INVERSE SPECTRAL PROBLEMS AND CLOSED EXPONENTIAL SYSTEMS

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Abstract. Consider the inverse eigenvalue problem of the Schrödinger operator defined on a finite interval. We give optimal and almost optimal conditions for a set of eigenvalues to determine the Schrödinger operator. These conditions are simple closedness properties of the exponential system corresponding to the known eigenvalues. The statements contain nearly all former results of this topic. We give also conditions for the recovering of the Weyl-Titchmarsh m-function from its values $m(\lambda_n)$.

1. INTRODUCTION

Consider the Schrödinger operator

$$Ly = -y'' + q(x)y \tag{1.1}$$

over the segment $[0, \pi]$ with a potential

$$q \in L_1(0, \pi) \quad \text{real-valued.} \tag{1.2}$$

The eigenvalue problem

$$Ly = \lambda y \quad \text{on} \quad [0, \pi], \tag{1.3}$$

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0, \tag{1.4}$$

$$y(\pi) \cos \beta + y'(\pi) \sin \beta = 0 \tag{1.5}$$

defines a sequence of eigenvalues

$$\lambda_0 < \lambda_1 < \dots < \lambda_n < \dots, \quad \lambda_n \in \mathbf{R}, \ \lambda_n \to +\infty; \tag{1.6}$$

they form together the spectrum $\sigma(q, \alpha, \beta)$.

In the inverse eigenvalue problems we aim to recover the potential q from a given set of eigenvalues (not necessarily taken from the same spectrum). The first result of this type is given in

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Theorem. A *Ambarzumian* [1]
Let $q \in C[0,\pi]$ and consider the Neumann eigenvalue problem

$$y'(0) = y'(\pi) = 0 \quad (\text{ i.e. } \alpha = \beta = \pi/2).$$

If the eigenvalues are $\lambda_n = n^2,\ n \geq 0$ then $q \equiv 0$.

Later it has been observed by G. Borg that the knowledge of the first eigenvalue $\lambda_0 = 0$ plays a crucial role here; he also found the general rule that in most cases two spectra are needed to recover the potential:

Theorem. B *Borg* [5]
Let $q \in L_1(0,\pi)$, $\sigma_1 = \sigma(q,0,\beta),\ \sigma_2 = \sigma(q,\alpha_2,\beta),\ \sin\alpha_2 \neq 0$ and

$$\tilde{\sigma}_2 = \begin{cases} \sigma_2 & \text{if } \sin\beta = 0 \\ \sigma_2 \setminus \{\lambda_0\} & \text{if } \sin\beta \neq 0 \end{cases}$$

Then $\sigma_1 \cup \tilde{\sigma}_2$ determines the potential a.e. and no proper subset has the same property.

Here determination means that there is no other potential $q^* \in L_1(0,\pi)$ with $\sigma_1 = \sigma_1^*,\ \tilde{\sigma}_2 = \tilde{\sigma}_2^*$. A related extension:

Theorem. C *Levinson* [16]
Let $q \in L_1(0,\pi)$. If $\sin(\alpha_1 - \alpha_2) \neq 0$ then the two spectra $\sigma(q,\alpha_1,\beta)$ and $\sigma(q,\alpha_2,\beta)$ determine the potential a.e.

By an interesting observation of Hochstadt and Lieberman, if half of the potential is known then one spectrum is enough to recover the other half of q:

Theorem. D *Hochstadt and Lieberman* [11]
If $q \in L_1(0,\pi)$, then q on $(0,\pi/2)$ and the spectrum $\sigma(q,\alpha,\beta)$ determine q a.e on $(0,\pi)$.

This idea has been further developed by Gesztesy and Simon:

Theorem. E *Gesztesy, Simon* [10]
Let $q \in L_1(0,\pi)$ and $\pi/2 < a < \pi$. Then q on $(0,a)$ and a subset $S \subset \sigma = \sigma(q,\alpha,\beta)$ of eigenvalues satisfying

$$\#\{\lambda \in S : \lambda \leq t\} \geq 2(1 - a/\pi)\#\{\lambda \in \sigma : \lambda \leq t\} + a/\pi - 1/2$$

for sufficiently large $t > 0$, uniquely determine q a.e. on $(0,\pi)$.

Another statement of this type is given in

Theorem. F *del Rio, Gesztesy, Simon* [7]
Let $q \in L_1(0,\pi)$, let $\sigma_i = \sigma(q,\alpha_i,\beta)$ be three different spectra and $S \subset \sigma_1 \cup \sigma_2 \cup \sigma_3$. If

$$\#\{\lambda \in S : \lambda \leq t\} \geq 2/3\#\{\lambda \in \sigma_1 \cup \sigma_2 \cup \sigma_3 : \lambda \leq t\}$$

for large t then the eigenvalues in S determine q.

In Horváth [12] a similar but more general sufficient condition is given for the case when the known eigenvalues are taken from N different spectra.

The following statement provides a necessary and sufficient condition for a set of eigenvalues to determine the potential; it is one of the major new results of this paper. Before its formulation it is useful to fix some terminology. Let $1 \le p \le \infty$ and $1/p + 1/p' = 1$. A system $\{\varphi_n : n \ge 1\}$, $\varphi_n \in L_{p'}(0,\pi)$ is called closed in $L_p(a,b)$ if $h \in L_p(a,b)$, $\int_0^\pi h\varphi_n = 0 \quad \forall n$ implies $h = 0$. This is equivalent to the completeness of the φ_n in $L_{p'}(0,\pi)$ if $p > 1$. Let $\beta \in \mathbf{R}$ be given and let q^*, $q \in L_p(0,\pi)$. We say that the (different) values $\lambda_n \in \mathbf{R}$ are *common eigenvalues of q^* and q* if there exist $\alpha_n \in \mathbf{R}$ with

$$\lambda_n \in \sigma(q,\alpha_n,\beta) \cap \sigma(q^*,\alpha_n,\beta).$$

So every eigenvalue λ_n is allowed to belong to different spectra. The values $\cot\alpha_n$ are defined by q, λ_n and β, see (1.12) below. In the above cited theorems the eigenvalues are taken from at most three spectra; in [12] the λ_n belong to finitely many spectra.

