

then the potential $q(r)$ *is uniquely determined by the shifts* $\delta(\lambda_n)$.

Theorem 1.3. *Let* $0 < \sigma < 2$ *be arbitrarily fixed and consider the class*

$$B_\sigma = \{q(r) : q(r) = 0 \, for \, r > a, \, r^{1-\sigma} q(r) \in L_1(0, a)\}.$$

Finally let $\lambda_n \in \mathbf{C}$ *be arbitrary numbers with* $\Re\lambda_n \geq \gamma > 0$. *Now if*

$$\sum \frac{\Re\lambda_n}{|\lambda_n|^2} < \infty \tag{1.7}$$

then the shifts $\delta(\lambda_n)$ *are not enough to recover the potential in* B_σ; *in other words for every* $q \in B_\sigma$ *there exists a different* $q^* \in B_\sigma$ *such that* $\delta(q^*, \lambda_n) = \delta(q, \lambda_n)$ $\forall n$.

Remark. If the numbers λ_n are of the form $l + 1/2$ then (1.5) and (1.6) are equivalent. Thus Theorem 1.2 extends Theorem 1.1.

Remark. The potentials of Theorem 1.1 belong to B_σ for every $0 < \sigma < 1/2$. Hence if (1.5) does not hold then there exists another potential $q^* \in B_\sigma$ with the same data δ_l, $l \in L$ as for q. Whether there exists a potential q^* with $rq^*(r) \in L_2(0, a)$ and $\delta_l(q^*) = \delta_l(q)$, $l \in L$ is an open question. Analogous L_p-problems are neither investigated here.

Next we consider the inverse eigenvalue problem for the Schrödinger operators

$$-y" + Q(x)y = \lambda^2 y \quad x \geq 0. \tag{1.8}$$

It is known that for $Q \in L_1(0, \infty)$ the operator is limit point at infinity and the (essentially unique) L_2-solution satisfies the asymptotical formula

$$y(x, \lambda) = c(\lambda) e^{i\lambda x}(1 + \mathbf{o}(1)) \quad x \to \infty, \quad \Im\lambda > 0, \tag{1.9}$$

see Theorem 2.1 below. Consider the boundary condition

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0 \tag{1.10}$$

for some $0 \le \alpha < \pi$. The values λ^2 for which the system (1.8), (1.10) has a nontrivial L_2-solution are called eigenvalues of the Schrödinger operator (1.8) with boundary condition (1.10). It is known that the eigenvalues are negative. We apply the notation

$$\lambda^2 \in \sigma(Q,\alpha)$$

for the eigenvalues of (1.8), (1.10).

We say that the values $\lambda_n^2 < 0$ are *common eigenvalues* of the potentials Q^* and Q if there exist numbers $0 \le \alpha_n < \pi$ with

$$\lambda_n^2 \in \sigma(Q^*,\alpha_n) \cap \sigma(Q,\alpha_n) \quad \forall n. \tag{1.11}$$

In other words, the boundary conditions can be different for every eigenvalue λ_n^2.

Remark. By the above setting every negative value $\lambda_n^2 < 0$ can be considered as an "eigenvalue" of Q if we define α_n correspondingly. However Q and λ_n^2 define α_n and hence (1.11) contains real information, namely that the parameter α_n is the same for Q^* and Q. This idea is useful since it is intimately connected with the problem of recovering the potential from phase shifts, see at the end of Section 3. below.

Theorem 1.4. *Let $Q \in L_1(0,\infty)$ and consider the different numbers $\lambda_n = ik_n$, $\inf k_n > 0$. If*

$$\sum \frac{1}{k_n} = \infty \tag{1.12}$$

then the eigenvalues λ_n^2 determine Q, i.e. there are no other potentials $Q^* \in L_1(0,\infty)$ such that the λ_n^2 are common eigenvalues of Q^* and Q in the above-defined sense.

Theorem 1.5. *For $0 < \delta$ define*

$$C_\delta = \{Q : \|Q\|_\delta = \int_0^\infty |Q(x)|e^{\delta x}dx < \infty\}.$$

Consider the numbers $\lambda_n = ik_n$, $\inf k_n > 0$. If

$$\sum \frac{1}{k_n} < \infty \tag{1.13}$$

then the eigenvalues $\lambda_n^2 = -k_n^2$ do not determine the potential in C_δ, i.e. for every $Q \in C_\delta$ there exists $Q^* \in C_\delta$, $Q^* \ne Q$ such that the λ_n^2 are common eigenvalues of Q^* and Q.

Remark. As we shall check by a Liouville transformation, the statements in Theorems 1.4 and 1.5 are special cases of Theorems 1.2 and 1.3 respectively.

Remark. Condition (1.6) is necessary and sufficient for the system $\{r^{\lambda_n}\}$ to be closed in $L_1(0,a)$. Analogously (1.12) holds if and only if $\{e^{-k_n x}\}$ is closed in $L_1(0,\infty)$, see Lemma 3.2 below.

The statements corresponding to Theorems 1.4 and 1.5 for Schrödinger operators over a finite interval has been formerly obtained by the author. Namely define $\sigma(Q,\alpha)$ as the spectrum of the operator

$$-y'' + Q(x)y = \lambda^2 y \quad x \in [0,\pi], \tag{1.14}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0,\ \ y(\pi) = 0. \tag{1.15}$$

We say that the values λ_n^2 are common eigenvalues of the potentials Q^* and Q if there are numbers $0 \le \alpha_n < \pi$ satisfying

$$\lambda_n^2 \in \sigma(Q^*,\alpha_n) \cap \sigma(Q,\alpha_n). \tag{1.16}$$

The potential $Q \in L_1(0,\pi)$ is said to be determined by the eigenvalues $\lambda_n^2 \in \mathbf{R}$ if there is no other potential $Q^* \in L_1(0,\pi)$ satisfying (1.16) for all n.

Theorem 1.6. [5] *Let λ_n be arbitrary real numbers satisfying $\lambda_n \not\to -\infty$. The potential $Q \in L_1(0,\pi)$ is defined by the eigenvalues λ_n^2 if and only if the system $e(\Lambda) = \{e^{\pm 2i\mu x}, e^{\pm 2i\lambda_n x} : n \ge 1\}$ is closed in $L_1(0,2\pi)$; here $\mu \ne \pm\lambda_n$ is an arbitrary number.*

Let $Q \in L_1^{\text{loc}}(0,\infty)$ be a potential which is limit point at infinity and for $\Im\lambda > 0$ let $y_1(x,\lambda)$ be the (essentially unique) solution of $-y'' + Qy = \lambda^2 y$, $0 \ne y \in L_2(0,\infty)$. Define the *m-function* of Q by

$$m(\lambda) = \frac{y_1'(0,\lambda)}{y_1(0,\lambda)} \quad \Im\lambda > 0.$$

As it is well-known, the m-function defines the potential:

Theorem 1.7. *Borg* [1], *Marchenko* [8] *If the m-functions of Q and Q^* are the same, $m(Q^*,\lambda) \equiv m(Q,\lambda)$ then $Q^* = Q$ a.e.*

Using the notion of m-functions we can generalize Theorems 1.4 and 1.5 a bit further in the following form:

Theorem 1.8. *Let λ_n be arbitrary different complex numbers with $\Im\lambda_n \ge \gamma > 0$ and suppose that the m-function $m(\lambda)$ is generated by a potential $Q \in L_1(0,\infty)$. If*

$$\sum \frac{\Im\lambda_n}{|\lambda_n|^2} = \infty \tag{1.17}$$

then the values $m(\lambda_n)$ define the m-function (and the potential). In other words, if $m(Q^*, \lambda_n) = m(Q, \lambda_n)\,\forall n$ (allowing that both sides be infinite) then $m(Q^*, \lambda) \equiv m(Q, \lambda)$ and $Q^* = Q$ a.e.

Theorem 1.9. *Let $\delta > 0$ and consider different numbers λ_n, $\Im\lambda_n \geq \gamma > 0$. The values $m(\lambda_n)$ of the m-function generated by $Q \in C_\delta$ defines $m(\lambda)$ if and only if (1.17) holds. In other words, if (1.17) is not valid then for every $Q \in C_\delta$ there exists a different $Q^* \in C_\delta$ satisfying $m(Q^*, \lambda_n) = m(Q, \lambda_n)\,\forall n$.*

For related results on finite intervals see [5].

Remark. Theorems 1.8 and 1.2, respectively 1.9 and 1.3 are identical, see (3.23) below.

In the proofs below we borrow some ideas and methods from [5]. Section 2. is devoted to collect the preliminary material needed while Section 3. contains the proofs of Theorems 1.2 to 1.9.

2. Preliminaries

We first recall the following known result:

Theorem 2.1. *Berezin, Shubin* [3] *Let $Q \in L_1(0, \infty)$ and $\lambda \in \mathbf{C} \setminus \{0\}$. Then the equation $-y'' + Qy = \lambda^2 y$ has two solutions*

$$y_1(x) = e^{i\lambda x}(1 + \mathbf{o}(1)), \quad y_2(x) = e^{-i\lambda x}(1 + \mathbf{o}(1)) \quad x \to \infty. \tag{2.1}$$

If $\Im\lambda \geq 0$ then the solution y_1 "regular at infinity" satisfies the integral equation

$$y_1(x) = e^{i\lambda x} + \int_x^\infty \frac{\sin\lambda(t-x)}{\lambda} Q(t) y_1(t)\, dt. \tag{2.2}$$

The function $y_1(x) = y_1(x, \lambda)$ is holomorphic in $\lambda \in \mathbf{C}^+$.

Consider two potentials Q^*, $Q \in L_1(0, \infty)$ and denote $y_1^*(x, \lambda)$, $y_1(x, \lambda)$ the corresponding y_1-solutions. Define the functions

$$F(x, \lambda) = y_1'(x, \lambda) y_1^*(x, \lambda) - y_1^{*\prime}(x, \lambda) y_1(x, \lambda), \tag{2.3}$$

$$F(\lambda) = F(0, \lambda). \tag{2.4}$$

By (1.11) the common eigenvalues of Q^* and Q are precisely the values $-k^2$, $k > 0$ where $F(ik) = 0$. The analogous functions in case of finite intervals are used e.g. in Gesztesy and Simon [4].

Lemma 2.2.

$$F(\lambda) = \int_0^\infty (Q^*(x) - Q(x)) y_1^*(x, \lambda) y_1(x, \lambda)\, dx \text{ for } \Im\lambda \geq 0,\ \lambda \neq 0. \tag{2.5}$$

Proof. (2.2) implies

$$y_1'(x,\lambda) = i\lambda e^{i\lambda x} - \int_x^\infty \cos\lambda(t-x)Q(t)y_1(t,\lambda)\,dt \tag{2.6}$$

and here the integral can be estimated by

$$\int_x^\infty \mathbf{O}\left(e^{\Im\lambda(t-x)}|Q(t)|e^{-\Im\lambda t}\right)dt = e^{-\Im\lambda x}\mathbf{O}\left(\int_x^\infty |Q|\right).$$

Consequently (for fixed λ)

$$y_1'(x,\lambda) = i\lambda e^{i\lambda x}(1+\mathbf{o}(1)), \quad x\to\infty. \tag{2.7}$$

Comparing this with (2.1) and (2.3) gives

$$F(x,\lambda)\to 0 \quad x\to\infty,\ \lambda \text{ fixed },\ \Im\lambda\ge 0,\ \lambda\ne 0. \tag{2.8}$$

Now we have

$$\begin{aligned} F'(x,\lambda) &= y_1''(x,\lambda)y_1^*(x,\lambda) - y_1(x,\lambda)y_1^{*\prime\prime}(x,\lambda) = \\ &= (Q(x)-Q^*(x))y_1^*(x,\lambda)y_1(x,\lambda) \end{aligned}$$

whence

$$F(\lambda) - F(N,\lambda) = \int_0^N (Q^*(x)-Q(x))y_1^*(x,\lambda)y_1(x,\lambda)\,dx.$$

Taking the limit $N\to\infty$ gives (2.5). □

Lemma 2.3. *$F(\lambda)$ is bounded in every half-plane $\{\lambda : \Im\lambda \ge \gamma > 0\}$.*

Proof. The function

$$z(x,\lambda) = y_1(x,\lambda)e^{-i\lambda x} = 1 + \mathbf{o}(1)$$

is bounded for $x\ge 0$ by a bound depending on λ. From

$$z(x,\lambda) = 1 + \int_x^\infty \frac{e^{2i\lambda(t-x)}-1}{2i\lambda}Q(t)z(t,\lambda)\,dt$$

we obtain

$$|z(x,\lambda)| \le 1 + \frac{\|z\|_\infty}{|\lambda|}\int_x^\infty |Q|$$

i.e.

$$\|z\|_\infty \le 1 + \frac{\int_0^\infty |Q|}{|\lambda|}\|z\|_\infty.$$

This means that $\|z\|_\infty \le 2$ if $|\lambda| \ge 2\int_0^\infty |Q|$, or

$$|y_1(x,\lambda)| \le 2e^{-\Im\lambda x} \text{ if } |\lambda| \ge 2\int_0^\infty |Q|,\ \Im\lambda\ge 0. \tag{2.9}$$