Let $0 \le a < \pi$ and $\lambda_n \in \mathbf{R}$ be different values. By the statement
"β, q on $(0,a)$ and the eigenvalues λ_n determine q in L_p"
we mean that there are no two different potentials q^*, $q \in L_p(0,\pi)$ with $q^* = q$ a.e. on $(0,a)$ such that the λ_n are common eigenvalues of q^* and q. By
"β, q on $(0,a)$ and the eigenvalues λ_n does not determine q in L_p"
we mean that for every $q \in L_p(0,\pi)$ there exists a different potential $q^* \in L_p(0,\pi)$ with $q^* = q$ a.e. on $(0,a)$ such that the λ_n are common eigenvalues of q^* and q.

Theorem 1.1. *Let $1 \le p \le \infty$, $q \in L_p(0,\pi)$, $0 \le a < \pi$ and let $\lambda_n \in \sigma(q,\alpha_n,0)$ be real numbers with $\lambda_n \not\to -\infty$. Then $\beta = 0$, q on $(0,a)$ and the eigenvalues λ_n determine q in L_p if and only if the system*

$$e(\Lambda) = \left\{ e^{\pm 2i\mu x}, e^{\pm 2i\sqrt{\lambda_n}x} : n \ge 1 \right\} \tag{1.7}$$

is closed in $L_p(a-\pi,\pi-a)$ for some (for any) $\mu \ne \pm\sqrt{\lambda_n}$.

In case $\sin\beta \ne 0$ we find a different situation. First we state a sufficient condition:

Theorem 1.2. *Let $1 \le p \le \infty$, $q \in L_p(0,\pi)$, $\sin\beta \ne 0$, $\lambda_n \in \sigma(q,\alpha_n,\beta)$, $\lambda_n \not\to -\infty$ and $0 \le a < \pi$. If the set*

$$e_0(\Lambda) = \left\{ e^{\pm 2i\sqrt{\lambda_n}x} : n \ge 1 \right\} \tag{1.8}$$

is closed in $L_p(a-\pi, \pi-a)$ *then* q *on* $(0,a)$ *and the eigenvalues* λ_n *determine* q *in* L_p.

The following example shows that the above closedness condition (1.8) is sharp in some cases:

Proposition 1.3. *Let* $\beta = \pi/2$,

$$q(x) = \begin{cases} 0 & \text{on } (0, \pi/2) \\ 1 & \text{on } (\pi/2, \pi), \end{cases}$$

$$q^*(x) = \begin{cases} 1 & \text{on } (0, \pi/2) \\ 0 & \text{on } (\pi/2, \pi). \end{cases}$$

Then for the set of all common eigenvalues of q^* *and* q*, the system* $e_0(\Lambda)$ *has deficiency 1 in* $L_p(-\pi,\pi)$, $1 \le p < \infty$. *In other words, the system* $e_1(\Lambda) = \left\{ e^{2i\mu x}, e^{\pm 2i\sqrt{\lambda_n} x} : n \ge 1 \right\}$ *with* $\mu \ne \pm\sqrt{\lambda_n}$ *is closed in* $L_p(-\pi,\pi)$.

Remark. In the important special cases considered by Borg in Theorem B, however, the closedness of $e_0(\Lambda)$ is not an optimal condition in Theorem 1.2; in those situations the codimension of $e_0(\Lambda)$ is 1 for the set of eigenvalues defining the potential (see Section 4).

Remark. Denote by $v(x,\lambda)$ the solution of

$$-v'' + q(x)v = \lambda v \quad \text{on } (0,\pi), \tag{1.9}$$

$$v(\pi,\lambda) = \sin\beta, \quad v'(\pi,\lambda) = -\cos\beta \tag{1.10}$$

and let $v^*(x,\lambda)$ be the same function defined by q^* instead of q. Then the common eigenvalues of q^* and q under the boundary condition (1.5) are precisely the solutions $\lambda_n \in \mathbf{R}$ of the equation

$$v(0,\lambda)v^{*\prime}(0,\lambda) = v'(0,\lambda)v^*(0,\lambda). \tag{1.11}$$

In this case $\lambda_n \in \sigma(q^*, \alpha_n, \beta) \cap \sigma(q, \alpha_n, \beta)$ with

$$\cot\alpha_n = -\frac{v'(0,\lambda_n)}{v(0,\lambda_n)} = -\frac{v^{*\prime}(0,\lambda_n)}{v^*(0,\lambda_n)}. \tag{1.12}$$

In looking for a necessary condition for $\sin\beta \ne 0$ we have to avoid the Ambarzumian-type exceptional cases where less than two spectra are enough to determine the potential. To this end, introduce the following *minimality condition*

(M) There exists $h \in L_p(a,\pi)$ such that

$$\int_a^\pi h \ne 0 \quad \text{but} \quad \int_a^\pi h(x)[v^2(x,\lambda_n) - 1/2\sin^2\beta]\,dx = 0 \quad \forall n.$$

For $1 < p$ this condition can also be formulated in the following form: the closed subspace generated in $L_{p'}(a, \pi)$ by the functions $v^2(x, \lambda_n) - 1/2 \sin^2 \beta$ does not contain the constant function 1; here $1/p + 1/p' = 1$.

Theorem 1.4. *Let* $\sin \beta \neq 0$, $0 \leq a < \pi$, $1 \leq p \leq \infty$ *and* λ_n, $n \geq 1$ *be different real numbers with* $\lambda_n \not\to -\infty$. *Suppose (M) and that*

$$e(\Lambda) = \left\{ e^{\pm 2i\mu x}, e^{\pm 2i\sqrt{\lambda_n} x} \right\}$$

is not closed in $L_p(a - \pi, \pi - a)$, *where* $\mu \neq \pm\sqrt{\lambda_n}$. *Then* q *on* $(0, a)$ *and the eigenvalues* λ_n *do not determine* q *in* L_p.

Define the *Weyl-Titchmarsh m-function* corresponding to the problem (1.3), (1.5) by

$$m_\beta(\lambda) = \frac{v'(0, \lambda)}{v(0, \lambda)} \tag{1.13}$$

where $v(x, \lambda)$ is given in (1.9), (1.10). It is a meromorphic function having poles at the zeros of $v(0, \lambda)$.