Consequently from (2.5)

$$|F(\lambda)| \le 4\int_0^\infty |Q^* - Q| \text{ if } |\lambda| \ge 2\max\left(\int_0^\infty |Q|, \int_0^\infty |Q|\right), \ \Im\lambda \ge 0.$$

This estimate with the continuity of F shows its boundedness on $\Im\lambda \ge \gamma$. ☐

From now on we consider more rapidly decaying potentials satisfying

$$\int_0^\infty x|Q(x)|\,dx < \infty. \tag{2.10}$$

This means that the function

$$\sigma(x) = \int_x^\infty |Q|$$

belongs to $L_1(0,\infty)$; indeed,

$$\int_0^\infty \sigma(x)\,dx = \int_0^\infty \int_x^\infty |Q(t)|\,dt\,dx = \int_0^\infty t|Q(t)|\,dt.$$

Define

$$\sigma_1(x) = \int_x^\infty \sigma.$$

We need the following well-known facts:

Lemma 2.4. *Marchenko* [7], *Chapter III. Suppose (2.10). Then there exists a function $K(x,t)$, continuous for $0 \le x \le t < \infty$, satisfying the properties*

$$y_1(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt \quad \textit{for} \quad x \ge 0,\, \Im\lambda \ge 0, \tag{2.11}$$

$$K(x,x) = 1/2\int_x^\infty Q, \tag{2.12}$$

$$|K(x,t)| \le 1/2\sigma\left(\frac{x+t}{2}\right)e^{\sigma_1(x)-\sigma_1(\frac{x+t}{2})}. \tag{2.13}$$

Define further

$$H(u,v) = K(u-v, u+v) \quad 0 \le v \le u < \infty. \tag{2.14}$$

Then

$$H(u,v) = \sum_{n=0}^\infty H_n(u,v) \tag{2.15}$$

where

$$H_0(u,v) = 1/2 \int_u^\infty Q, \tag{2.16}$$

$$H_n(u,v) = \int_u^\infty \int_0^v Q(\alpha-\beta) H_{n-1}(\alpha,\beta)\, d\beta\, d\alpha, \quad n \ge 1. \tag{2.17}$$

Finally we have

$$|H_n(u,v)| \le 1/2\sigma(u) \frac{[\sigma_1(u-v) - \sigma_1(v)]^n}{n!} \quad n \ge 0. \tag{2.18}$$

Let Q, $Q^* \in C_\delta$ (as in Theorem 1.5) and define the kernels $K(x,t)$, $K^*(x,t)$ corresponding to Lemma 2.4. Introduce the third kernel (2.19)

$$K_1(x,t,Q,Q^*) = 2K(t,2x-t) + 2K^*(t,2x-t) + \\ + 2\int_t^{2x-t} K(t,u)K^*(t,2x-u)\, du, \quad 0 \le t \le x < \infty$$

and the corresponding Volterra type integral operator

$$A_{Q^*} : C_\delta \to C_\delta$$

$$(A_{Q^*}h)(x) = h(x) + \int_0^x K_1(x,t,Q,Q^*) h(t)\, dt.$$

Its relevance is justified by the following formula:

Lemma 2.5.

$$F(\lambda) = \int_0^\infty e^{2i\lambda x} \left[A_{Q^*}(Q^* - Q)\right](x)\, dx. \tag{2.20}$$

Proof. It is a simple calculation by substituting (2.11) into (2.5):

$$F(\lambda) = \int_0^\infty (Q^*(x) - Q(x)) \left[e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t} dt\right] \cdot \\ \cdot \left[e^{i\lambda x} + \int_x^\infty K^*(x,t)e^{i\lambda t} dt\right] dx.$$

By interchanging the order of integrations we get

$$\int_0^\infty (Q^*(x) - Q(x)) \int_x^\infty K(x,t)e^{i\lambda(x+t)} dt\, dx = \\ 2\int_0^\infty (Q^*(x) - Q(x)) \int_x^\infty K(x,2\tau - x)e^{2i\lambda\tau} d\tau\, dx = \\ 2\int_0^\infty e^{2i\lambda\tau} \int_0^\tau (Q^*(x) - Q(x)) K(x,2\tau - x)\, dx\, d\tau$$

and

$$\int_0^\infty (Q^*(x)-Q(x))\int_x^\infty K(x,t)\int_x^\infty K^*(x,\tau)e^{i\lambda(t+\tau)}d\tau\,dt\,dx =$$
$$2\int_0^\infty (Q^*(x)-Q(x))\int_x^\infty K(x,t)\int_{\frac{x+t}{2}}^\infty K^*(x,2u-t)e^{2i\lambda u}du\,dt\,dx =$$
$$2\int_0^\infty (Q^*(x)-Q(x))\int_x^\infty e^{2i\lambda u}\int_x^{2u-x} K(x,t)K^*(x,2u-t)\,dt\,du\,dx =$$
$$2\int_0^\infty e^{2i\lambda u}\int_0^u (Q^*(x)-Q(x))\int_x^{2u-x} K(x,t)K^*(x,2u-t)\,dt\,dx\,du.$$

We used the fact that $\int_x^\infty |K(x,t)|\,dt$ is bounded in x, see (2.13). Finally we get

$$\begin{aligned}F(\lambda) = \int_0^\infty e^{2i\lambda x}\Big[Q^*(x)-Q(x)+2\int_0^x (Q^*(t)-Q(t))K(t,2x-t)\,dt \\ +2\int_0^x (Q^*(t)-Q(t))K^*(t,2x-t)\,dt \\ +2\int_0^x (Q^*(t)-Q(t))\int_t^{2x-t} K(t,u)K^*(t,2x-u)\,du\,dt\Big]\,dx\end{aligned}$$

which is (2.20). □

Define the kernel

$$\tilde{K}(x,t,Q,Q^*) = e^{\delta(x-t)}K_1(x,t,Q,Q^*)$$

and the corresponding Volterra operator

$$\tilde{K} : L_1(0,\infty) \to L_1(0,\infty),$$

$$(\tilde{K}h)(x) = \int_0^x \tilde{K}(x,t,Q,Q^*)h(t)\,dt.$$

Lemma 2.6. *Let $Q, Q^* \in C_\delta$ for some $\delta > 0$. Then*
a) $\tilde{K} : L_1(0,\infty) \to L_1(0,\infty)$ is compact,
b) $A_{Q^*} : C_\delta \to C_\delta$ is an (onto) isomorphism.

Proof. From

$$\begin{aligned}\sigma_1(x) = \int_x^\infty\int_t^\infty |Q(\tau)|\,d\tau\,dt \le \int_x^\infty e^{-\delta t}\int_t^\infty e^{\delta\tau}|Q(\tau)|\,d\tau\,dt \\ \le \|Q\|_\delta \int_x^\infty e^{-\delta t}dt = \frac{1}{\delta}e^{-\delta x}\|Q\|_\delta\end{aligned}$$

and from (2.13) we get

$$|K(x,t)| \le c(D,\delta)\int_{\frac{x+t}{2}}^{\infty}|Q| \quad \text{i}f \quad \|Q\|_\delta \le D. \tag{2.21}$$

Consequently

$$|K_1(x,t,Q,Q^*)| \le c(D,\delta)\Bigg[\int_x^{\infty}(|Q^*|+|Q|) \\ + \int_t^{2x-t}\left(\int_{\frac{t+u}{2}}^{\infty}|Q|\right)\left(\int_{x+\frac{t-u}{2}}^{\infty}|Q^*|\right)du\Bigg] \quad \text{if} \quad \|Q^*\|_\delta, \|Q\|_\delta \le D \tag{2.22}$$

and a corresponding bound can be given for $\tilde{K}$ after the multiplication by $e^{\delta(x-t)}$.

The operator-norm of $\tilde{K}$ has the bound

$$\|\tilde{K}\|_{1,1} \le \sup_t \int_t^{\infty}|\tilde{K}(x,t,Q,Q^*)|\,dx \tag{2.23}$$

as it is seen from

$$\int_0^{\infty}\left|\int_0^x \tilde{K}(x,t)h(t)\,dt\right|dx \le \int_0^{\infty}|h(t)|\int_t^{\infty}|\tilde{K}(x,t)|\,dx\,dt$$

(the dependence of $\tilde{K}$ on Q and Q^* is not indicated).

In proving a) we will approximate $\tilde{K}$ in operator norm by operators of finite dimensional range. Let first

$$\tilde{K}_N(x,t) = \begin{cases} \tilde{K}(x,t) & \text{for} \quad x \le N \\ 0 & \text{for} \quad x > N. \end{cases}$$

We will check that

$$\|\tilde{K} - \tilde{K}_N\|_{1,1} \to 0 \quad \text{if} \quad N \to \infty. \tag{2.24}$$

Indeed,

$$\sup_t \int_t^{\infty}\left|\tilde{K}(x,t) - \tilde{K}_N(x,t)\right|dx = \max(I_1, I_2)$$

where

$$I_1 = \sup_{0\le t\le N}\int_N^{\infty}|\tilde{K}(x,t)|\,dx,$$
$$I_2 = \sup_{t>N}\int_t^{\infty}|\tilde{K}(x,t)|\,dx.$$

In I_1 we have by (2.22)

$$\int_N^\infty e^{\delta(x-t)} \int_x^\infty |Q(\tau)|\,d\tau\,dx = \int_N^\infty |Q(\tau)| \int_N^\tau e^{\delta(x-t)} dx\,d\tau \tag{2.25}$$
$$\leq 1/\delta e^{-\delta t} \int_N^\infty |Q(\tau)| e^{\delta\tau} d\tau \to 0 \quad \text{if} \quad N \to \infty$$

and

(2.26)
$$\int_N^\infty e^{\delta(x-t)} \int_t^{2x-t} \left(\int_{\frac{t+u}{2}}^\infty |Q|\right)\left(\int_{x+\frac{t-u}{2}}^\infty |Q^*|\right) du\,dx$$
$$= \int_t^\infty \left(\int_{\frac{t+u}{2}}^\infty |Q|\right) \int_{\max(N,\frac{u+t}{2})}^\infty e^{\delta(x-t)} \int_{x+\frac{t-u}{2}}^\infty |Q^*(\tau)|\,d\tau\,dx\,du$$
$$= \int_t^\infty \left(\int_{\frac{t+u}{2}}^\infty |Q|\right) \int_{\max(N+\frac{t-u}{2},t)}^\infty |Q^*(\tau)| \int_{\max(N,\frac{t+u}{2})}^{\tau+\frac{u-t}{2}} e^{\delta(x-t)} dx\,d\tau\,du$$
$$\leq 1/\delta \int_t^\infty \left(\int_{\frac{t+u}{2}}^\infty |Q|\right) \int_{\max(N+\frac{t-u}{2},t)}^\infty |Q^*(\tau)| e^{\delta\tau} d\tau \cdot e^{-3/2\delta t} e^{\delta u/2} du$$
$$= 1/\delta \int_t^\infty |Q(\nu)| \int_t^{2\nu-t} e^{\delta u/2} \int_{\max(N+\frac{t-u}{2},t)}^\infty |Q^*(\tau)| e^{\delta\tau} d\tau\,du\,d\nu \cdot e^{-3/2\delta t}$$
$$\leq 1/\delta e^{-3/2\delta t} \int_t^\infty |Q(\nu)| \int_{\max(N+t-\nu,t)}^\infty |Q^*(\tau)| e^{\delta\tau} d\tau \cdot \int_t^{2\nu-t} e^{\delta u/2} du\,d\nu$$
$$\leq 2/\delta^2 e^{-2\delta t} \int_t^\infty |Q(\nu)| e^{\delta\nu} \int_{\max(N+t-\nu,t)}^\infty |Q^*(\tau)| e^{\delta\tau} d\tau\,d\nu.$$

This expression is small (uniformly in t), if N is large. Indeed, if $\max(N+t-\nu,t) \geq N/2$ then the inner integral is small; if $\max(N+t-\nu,t) < N/2$ then $\nu > N/2$ and on this domain the outer integral is small. This verifies that $I_1 \to 0$ if $N \to \infty$. In I_2 we can apply similar manipulations (with t instead of N), namely for $N \to \infty$

$$\int_t^\infty e^{\delta(x-t)} \left(\int_x^\infty |Q|\right) dx \leq 1/\delta e^{-\delta t} \int_t^\infty |Q(\tau)| e^{\delta\tau} d\tau$$
$$\leq 1/\delta e^{-\delta N} \|Q\|_\delta \to 0,$$

$$\int_t^\infty e^{\delta(x-t)} \int_t^{2x-t} \left(\int_{\frac{t+u}{2}}^\infty |Q| \right) \left(\int_{x+\frac{t-u}{2}}^\infty |Q^*| \right) du\, dx$$
$$\leq 2/\delta^2 e^{-2\delta t} \int_t^\infty |Q(\nu)| e^{\delta\nu} \int_t^\infty |Q^*(\tau)| e^{\delta\tau} d\tau\, d\nu$$
$$\leq 2/\delta^2 e^{-2\delta N} \|Q\|_\delta \|Q^*\|_\delta \to \infty.$$

Thus (2.24) is verified.