Theorem. G *Borg* [6], *Marchenko* [19]
The potential and the value $\tan \beta$ *can be recovered from the m-function* $m_\beta(\lambda)$.

In the context of the m-function Theorem 1.1 and Theorem 1.2 can be generalized in the following way:

Theorem 1.5. *Let* $1 \leq p \leq \infty$ *and* λ_n, $n \geq 1$ *be arbitrary different real numbers with* $\lambda_n \not\to -\infty$. *Let* β_1, $\beta_2 \in \mathbf{R}$, q^*, $q \in L_p(0, \pi)$ *and consider the m-functions* m_{β_1} *and* $m^*_{\beta_2}$, *defined by* q *and* q^* *respectively.*

- *If the system* $e_0(\Lambda)$ *is closed in* $L_p(-\pi, \pi)$ *then*
$$m_{\beta_1}(\lambda_n) = m^*_{\beta_2}(\lambda_n), \quad n \geq 1 \tag{1.14}$$
implies $m_{\beta_1} \equiv m^*_{\beta_2}$ *(so* $\tan \beta_1 = \tan \beta_2$ *and* $q^* = q$*).*
- *Let* $\sin \beta_1 \cdot \sin \beta_2 = 0$. *Then (1.14) implies* $\sin \beta_1 = \sin \beta_2 = 0$. *In this case (1.14) implies* $m^*_0 \equiv m_0$ *if and only if the system* $e(\Lambda)$ *is closed in* $L_p(-\pi, \pi)$.

Remark. We allow in (1.14) that both sides be infinite.

A former result of this type is given in

Theorem. H *del Rio, Gesztesy, Simon* [7]
Denote $c_+ = \max(c, 0)$ *and let* $q \in L_1(0, \pi)$. *If* $\lambda_n > 0$ *are distinct numbers satisfying*

$$\sum_{n=0}^{\infty} \frac{(\lambda_n - n^2/4)_+}{1 + n^2} < \infty \tag{1.15}$$

then the values $m_\beta(\lambda_n)$ *determine* m_β *(and* $\tan\beta$ *).*

Since (1.15) implies the closedness of $e_0(\Lambda)$, this statement is a special case of Theorem 1.5; see Section 4.

Finally we mention the following localized version of Theorem G. It was first given in Simon [20]; see also Gesztesy and Simon [9], [8] and Bennewitz [4].

Theorem. I [20], [9], [8], [4]
Let $\beta_1,\ \beta_2 \in \mathbf{R}$, $q^*,\ q \in L_1(0,\pi)$, $0 \le a < \pi$. *Then* $q^* = q$ *a.e on* $(0,a)$ *if and only if for every* $\varepsilon > 0$

$$m_{\beta_1}(\lambda) - m^*_{\beta_2}(\lambda) = \mathbf{O}\left(e^{-2(a-\varepsilon)|\Im\sqrt{\lambda}|}\right) \tag{1.16}$$

holds along a nonreal ray $\arg\lambda = \gamma$, $\sin\gamma \neq 0$.

Using this statement the following generalization of Theorem 1.5 can be given:

Theorem 1.6. *Let* $1 \le p \le \infty$ *and* $\lambda_n,\ n \ge 1$ *be arbitrary different real numbers with* $\lambda_n \not\to -\infty$. *Let* $\beta_1,\ \beta_2 \in \mathbf{R}$, $q^*,\ q \in L_p(0,\pi)$ *and suppose that (1.16) holds for every* $\varepsilon > 0$ *along a nonreal ray.*

- *If the system* $e_0(\Lambda)$ *is closed in* $L_p(a-\pi,\pi-a)$ *then (1.14) implies* $m_{\beta_1} \equiv m^*_{\beta_2}$.
- *Let* $\sin\beta_1 \cdot \sin\beta_2 = 0$. *Then (1.14) yields* $\sin\beta_1 = \sin\beta_2 = 0$. *In this case (1.14) implies* $m^*_0 \equiv m_0$ *if and only if the system* $e(\Lambda)$ *is closed in* $L_p(a-\pi,\pi-a)$.

Remark. The statements of Theorems 1.1 and 1.5 for the Schrödinger operators on the half-line are investigated in the forthcoming paper [13]. It turns out that the inverse eigenvalue problem is closely related to the inverse scattering problem with fixed energy.

The organization of the paper is as follows. In Section 2 we provide the proof of Theorem 1.1; the main ingredient is Lemma 2.1. Some technical background needed in the proof is given only in Section 5. Section 3 is devoted to prove Theorems 1.2, 1.4, 1.5 and 1.6 by modifying the procedure presented in Section 2. The applications of the new results are collected in Section 4; we show how the above-mentioned former results can be presented as special cases of Theorems 1.1 to 1.6. It requires the use of some standard tools from the theory of nonharmonic Fourier series, more precisely, some closedness and basis tests for exponential systems. Finally at the end of Section 4 we check the properties of the counterexample formulated in Proposition 1.3.

2. PROOF OF THEOREM 1.1

In this section we provide the proof of Theorem 1.1. We start with some lemmas.

Lemma 2.1. *Let B_1 and B_2 be Banach spaces. For every $q \in B_1$ a continuous linear operator*

$$A_q : B_1 \to B_2$$

be defined so that for some $q_0 \in B_1$

$$A_{q_0} : B_1 \to B_2 \text{ is an (onto) isomorphism,} \tag{2.1}$$

and the mapping $q \to A_q$ be Lipschitzian in the sense that

$$\|(A_{q^*} - A_q)h\| \le c(q_0)\|q^* - q\|\|h\| \quad \forall h, q, q^* \in B_1,\ \|q\|, \|q^*\| \le 2\|q_0\|, \tag{2.2}$$

the constant $c(q_0)$ being independent of q, q^* and h. Then the set $\{A_q(q - q_0) : q \in B_1\}$ contains a ball in B_2 with center at the origin.