We extend the kernel $\tilde{K}_N$ by zero to $[0,N]^2$ and (for fixed N and large M) divide $[0,N]^2$ into M^2 small squares of edge length N/M. In every small squares define $\tilde{K}_{N,M}$ to be constant, namely the value of $\tilde{K}_N$ at the left upper vertex. Since $\tilde{K}_N$ is continuous on the compact set $0 \leq t \leq x \leq N$, $\tilde{K}_N - \tilde{K}_{N,M}$ is uniformly small for large M, except for the small squares along the diagonal $x = t$. In these exceptional squares $\tilde{K}_N - \tilde{K}_{N,M} = \tilde{K}_N$ is bounded (by a bound depending on N). Consequently

$$\sup_{t\leq N} \int_t^\infty |\tilde{K}_N(x,t) - \tilde{K}_{N,M}(x,t)|\, dx \to 0 \quad M \to \infty.$$

With (2.24) this means that $\tilde{K}$ can be approximated by the $\tilde{K}_{N,M}$ in operator norm. Since the $\tilde{K}_{N,M}$ have finite dimensional range, $\tilde{K}$ is compact, so the statement a) is proved. The point b) is a corollary of a). Indeed, a) implies the compactness of the integral operator with kernel K_1. We know $A_{Q^*} = I + K_1$. If A_{Q^*} is not an isomorphism then -1 must be an eigenvalue of K_1 i.e. there exists $0 \neq h \in C_\delta$ such that $-h = K_1 h$. But this is impossible for a Volterra operator with continuous kernel. So A_{Q^*} is isomorphism as asserted. □

Lemma 2.7. *Let $Q, Q^*, Q^{**} \in C_\delta$ for some $\delta > 0$ and suppose that $\|Q\|_\delta, \|Q^*\|_\delta, \|Q^{**}\|_\delta \leq D$. Then*

$$\|(A_{Q^{**}} - A_{Q^*})h\|_\delta \leq c(D,\delta)\|Q^{**} - Q^*\|_\delta \|h\|_\delta \quad \forall\ h \in C_\delta. \tag{2.27}$$

The constant $c(D,\delta)$ depends only on its arguments.

Proof. Consider the functions $H_n(u,v)$ defined in Lemma 2.4. Denote

$$\varrho(u,v) = \sigma_1(u-v) - \sigma_1(v) \quad 0 \leq v \leq u;$$

it is increasing in v for fixed u and decreasing in u for fixed v since

$$\frac{\partial\varrho}{\partial u} = -\int_{u-v}^u |Q| < 0.$$

Analogously define ϱ^* with Q^* instead of Q. Finally let

$$\tilde\sigma(u)=\int_u^\infty |Q^*-Q|,\quad \tilde\sigma_1(u)=\int_u^\infty \tilde\sigma,\quad \tilde\varrho(u,v)=\tilde\sigma_1(u-v)-\tilde\sigma_1(u).$$

The proof of (2.27) is based on the estimate

$$\begin{aligned}(2.28)\quad |H_n^*(u,v)-H_n(u,v)| &\le 1/2\sigma^*(u)\tilde\varrho(u,v)\frac{[\varrho(u,v)+\varrho^*(u,v)]^{n-1}}{(n-1)!}\\ &+1/2\tilde\sigma(u)\frac{\varrho(u,v)^n}{n!};\end{aligned}$$

for $n=0$ only the second summand is considered on the right. We apply induction on n. For $n=0$

$$|H_0^*(u,v)-H_0(u,v)|=1/2\Big|\int_u^\infty (Q^*-Q)\Big|\le 1/2\tilde\sigma(u).$$

Suppose (2.28) for a value of n. Then

$$\begin{aligned}(2.29)\quad & H_{n+1}^*(u,v)-H_{n+1}(u,v)=\\ &=\int_u^\infty\int_0^v (Q^*(\alpha-\beta)-Q(\alpha-\beta))\,H_n^*(\alpha,\beta)\,d\beta\,d\alpha\\ &+\int_u^\infty\int_0^v Q(\alpha-\beta)\,(H_n^*(\alpha,\beta)-H_n(\alpha,\beta))\,d\beta\,d\alpha \overset{def}{=} I_1+I_2.\end{aligned}$$

We use the identity

$$\begin{aligned}(2.30)\qquad &\int_u^\infty\int_0^v \big|Q^*(\alpha-\beta)-Q(\alpha-\beta)\big|\,d\beta\,d\alpha\\ &=\int_u^\infty (\tilde\sigma(\alpha-v)-\tilde\sigma(\alpha))\,d\alpha=\tilde\varrho(u,v)\end{aligned}$$

and (2.18) to obtain

$$\begin{aligned}(2.31)\quad |I_1| &\le 1/2\int_u^\infty\int_0^v \big|Q^*(\alpha-\beta)-Q(\alpha-\beta)\big|\sigma^*(\alpha)\frac{\varrho^*(\alpha,\beta)^n}{n!}\,d\beta\,d\alpha\\ &\le 1/2\sigma^*(u)\frac{\varrho^*(u,v)^n}{n!}\int_u^\infty\int_0^v \big|Q^*(\alpha-\beta)-Q(\alpha-\beta)\big|\,d\beta\,d\alpha\\ &=1/2\sigma^*(u)\tilde\varrho(u,v)\frac{\varrho^*(u,v)^n}{n!}.\end{aligned}$$

On the other hand

$$|I_2| \le 1/2 \int_u^\infty \int_0^v |Q(\alpha-\beta)| \cdot \tilde\sigma(\alpha) \frac{\varrho(\alpha,\beta)^n}{n!} d\beta\, d\alpha \tag{2.32}$$

$$+1/2 \int_u^\infty \int_0^v |Q(\alpha-\beta)| \cdot \sigma^*(\alpha) \tilde\varrho(\alpha,\beta) \frac{[\varrho(\alpha,\beta)+\varrho^*(\alpha,\beta)]^{n-1}}{(n-1)!} d\beta\, d\alpha$$

$$\stackrel{def}{=} I_{21} + I_{22}.$$

Apply the identity

$$\int_0^v |Q(\alpha-\beta)|\, d\beta = -\frac{\partial\varrho}{\partial\alpha}(\alpha, v)$$

to get

$$\begin{aligned} I_{21} &\le 1/2\tilde\sigma(u) \int_u^\infty \frac{\varrho(\alpha,v)^n}{n!} \int_0^v |Q(\alpha-\beta)|\, d\beta\, d\alpha \\ &= 1/2\tilde\sigma(u) \frac{\varrho(u,v)^{n+1}}{(n+1)!}, \end{aligned} \tag{2.33}$$

$$\begin{aligned} I_{22} &\le 1/2\sigma^*(u)\tilde\varrho(u,v) \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{(n-1)!} \varrho^*(u,v)^{n-1-k} \cdot \\ &\cdot \int_u^\infty \varrho(\alpha,v)^k \int_0^v |Q(\alpha-\beta)|\, d\beta\, d\alpha \\ &= 1/2\sigma^*(u)\tilde\varrho(u,v) \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{(n-1)!} \frac{1}{k+1} \varrho^*(u,v)^{n-1-k} \varrho(u,v)^{k+1} \\ &= 1/2\sigma^*(u)\tilde\varrho(u,v) \sum_{k=1}^{n} \frac{\binom{n}{k}}{n!} \varrho^*(u,v)^{n-k} \varrho(u,v)^k. \end{aligned} \tag{2.34}$$

Summing up (2.31)-(2.34) gives

$$\begin{aligned} |H^*_{n+1}(u,v) - H_{n+1}(u,v)| &\le 1/2\sigma^*(u)\tilde\varrho(u,v) \frac{[\varrho(u,v)+\varrho^*(u,v)]^n}{n!} \\ &+ 1/2\tilde\sigma(u) \frac{\varrho(u,v)^{n+1}}{(n+1)!} \end{aligned}$$

which verifies (2.28) for $n+1$. Consequently

$$|H^*(u,v) - H(u,v)| \le 1/2\sigma^*(u)\tilde\varrho(u,v) e^{\varrho(u,v)+\varrho^*(u,v)} + 1/2\tilde\sigma(u) e^{\varrho(u,v)}. \tag{2.35}$$

Since we have

$$\begin{aligned}\tilde{\varrho}(u,v) &= \tilde{\sigma}_1(u-v) - \tilde{\sigma}_1(u) \le \tilde{\sigma}_1(0)\\ &= \int_0^\infty t|Q^*(t)-Q(t)|\,dt \le c(\delta)\|Q^*-Q\|_\delta,\end{aligned}$$

the estimate (2.35) can be continued as follows:

$$|H^*(u,v)-H(u,v)| \le c(D,\delta)\left[\int_u^\infty |Q^*-Q| + \|Q^*-Q\|_\delta \cdot \int_u^\infty |Q^*|\right] \tag{2.36}$$

or, from (2.14)

$$\begin{aligned}&|K^*(x,t)-K(x,t)| \le\\ &\le c(D,\delta)\left[\int_{\frac{x+t}{2}}^\infty |Q^*-Q| + \|Q^*-Q\|_\delta \cdot \int_{\frac{x+t}{2}}^\infty |Q^*|\right].\end{aligned} \tag{2.37}$$

Now we are able to prove the estimate

$$\begin{aligned}&\int_t^\infty e^{\delta x}\big|K_1(x,t,Q,Q^{**}) - K_1(x,t,Q,Q^*)\big|\,dx\\ &\qquad \le C(D,\delta)\|Q^{**}-Q^*\|_\delta \quad \text{for all} \quad t \ge 0.\end{aligned} \tag{2.38}$$

This implies the Lipschitz property (2.27) because

$$\|A_{Q^{**}} - A_{Q^*}\| \le \sup_{t\ge 0}\int_t^\infty e^{\delta(x-t)}|K_1(x,t,Q,Q^{**}) - K_1(x,t,Q,Q^*)|\,dx.$$

Now use the decomposition (2.19) in

$$\begin{aligned}&\int_t^\infty e^{\delta x}|K_1(x,t,Q,Q^{**}) - K_1(x,t,Q,Q^*)|\,dx\\ &\le 2\int_t^\infty e^{\delta x}|K^{**}(t,2x-t) - K^*(t,2x-t)|\,dx\\ &+2\int_t^\infty e^{\delta x}\int_t^{2x-t}|K(t,u)||K^{**}(t,2x-u) - K^*(t,2x-u)|\,du\,dx\\ &\stackrel{def}{=} 2I_1 + 2I_2.\end{aligned} \tag{2.39}$$

We estimate I_1 using (2.37) as follows:

$$\begin{aligned}
&\int_t^\infty e^{\delta x} \int_x^\infty |Q^{**}(\tau) - Q^*(\tau)|\, d\tau\, dx \\
=&\int_t^\infty |Q^{**}(\tau) - Q^*(\tau)| \int_t^\tau e^{\delta x} dx\, d\tau \\
\leq& 1/\delta \int_t^\infty |Q^{**}(\tau) - Q^*(\tau)| e^{\delta\tau} d\tau = 1/\delta \|Q^{**} - Q^*\|_\delta
\end{aligned}$$

and analogously

$$\|Q^{**} - Q^*\|_\delta \int_t^\infty e^{\delta x} \left(\int_x^\infty |Q^*| \right) dx \leq D/\delta \|Q^{**} - Q^*\|_\delta$$

hence

$$|I_1| \leq c(D, \delta) \|Q^{**} - Q^*\|_\delta.$$

In I_2 we have, as in verifying (2.24) above,

$$\begin{aligned}
&\int_t^\infty e^{\delta x} \int_t^{2x-t} |K(t,u)| \left(\int_{x+\frac{t-u}{2}}^\infty |Q^{**} - Q^*| \right) du\, dx \\
\leq& c(D,\delta) \int_t^\infty e^{\delta x} \int_t^{2x-t} \left(\int_{\frac{t+u}{2}}^\infty |Q| \right) \left(\int_{x+\frac{t-u}{2}}^\infty |Q^{**} - Q^*| \right) du\, dx \\
\leq& c(D,\delta) \int_t^\infty |Q(\nu)| e^{\delta\nu} \int_t^\infty |Q^*(\xi) - Q(\xi)| e^{\delta\xi} d\xi\, d\nu \\
\leq& c(D,\delta) \|Q^* - Q\|_\delta
\end{aligned}$$

and by the same way we obtain

$$\begin{aligned}
&\|Q^{**} - Q^*\|_\delta \cdot \int_t^\infty e^{\delta x} \int_t^{2x-t} |K(t,u)| \left(\int_{x+\frac{t-u}{2}}^\infty |Q^*| \right) du\, dx \\
&\leq c(D,\delta) \|Q^{**} - Q^*\|_\delta.
\end{aligned}$$