Proof. Let $G_0 \in B_2$ be an arbitrary element, the norm of which is small in a sense to be specified later. Our task is to find an element $q^* \in B_1$ such that

$$A_{q^*}(q^* - q_0) = G_0. \tag{2.3}$$

This will be done by the following iteration. The vector q_0^* be defined by

$$A_{q_0}(q_0^* - q_0) = G_0 \tag{2.4}$$

and q_{k+1}^* by

$$A_{q_0}(q_{k+1}^* - q_0) = G_0 - (A_{q_k^*} - A_{q_0})(q_k^* - q_0), \quad k \ge 0. \tag{2.5}$$

This is justified by (2.1). We state that $q_k^* \to q^*$, a solution of (2.3). Indeed, consider the following corollary of (2.5):

$$A_{q_0}(q_{k+1}^* - q_k^*) = -(A_{q_k^*} - A_{q_0})(q_k^* - q_{k-1}^*) - (A_{q_k^*} - A_{q_{k-1}^*})(q_{k-1}^* - q_0); \tag{2.6}$$

if $k = 0$, we use instead

$$A_{q_0}(q_1^* - q_0^*) = -(A_{q_0^*} - A_{q_0})(q_0^* - q_0). \tag{2.6'}$$

Using the conditions (2.1), (2.2) we get from the formulae (2.4), (2.6') and (2.6) that

$$\|q_0^* - q_0\| \le c_1\|G_0\|, \tag{2.7}$$

$$\|q_1^* - q_0^*\| \le c_1\|q_0^* - q_0\|^2 \text{ if } \|q_0^*\| \le 2\|q_0\|, \tag{2.8}$$

$$\|q^*_{k+1}-q^*_k\| \le c_1\|q^*_k-q^*_{k-1}\|(\|q^*_k-q_0\|+\|q^*_{k-1}-q_0\|), \tag{2.9}$$
$$\text{if } \|q^*_k\|\le 2\|q_0\|,\ \|q^*_{k-1}\|\le 2\|q_0\|,\ k\ge 1$$

with a constant c_1 independent of the q^*_k, $k \ge 0$ and of G_0. We suppose that G_0 is small enough to ensure

$$8c_1^2\|G_0\| \le 1, \quad c_1\|G_0\| \le 1/2\|q_0\| \tag{2.10}$$

and we prove that

$$\|q^*_{k+1}-q^*_k\| \le 1/2\|q^*_k-q^*_{k-1}\|, \quad \|q^*_k\| \le 2\|q_0\| \text{ if } k\ge 1. \tag{2.11}$$

Indeed, (2.7) and (2.10) imply $\|q^*_0\| \le 3/2\|q_0\|$ and then by (2.8)

$$\|q^*_1-q^*_0\| \le c_1\|q^*_0-q_0\|^2 \le c_1^2\|G_0\|\cdot\|q^*_0-q_0\| \le 1/2\|q^*_0-q_0\| \le 1/4\|q_0\|$$

and then

$$\|q^*_1\| \le \|q^*_1-q^*_0\|+\|q^*_0-q_0\|+\|q_0\| \le (1/4+1/2+1)\|q_0\|.$$

Consequently by (2.9)

$$\begin{aligned}\|q^*_2-q^*_1\| &\le c_1\|q^*_1-q^*_0\|(\|q^*_1-q_0\|+\|q^*_0-q_0\|) \le\\ &\le c_1\|q^*_1-q^*_0\|(\|q^*_1-q^*_0\|+2\|q^*_0-q_0\|) \le\\ &\le \|q^*_1-q^*_0\|(c_1^2\|q^*_0-q_0\|^2+2c_1\|q^*_0-q_0\|) \le\\ &\le \|q^*_1-q^*_0\|(c_1^4\|G_0\|^2+2c_1^2\|G_0\|) \le 1/2\|q^*_1-q^*_0\|\end{aligned}$$

which is (2.11) for $k=1$. Now suppose (2.11) below a fixed value of k and prove it for that k. We have

$$\begin{aligned}\|q^*_i-q_0\| &\le \|q^*_i-q^*_{i-1}\|+\cdots+\|q^*_1-q^*_0\|+\|q^*_0-q_0\| \le\\ &\le 2\|q^*_1-q^*_0\|+\|q^*_0-q_0\| \le 2c_1\|q^*_0-q_0\|^2+\|q^*_0-q_0\| \le\\ &\le 2c_1^3\|G_0\|^2+c_1\|G_0\| \le 2c_1\|G_0\| \le \|q_0\|\end{aligned}$$

for $i \le k$ and then

$$\|q^*_k\| \le \|q^*_k-q_0\|+\|q_0\| \le 2\|q_0\|.$$

Consequently

$$\begin{aligned}\|q^*_{k+1}-q^*_k\| &\le c_1\|q^*_k-q^*_{k-1}\|(\|q^*_k-q_0\|+\|q^*_{k-1}-q_0\|) \le\\ &\le \|q^*_k-q^*_{k-1}\|(4c_1^4\|G_0\|^2+2c_1^2\|G_0\|) \le 1/2\|q^*_k-q^*_{k-1}\|\end{aligned}$$

so (2.11) is proved and then $q^*_k \to q^*$ in B_1. Now

(2.12)
$$A_{q_0}(q^*_{k+1}-q_0) = G_0+(A_{q^*}-A_{q^*_k})(q^*_k-q_0)-(A_{q^*}-A_{q_0})(q^*_k-q_0).$$

Since

$$\|(A_{q^*}-A_{q^*_k})(q^*_k-q_0)\| \le c\|q^*-q^*_k\|\cdot\|q^*_k-q_0\| \to 0 \quad k\to\infty,$$

we can take the limit in (2.12) to obtain

$$A_{q_0}(q^* - q_0) = G_0 - (A_{q^*} - A_{q_0})(q^* - q_0).$$

This is (2.3) so the proof is complete. □

In the following statement the point a)(in a less general situation) and the formula (2.16) are due to Gesztesy and Simon [10], [8]. We give the whole proof for the sake of completeness.

Lemma 2.2. *Let $0 \le a < \pi$, $q, q^* \in L_1(0,\pi)$, $q^* = q$ a.e. on $(0,a)$. Consider the function*

$$F(z) = v^*(a,z)v'(a,z) - v(a,z)v^{*\prime}(a,z) \tag{2.13}$$

where v and v^* are defined by q and q^* respectively in (1.9), (1.10) with $\beta = 0$. The derivatives in (2.13) refer to x. Then
a) The real zeros of $F(z)$ are precisely the common eigenvalues of q and q^*; in other words, all values $z = \lambda \in \mathbf{R}$ for which there exists $\alpha \in \mathbf{R}$ with $\lambda \in \sigma(q^*,\alpha,0) \cap \sigma(q,\alpha,0)$.
b) If $\lambda_n \not\to -\infty$ holds for the (infinitely many) common eigenvalues of q^* and q then

$$\int_a^\pi (q^* - q) = 0. \tag{2.14}$$

Proof. $F(\lambda) = 0$ if and only if the initial condition vectors $(v(a,\lambda), v'(a,\lambda))$ and $(v^*(a,\lambda), v^{*\prime}(a,\lambda))$ are parallel. Since $q^* = q$ a.e. on $(0,a)$, this means that v^* and v are identical on $[0,a]$ up to a constant factor. In other words we have $\lambda \in \sigma(q^*,\alpha,0) \cap \sigma(q,\alpha,0)$ with $\tan\alpha = -\frac{v(0,\lambda)}{v'(0,\lambda)} = -\frac{v^*(0,\lambda)}{v^{*\prime}(0,\lambda)}$. This proves a). To show b) take the function

$$F(x,z) = v^*(x,z)v'(x,z) - v(x,z)v^{*\prime}(x,z). \tag{2.15}$$

Now

$$\begin{aligned}\frac{\partial F}{\partial x}(x,z) &= v^*(x,z)v''(x,z) - v(x,z)v^{*\prime\prime}(x,z) = \\ &= (q(x) - q^*(x))v(x,z)v^*(x,z)\end{aligned}$$

which implies

(2.16)
$$F(z) = -\int_a^\pi \frac{\partial F}{\partial x}(x,z)\,dx = \int_a^\pi (q^*(x) - q(x))v(x,z)v^*(x,z)\,dx.$$

If the zeros λ_n have a finite accumulation point then the entire function $F(z)$ is identically zero, which implies $m^* = m$ and $q^* = q$; in this case

(2.14) is obvious. Otherwise the λ_n have a subsequence tending to $+\infty$. By Lemma 5.2

$$
\begin{aligned}
(2.17)\quad & 2(z^2-\mu^2)F(z^2) = \\
&= 2(z^2-\mu^2)\int_a^\pi (q^*(x)-q(x))v(x,z^2)v^*(x,z^2)\,dx = \\
&= \int_a^\pi (q^*-q) - \int_a^\pi (q^*(x)-q(x))\cos 2z(\pi-x)\,dx \\
&- \int_a^\pi (q^*(x)-q(x)) - \int_0^{2(\pi-x)} \cos z\tau M(\pi-x,\tau,\mu^2)\,d\tau\,dx \\
&= I_1 - I_2 - I_3.
\end{aligned}
$$

Now I_3 has the form

(2.18)

$$I_3 = \int_0^{2(\pi-a)} \cos z\tau \int_a^{\pi-\tau/2} (q^*(x)-q(x))M(\pi-x,\tau,\mu^2)\,dx\,d\tau.$$

This means that for the subsequence of values $z=\sqrt{\lambda_n}$ tending to $+\infty$ we have $I_3 \to 0$. Since $I_2 \to 0$ is obvious, from $F(\lambda_n)=0$ we infer (2.14) as asserted. $\square$

Proof of Theorem 1.1

We consider the closedness of the system

$$C(\Lambda) = \{\cos 2\mu x, \cos 2\sqrt{\lambda_n}x : n \geq 1\} \tag{2.19}$$

in $L_p(0,\pi-a)$ instead of that of $e(\Lambda)$ in $L_p(a-\pi,\pi-a)$; this is justified in Lemma 5.4.

The if part: if the system $C(\Lambda)$ is closed in $L_p(0,\pi-a)$ then the eigenvalues λ_n and $q|_{(0,a)}$ determine q on the whole $(0,\pi)$. Suppose indirectly that there exists another potential $q^* \in L_p$ with $q^*=q$ a.e. on $(0,a)$ and $\lambda_n \in \sigma(q^*,\alpha_n,0) \cap \sigma(q,\alpha_n,0)$ for some $\alpha_n \in \mathbf{R}$. Define $F(z)$ by (2.13); then $F(\lambda_n)=0$ $(n \geq 1)$ and $F \not\equiv 0$. The function

$$G(z) = -2(z^2-\mu^2)F(z^2)$$

has zeros at $\pm\mu$, $\pm\sqrt{\lambda_n}$. From (2.14) we get

$$G(z) = \int_a^\pi (q^*(x)-q(x))\left[1-2(z^2-\mu^2)v(x,z^2)v^*(x,z^2)\right]\,dx. \tag{2.20}$$

Define the linear operators

$$A_{q^*} : L_p(a,\pi) \to L_p(a,\pi)$$

$$(A_{q^*}h)(x) = h(x) + 2\int_a^x h(\tau)M(\pi-\tau,2(\pi-x),\mu^2,q,q^*)\,d\tau.$$

Then Lemma 5.2 gives after an interchange of integrations

$$\int_a^\pi (q^*(x)-q(x))\left[1-2(z^2-\mu^2)v(x,z^2)v^*(x,z^2)\right]\,dx = \tag{2.21}$$
$$=\int_a^\pi \cos 2z(\pi-x)\left[A_{q^*}(q^*-q)\right](x)\,dx.$$

Observe that

$$A_{q^*}:L_p(a,\pi)\to L_p(a,\pi) \text{ is an isomorphism.} \tag{2.22}$$

Indeed, the Volterra operator

$$h\mapsto 2\int_a^x h(\tau)M(\pi-\tau,2(\pi-x),\mu^2,q,q^*)\,d\tau$$

with continuous kernel is known to have the spectrum $\sigma=\{0\}$. In particular, $-1\notin\sigma$ i.e. A_{q^*} is isomorphism. Now if $q^*\neq q$ then $A_{q^*}(q^*-q)\neq 0$, hence by (2.20) and (2.21) the system $C(\Lambda)$ is not closed in $L_p(0,\pi-a)$. This contradiction proves the if part of Theorem 1.1.