Thus we have checked the estimate (2.38). The proof of Lemma 2.7 is complete. □

3. Proof of the main results

Proof of Theorem 1.8 Suppose indirectly that (1.17) holds, however there are two potentials $Q^*, Q \in L_1(0,\infty)$ with $m(Q^*, \lambda_n) = m(Q, \lambda_n)\ \forall n$. Introduce the function $F(\lambda)$ as in (2.4), then $F(\lambda_n) = 0\ \forall n$. Let $\mu > 0$ be arbitrary; by Lemma 2.3 the function $F(\lambda + i\mu)$

is bounded analytic in the upper half-plane $\Im\lambda > 0$. So if F is not identically vanishing, its zeros must satisfy the Blaschke condition

$$\sum_{F(\lambda+i\mu)=0,\Im\lambda>0} \frac{\Im\lambda}{1+|\lambda|^2} < \infty$$

see e.g. in Duren [2]. In particular, for the zeros $\lambda_n - i\mu$

$$\sum_{\Im\lambda_n>\mu} \frac{\Im\lambda_n - \mu}{1+|\lambda_n - i\mu|^2} < \infty \quad \text{for all} \quad \mu > 0.$$

We can suppose that $\gamma > 2\mu$ and then

$$\sum \frac{\Im\lambda_n}{|\lambda_n|^2} \leq c(\mu) \sum \frac{\Im\lambda_n - \mu}{1+|\lambda_n - i\mu|^2} < \infty,$$

a contradiction. Consequently $F(\lambda) \equiv 0$ i.e. $m^* \equiv m$ and by Theorem 1.7, $Q^* = Q$ a.e. . □

In order to prove Theorem 1.9 we need

Lemma 3.1. [5] *Let B_1 and B_2 be Banach spaces. For every $q \in B_1$ a continuous linear operator*

$$A_q : B_1 \to B_2$$

be defined so that for some $q_0 \in B_1$

$$A_{q_0} : B_1 \to B_2 \quad \textit{is an (onto) isomorphism} \tag{3.1}$$

and the mapping $q \mapsto A_q$ be Lipschitzian in the sense that

$$\|(A_{q^*} - A_q)h\|_2 \leq c(q_0)\|q^* - q\|_1\|h\|_1 \tag{3.2}$$
$$\textit{for all } h, q, q^* \in B_1 \textit{ with } \|q\|_1, \|q^*\|_1 \leq 2\|q_0\|_1,$$

the constant $c(q_0)$ being independent of q, q^* and h. Then the set $\{A_q(q - q_0) : q \in B_1\}$ contains a ball in B_2 with center at the origin.

The Müntz theorem about the closedness of exponential systems is known in several versions. The author didn't find a proper reference containing the version formulated below, so a proof is provided.

Lemma 3.2. *Let λ_n be different complex numbers with $\Im\lambda_n \geq 0$, $\liminf \Im\lambda_n > 0$. Then*

$$\left\{e^{i\lambda_n x}\right\} \quad \textit{is closed in } L_1(0,\infty) \Longleftrightarrow \sum \frac{\Im\lambda_n}{1+|\lambda_n|^2} = \infty. \tag{3.3}$$

Proof. The if part: Let $h \in L_1(0,\infty)$, $\int_0^\infty h(x)e^{i\lambda_n x}dx = 0$ for all n. Define $H(z) = \int_0^\infty h(x)e^{izx}dx$, $\Im z \geq 0$. It is bounded analytic in the upper half-plane, so if $H \neq 0$, its zeros satisfy the Blaschke condition and then

$$\sum_{\Im\lambda_n>0} \frac{\Im\lambda_n}{1+|\lambda_n|^2} < \infty,$$

a contradiction. This implies $H \equiv 0$ and then $h = 0$ i.e. the system $\{e^{i\lambda_n x}\}$ is closed in L_1.
The only if part: Let

$$\sum_{\Im\lambda_n>0} \frac{\Im\lambda_n}{1+|\lambda_n|^2} < \infty.$$

Since $\liminf \lambda_n > 0$, there exists $\delta > 0$ such that $\Im\lambda_n \geq 2\delta$ with finitely many exceptions. Thus

$$\sum \frac{\Im(\lambda_n + i\delta)}{1+|\lambda_n+i\delta|^2} < \infty. \tag{3.4}$$

This means that there exists a Blaschke product $B(z)$ with $B(\lambda_n+i\delta) = 0$ $\forall n$, see [2]. The function $\frac{B(z)}{z+i}$ belongs to the Hardy space $H^2(\mathbf{C}^+)$. By the Paley-Wiener theorem there exists $0 \neq g \in L_2(0,\infty)$ with

$$\frac{B(z)}{z+i} = \int_0^\infty g(x)e^{izx}dx.$$

Consequently

$$0 = \int_0^\infty g(x)e^{i(\lambda_n+i\delta)x}dx = \int_0^\infty g(x)e^{-\delta x}e^{i\lambda_n x}dx = \int_0^\infty h(x)e^{i\lambda_n x}dx$$

with

$$0 \neq h(x) = g(x)e^{-\delta x} \in L_1(0,\infty).$$

Thus $\{e^{i\lambda_n x}\}$ is not closed in L_1. □

Proof of Theorem 1.9 Let $Q \in C_\delta$ be arbitrary; our task is to find a different $Q^* \in C_\delta$ with $m(Q^*,\lambda_n) = m(Q,\lambda_n)$ i.e. $F(\lambda_n) = 0$. Suppose that (1.17) is not true. Then

$$\sum \frac{\Im(2\lambda_n + i\delta)}{|2\lambda_n+i\delta|^2} < \infty \tag{3.5}$$

(because $\Im\lambda_n \geq \gamma > 0$). By Lemma 3.2 there exists $0 \neq g \in L_1(0,\infty)$ with

$$0 = \int_0^\infty g(x)e^{(2i\lambda_n-\delta)x}dx = \int_0^\infty h(x)e^{2i\lambda_n x}dx, \tag{3.6}$$

$$h(x) = g(x)e^{-\delta x} \in C_\delta. \tag{3.7}$$

Recall from Lemma 2.5 the identity

$$F(\lambda) = \int_0^\infty e^{2i\lambda x} \left[A_{Q^*}(Q^* - Q)\right](x)\, dx.$$