The only if part: if $C(\Lambda)$ is not closed in $L_p(0,\pi-a)$ and if $\lambda_n\not\to-\infty$ then for every $q\in L_p(0,\pi)$ there exists $q^*\in L_p(0,\pi)$, $q^*\neq q$ but $q^*=q$ a.e. on $(0,a)$ and there exist values $\alpha_n\in\mathbf{R}$ with $\lambda_n=\sigma(q^*,\alpha_n,0)\cap\sigma(q,\alpha_n,0)$ for all $n\geq 1$. Indeed, since $C(\Lambda)$ is not closed, there exists a function $0\neq h\in L_p(0,\pi-a)$ such that

$$G_0(z)\overset{def}{=}\int_0^{\pi-a} h(x)\cos 2zx\,dx \tag{2.23}$$

has zeros at $\pm\mu$ and $\pm\sqrt{\lambda_n}$. Our task is to show that for every $q\in L_p(0,\pi)$ there exists $q^*\in L_p(0,\pi)$, $q^*\neq q$, $q^*=q$ a.e. on $(0,a)$ such that

$$\gamma G_0(z)=\int_a^\pi (q^*(x)-q(x))\left[1-2(z^2-\mu^2)v(x,z^2)v^*(x,z^2)\right]\,dx \tag{2.24}$$

holds for some constant $\gamma\neq 0$. Indeed, $G_0(\mu)=0$ and (2.24) gives (2.14) and then the function $F(z)$ defined in (2.13) has zeros $F(\lambda_n)=0$ i.e. the λ_n are common eigenvalues of q^* and q. Taking into account (2.21), (2.23) and (2.24), our task is to find q^* with

$$\gamma h(\pi-x)=A_{q^*}(q^*-q)(x) \text{ a.e. for some } \gamma\neq 0. \tag{2.25}$$

We check this representation by Lemma 2.1 applied with $B_1=B_2=L_p(a,\pi)$. The condition (2.1) is verified in (2.22) and (2.2) follows from

Lemma 5.2, since if q, q^*, $q^{**} \in L_p$ with norms $\le D$ then

$$\|(A_{q^{**}} - A_{q^*})h\| = 2\left\{ \int_a^\pi \Big| \int_a^x h(\tau)\big[M(\pi-\tau, 2(\pi-x), \mu^2, q, q^{**}) - \right.$$

$$\left. -M(\pi-\tau, 2(\pi-x), \mu^2, q, q^*)\big]\, d\tau\Big|^p dx \right\}^{1/p} \le$$

$$\le c(D)\|q^{**} - q^*\| \left\{ \int_a^\pi \left(\int_a^x |h| \right)^p dx \right\}^{1/p} \le c_1(D)\|q^{**} - q^*\| \cdot \|h\|$$

with straightforward modifications for $p = \infty$. So Lemma 2.1 applies and this shows the possibility of the representation (2.25) with sufficiently small $\gamma \ne 0$. The proof is complete. □

3. Proofs of Theorems 1.2 to 1.6

In this part of the paper we give the proofs of the remaining new results. They are modifications of the proof of Theorem 1.1 or consequences of already proved results. The proof of Proposition 1.3 is deferred to Section 4.

Lemma 3.1. *Let $1 \le p \le \infty$, $q, q^* \in L_p(0,\pi)$, $0 \le a < \pi$, $q^* = q$ a.e. on $(0,a)$. Let $F(z)$ be defined by (2.13), where the functions v and v^* are given in (1.9), (1.10) with q and q^*. Let $\sin\beta \ne 0$. Then*
a) The real zeros of $F(z)$ are precisely the common eigenvalues $\lambda_n \in \sigma(q^*, \alpha_n, \beta) \cap \sigma(q, \alpha, \beta)$ of q^* and q.
b) If $\lambda_n \not\to -\infty$ holds for the (infinitely many) common eigenvalues of q and q^* then (2.14) holds.

Proof. The verification of Lemma 2.2 can be repeated, only (2.17) is substituted by

(3.1)
$$F(z^2) = \frac{\sin^2\beta}{2}\int_a^\pi (q^* - q) + \frac{\sin^2\beta}{2}\int_a^\pi (q^*(x) - q(x))\cos 2z(\pi - x)\, dx$$
$$+ \int_a^\pi (q^*(x) - q(x)) \int_0^{2(\pi - x)} \cos z\tau L(\pi - x, \tau)\, d\tau\, dx,$$

see Lemma 5.3. Consequently

$$F(z^2) \to \frac{\sin^2\beta}{2}\int_a^\pi (q^* - q) \text{ if } z \to +\infty,\ z \in \mathbf{R}$$

and the proof of (2.14) is finished like in Lemma 2.2. □

Proof of Theorem 1.2
We have to show that if the system

$$C_0(\Lambda) = \{\cos 2\sqrt{\lambda_n}x : n \geq 1\} \tag{3.2}$$

is closed in $L_p(0, \pi - a)$ then $q|_{(0,a)}$ and the eigenvalues λ_n determine q. Indeed, let $q^* \in L_p(0,\pi)$ be another potential with $q^* = q$ a.e. on $(0,a)$ such that $\lambda_n \in \sigma(q^*, \alpha_n, \beta) \cap \sigma(q, \alpha, \beta)$, $n \geq 1$ for some $\alpha_n \in \mathbf{R}$. From Lemma 5.3 we infer for $h \in L_p(a,\pi)$

$$\begin{aligned}\int_a^\pi h(x)\left[v(x,z^2)v^*(x,z^2) - 1/2\sin^2\beta\right]\,dx = \\ = \int_a^\pi \cos 2z(\pi - x) A_{q^*}h(x)\,dx\end{aligned} \tag{3.3}$$

where

$$A_{q^*}h(x) = \frac{\sin^2\beta}{2}h(x) + \int_a^x h(\tau)2L(\pi - \tau, 2(\pi - x), q, q^*)\,d\tau. \tag{3.4}$$

We observe that

$$A_{q^*} : L_p(a,\pi) \to L_p(a,\pi) \text{ is an isomorphism} \tag{3.5}$$

just like in the proof of Theorem 1.1. Let $F(z)$ be defined by (2.13), (2.16). It follows from (2.14) that

$$F(z^2) = \int_a^\pi \cos 2z(\pi - x)\left[A_{q^*}(q^* - q)\right](x)\,dx. \tag{3.6}$$

Now if $q^* \neq q$ then $0 \neq h = A_{q^*}(q^* - q) \in L_p$ satisfies $\int_a^\pi h(x)\cos 2\sqrt{\lambda_n}(\pi - x)\,dx = 0 \quad \forall n$, in contradiction with the closedness of $C_0(\Lambda)$ in $L_p(0, \pi - a)$. □

The following statement is the counterpart of Lemma 2.1:

Lemma 3.2. *Let B_1 and B_2 be Banach spaces, let $\varphi : B_2 \to \mathbf{C}$ be a bounded linear functional and let B_{21} be a closed subspace of B_2. For every $q \in B_1$ define a continuous linear operator*

$$A_q : B_1 \to B_2.$$

Suppose (2.1), (2.2) and

$$\dim B_{21} \geq 2, \quad B_{21} \not\subset \mathrm{Ker}\varphi. \tag{3.7}$$

Then the set $\{A_q(q - q_0) : q \in B_1, q - q_0 \in A_{q_0}^{-1}(\mathrm{Ker}\varphi)\}$ contains a nonzero element of B_{21}.

Proof. Take an element $0 \neq G_0 \in B_{21} \cap \mathrm{Ker}\varphi$ and let $G_{00} \in B_{21} \setminus \mathrm{Ker}\varphi$ with $\varphi(G_{00}) = 1$. Define the operator $P : B_2 \to B_2$ by

$$PG = G - \varphi(G)G_{00}. \tag{3.8}$$

By definition we have

$$\mathrm{ImP} \subset \mathrm{Ker}\varphi. \tag{3.9}$$

The vector q_0^* be defined by

$$A_{q_0}(q_0^* - q_0) = G_0 \tag{3.10}$$

and q_{k+1}^* by

$$A_{q_0}(q_{k+1}^* - q_0) = G_0 - P((A_{q_k^*} - A_{q_0})(q_k^* - q_0)), \quad k \geq 0. \tag{3.11}$$

Then we have for $k \geq 1$

$$\begin{aligned}&A_{q_0}(q_{k+1}^* - q_k^*) = \\ &= -P\left[(A_{q_k^*} - A_{q_0})(q_k^* - q_{k-1}^*) - (A_{q_k^*} - A_{q_{k-1}^*})(q_{k-1}^* - q_0)\right];\end{aligned}$$

if $k = 0$, we use instead

$$A_{q_0}(q_1^* - q_0^*) = -P((A_{q_0^*} - A_{q_0})(q_0^* - q_0)).$$

These correspond to the formulae (2.6), (2.6'). Since the operator P is bounded, the same estimation procedure can be executed as in Lemma 2.1. So (2.11) holds and then $q_k^* \to q^* \in B_1$. Taking the limit in (3.11) we can verify again as in Lemma 2.2 that

$$A_{q_0}(q^* - q_0) = G_0 - P((A_{q^*} - A_{q_0})(q^* - q_0)) \tag{3.12}$$

,i.e.,

(3.13)
$$A_{q^*}(q^* - q_0) = G_0 + (I - P)((A_{q^*} - A_{q_0})(q^* - q_0)) = G_0 + cG_{00}$$

with some constant c. This shows that $0 \neq A_{q^*}(q^* - q_0) \in B_{21}$. From (3.12) and (3.9) we finally get $q^* - q_0 \in A_{q_0}^{-1}(\mathrm{Ker}\varphi)$. Lemma 3.2 is proved. $\square$

Proof of Theorem 1.4

Let $q \in L_p(0, \pi)$; our task is to find a different $q^* \in L_p(0, \pi)$, $q^* = q$ on $(0, a)$ such that the λ_n are common eigenvalues of q^* and q. This will be done by applying Lemma 3.2 with $B_1 = B_2 = L_p(a, \pi)$,

$$\varphi : L_p(a, \pi) \to \mathbf{C}, \quad \varphi(h) = \int_a^\pi A_q^{-1} h$$

and

$$B_{21} = \{h \in L_p(a, \pi) : \int_a^\pi h(x) \cos 2\sqrt{\lambda_n}(\pi - x)\, dx = 0 \ \forall n\}.$$

Now the condition (2.1) is given in (3.5), (2.2) follows from Lemma 5.3. In order to check $\dim B_{21} \geq 2$ recall the following identity (see Young [21], Chapter III.):
Let $\alpha(t)$ belong to $L_p(-d,d)$ and suppose that

$$f(z) = \int_{-d}^{d} \alpha(t) e^{izt} dt \quad \text{ satisfies } f(\mu) = 0.$$

Then for every $\lambda \neq \mu,\ \lambda \in \mathbf{C}$ there exists $\beta(t) \in L_p(-d,d)$ with

$$\frac{z-\lambda}{z-\mu} f(z) = \int_{-d}^{d} \beta(t) e^{izt} dt,$$

namely

$$\beta(t) = \alpha(t) + i(\lambda-\mu) e^{-i\mu t} \int_{-d}^{t} \alpha(s) e^{i\mu s} ds.$$

This can be verified by direct substitution. A repeated application of this idea gives that if $f(\pm\mu) = 0$ (or $f(0) = f'(0) = 0$ for $\mu = 0$), then for every $\lambda \neq \pm\mu$ there exists $\gamma(t) \in L_p(-d,d)$ with

$$\frac{z^2-\lambda^2}{z^2-\mu^2} f(z) = \int_{-d}^{d} \gamma(t) e^{izt} dt.$$

Supposing that $\alpha(t)$ is even, $\alpha(-t) = \alpha(t)$, we see that $f(z)$ and thus $\gamma(t)$ is even. In other words, $f(z) = \int_0^d 2\alpha(t) \cos zt\, dt$, $f(\mu^2) = 0$ implies

$$\frac{z^2-\lambda^2}{z^2-\mu^2} f(z) = \int_0^d 2\gamma(t) \cos zt\, dt.$$

Since $C(\Lambda)$ is not closed in $L_p(0,\pi-a)$, there exists $0 \neq h \in L_p(0,\pi-a)$ with

$$f(z) = \int_0^{\pi-a} h(t) \cos zt\, dt, \quad f(2\sqrt{\lambda_n}) = 0\ \forall n, \ \ f(2\mu) = 0.$$