Comparing this with (3.6) we see that $F(\lambda_n) = 0$ is guaranteed if

$$A_{Q^*}(Q^* - Q) = ch \quad \text{for some } c \neq 0. \tag{3.8}$$

To check (3.8) we apply Lemma 3.1 with $B_1 = B_2 = C_\delta$. The conditions (3.1) and (3.2) are verified in Lemmas 2.6 and 2.7 respectively. Since the set $\{A_{Q^*}(Q^* - Q) : Q^* \in C_\delta\}$ contains all elements of C_δ of sufficiently small norm, there exists a potential $Q^* \in C_\delta$, $Q^* \neq Q$ satisfying (3.8). Now from $F(\lambda_n) = 0$ it follows that $m^*(\lambda_n) = m(\lambda_n)$, though $m^* \neq m$. □

Proof of Theorems 1.4 and 1.5 If $\lambda_n = ik_n$, $\inf k_n = \gamma > 0$ then $m^*(\lambda_n) = m(\lambda_n)$ if and only if the values $\lambda_n^2 = -k_n^2$ are common eigenvalues of Q^* and Q. Since $\Im\lambda_n/|\lambda_n|^2 = 1/k_n$, Theorems 1.4 and 1.5 are contained in Theorems 1.8 and 1.9 respectively. □

Proof of Theorems 1.2 and 1.3 Let $Q \in L_1(0,\infty)$ and consider the solution $y_1(x,\lambda) = e^{i\lambda x}(1 + \mathbf{o}(1))$ of $-y'' + Qy = \lambda^2 y$ with $\Im\lambda \geq 0$. Introduce the function

$$\varphi(r,\lambda) = \sqrt{r}\, y_1(\ln \tfrac{a}{r}, i\lambda) \quad \text{for } 0 < r \leq a,\ \Re\lambda \geq 0. \tag{3.9}$$

A short calculation gives that

$$\varphi''(r,\lambda) - \frac{\lambda^2 - 1/4}{r^2}\varphi(r,\lambda) - 1/r^2 Q(\ln \tfrac{a}{r})\varphi(r,\lambda) = 0,\ 0 < r \leq a. \tag{3.10}$$

This means that φ satisfies (1.1) (for $r \leq a$) if

$$q(r) = 1/r^2 Q(\ln \tfrac{a}{r}) + 1. \tag{3.11}$$

We observe that

$$rq(r) \in L_1(0,a) \Longleftrightarrow Q(x) \in L_1(0,\infty). \tag{3.12}$$

Indeed, substituting $x = \ln \frac{a}{r}$ gives

$$\int_0^\infty |Q(x)|\, dx = \int_0^a \frac{1}{r}|Q(\ln \tfrac{a}{r})|\, dr = \int_0^a r|q(r) - 1|\, dr$$

and this is finite if and only if $rq(r) \in L_1(0,a)$. Analogously for $0 < \delta < 2$

$$r^{1-\delta}q(r) \in L_1(0,a) \Longleftrightarrow Q \in C_\delta \tag{3.13}$$

since

$$\int_0^\infty |Q(x)|e^{\delta x}dx = a^\delta \int_0^a r^{1-\delta}|q(r)-1|\,dr.$$

From $y_1(x,\lambda) = e^{i\lambda x}(1+\mathbf{o}(1))$ we infer

$$\varphi(r,\lambda) = a^{-\lambda}r^{\lambda+1/2}(1+\mathbf{o}(1)) \quad r\to 0+ \tag{3.14}$$

which corresponds to (1.2).

In the potential-free domain $r\geq a$ (1.1) has the form

$$\varphi'' + \varphi = \frac{\lambda^2-1/4}{r^2}\varphi \quad r\geq a. \tag{3.15}$$

Its general solution can be given by Bessel functions

$$\varphi(r,\lambda) = c(\lambda)\sqrt{r}\left[\cos\alpha(\lambda)\cdot J_\lambda(r) + \sin\alpha(\lambda)\cdot Y_\lambda(r)\right], \tag{3.16}$$

with $\alpha(\lambda)\in\mathbf{C}$, see Watson [10]. Taking into account the asymptotic forms

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}}\cos(r-\nu\pi/2-\pi/4) + \mathbf{O}(r^{-3/2}) \quad r\to\infty, \tag{3.17}$$

$$Y_\nu(r) = \sqrt{\frac{2}{\pi r}}\sin(r-\nu\pi/2-\pi/4) + \mathbf{O}(r^{-3/2}) \quad r\to\infty \tag{3.18}$$

we get

$$\begin{aligned}\varphi(r,\lambda) &= c(\lambda)\sqrt{\frac{2}{\pi}}\cos(r-\lambda\pi/2-\pi/4-\alpha(\lambda)) + \mathbf{O}(1/r) = \\ &= c(\lambda)\sqrt{\frac{2}{\pi}}\sin(r-\pi/2(\lambda-1/2)-\alpha(\lambda)) + \mathbf{O}(1/r)\end{aligned} \tag{3.19}$$

which is compatible with (1.3) if and only if

$$\delta(\lambda) \equiv -\alpha(\lambda) \quad (\mod \pi). \tag{3.20}$$

On the other hand

$$\frac{\varphi'(a,\lambda)}{\varphi(a,\lambda)} = \frac{\cos\alpha(\lambda)\frac{d(\sqrt{r}J_\lambda(r))}{dr}(a) + \sin\alpha(\lambda)\frac{d(\sqrt{r}Y_\lambda(r))}{dr}(a)}{\sqrt{r}\left[\cos\alpha(\lambda)J_\lambda(a)+\sin\alpha(\lambda)Y_\lambda(a)\right]}. \tag{3.21}$$

Finally from (3.9) it follows that

$$\frac{\varphi'(a,\lambda)}{\varphi(a,\lambda)} = \frac{1/2a^{-1/2}y_1(0,i\lambda) - a^{-1/2}y_1'(0,i\lambda)}{a^{1/2}y_1(0,i\lambda)}. \tag{3.22}$$

Now (3.20)-(3.22) shows that the knowledge of the shift $\delta(\lambda)$ (for a fixed λ, $\Re\lambda\geq 0$) means the knowledge of $\frac{\varphi'(a,\lambda)}{\varphi(a,\lambda)}$ and of $\frac{y_1'(0,i\lambda)}{y_1(0,i\lambda)}$. In other words for Q, $Q^* \in L_1(0,\infty)$ and for q, q^* defined by (3.11) we have

$$\delta(q^*,\lambda) \equiv \delta(q,\lambda)(mod\pi) \iff m(Q^*,i\lambda) = m(Q,i\lambda) \tag{3.23}$$

for the corresponding m-functions. Taking into account (3.12) and (3.13) we see that Theorem 1.2 and 1.8, respectively Theorem 1.3 and 1.9 are the same statements. The proof is complete. □

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