Take any number $\mu_1 \neq \pm\mu,\ \mu_1 \neq \pm\sqrt{\lambda_n}$, then

$$\frac{z^2-4\mu_1^2}{z^2-4\mu^2} f(z) = \int_0^{\pi-a} h_1(t) \cos zt\, dt \text{ for some } h_1 \in L_p(0,\pi-a).$$

Consequently $h(\pi-t)$ and $h_1(\pi-t)$ are linearly independent elements of B_{21}, thus $\dim B_{21} \geq 2$ as asserted. Finally the minimality condition (M) implies by (3.3) that there exists a function $h \in L_p(a,\pi)$ satisfying

$$\int_a^{\pi} A_q h(x) \cos 2\sqrt{\lambda_n}(\pi-x)\, dx = 0 \quad \forall n \text{ but } \int_a^{\pi} h \neq 0.$$

Let $h_1 = A_q h$, then $h_1 \in B_{21} \setminus \mathrm{Ker}\varphi$ showing that $B_{21} \not\subset \mathrm{Ker}\varphi$. Thus all conditions formulated in Lemma 3.2 are fulfilled, so there exists $q^* \neq q$, $q^* \in L_p(a, \pi)$ such that

$$A_{q^*}(q^* - q) \in B_{21} \text{ and } A_q(q^* - q) \in \mathrm{Ker}\varphi \text{ i.e. } \int_a^\pi (q^* - q) = 0. \tag{3.14}$$

Define $F(z)$ corresponding to q^* and q. Putting together the formulae (2.16), (3.3) and (3.14) gives

$$\begin{aligned} F(z^2) &= \int_a^\pi (q^*(x) - q(x)) \left[v(x, z^2) v^*(x, z^2) - 1/2 \sin^2 \beta \right] dx = \\ &= \int_a^\pi \cos 2z(\pi - x) \left[A_{q^*}(q^* - q) \right](x) \, dx \end{aligned}$$

and then $F(\lambda_n) = 0$,i.e., the λ_n are common eigenvalues of q^* and q. The proof of Theorem 1.4 is complete. ☐

Proof of Theorems 1.5 and 1.6

To make explicit the dependence on the parameter β we denote by $v(x, \lambda, \beta)$ the solution of (1.9), (1.10). Let

$$F(x, z) = v'(x, z, \beta_1) v^*(x, z, \beta_2) - v(x, z, \beta_1) v^{*\prime}(x, z, \beta_2).$$

We have $F(0, \lambda_n) = 0$ by (1.14). The condition (1.16) means that $q^* = q$ a.e. on $(0, a)$ and then

$$F(\lambda_n) = 0 \text{ if } F(z) = F(a, z).$$

If the values λ_n have a finite accumulation point then $F(0, z) \equiv 0$ and $m^* = m$ follows. In this case $e_0(\Lambda)$ is also closed in $L_p(a - \pi, \pi - a)$. Indeed, if $G(\sqrt{\lambda_n}) = 0$ with $G(z) = \int_0^{\pi - a} h(x) \cos 2zx \, dx$ where $h \in L_p(0, \pi - a)$ then $G \equiv 0$ and $h = 0$. So in what follows we can suppose that $\lambda_{n_k} \to \infty$ for a subsequence. As in Lemma 2.2 we can verify that

$$F(z) = \int_a^\pi (q^*(x) - q(x)) v(x, z, \beta_1) v^*(x, z, \beta_2) \, dx + F(\pi, z) \tag{3.15}$$

where

$$F(\pi, z) = -\cos \beta_1 \sin \beta_2 + \cos \beta_2 \sin \beta_1 = \sin(\beta_1 - \beta_2).$$

Suppose first that

$$\sin \beta_1 \cdot \sin \beta_2 \neq 0. \tag{3.16}$$

Analogously as in (3.3) we can check by Lemma 5.3' that

$$\int_a^\pi h(x)\left[v(x,z^2,\beta_1)v^*(x,z^2,\beta_2)-1/2\sin\beta_1\sin\beta_2\right]\,dx = \tag{3.17}$$

$$= \int_a^\pi \cos 2z(\pi-x)B_{q^*}h(x)\,dx$$

where

$$B_{q^*}h(x) = \frac{\sin\beta_1\sin\beta_2}{2}h(x)+ \tag{3.18}$$

$$+\int_a^x h(\tau)2L(\pi-\tau,2(\pi-x),q,q^*,\beta_1,\beta_2)\,d\tau.$$

Consequently

$$F(z^2)=\sin(\beta_1-\beta_2)+1/2\sin\beta_1\sin\beta_2\int_a^\pi(q^*-q)+$$

$$+\int_a^\pi \cos 2z(\pi-x)B_{q^*}(q^*-q)(x)\,dx.$$

From $\lambda_{n_k}\to+\infty$, $F(\lambda_{n_k})=0$ it follows that

$$\sin(\beta_1-\beta_2)+1/2\sin\beta_1\sin\beta_2\int_a^\pi(q^*-q)=0$$

and then

$$0=F(\lambda_n)=\int_a^\pi \cos 2\sqrt{\lambda_n}(\pi-x)B_{q^*}(q^*-q)(x)\,dx \quad \forall n.$$

Since $C_0(\Lambda)$ is closed in $L_p(0,\pi-a)$, we infer $B_{q^*}(q^*-q)=0$,i.e., $F\equiv 0$, and hence $m_{\beta_1}\equiv m^*_{\beta_2}$.
Now consider the case

$$\sin\beta_1\cdot\sin\beta_2=0. \tag{3.19}$$

We see from (5.14) that for $\sin\beta=0$

$$v(\pi-x,z^2,\beta)=\mathbf{O}\left(\frac{1}{|z|}e^{|\Im z|x}\right)$$

uniformly in $z\in\mathbf{C},\ z\neq 0$ and $0\le x\le\pi-a$. Analogously from (5.25) we get for $\sin\beta\neq 0$

$$v(\pi-x,z^2,\beta)=\mathbf{O}\left(e^{|\Im z|x}\right).$$

This implies in (3.15) that

$$F(z^2)=\sin(\beta_1-\beta_2)+\mathbf{O}\left(\frac{1}{|z|}e^{|\Im z|x}\right).$$

Now from $F(\lambda_{n_k}) = 0$, $\lambda_{n_k} \to +\infty$ it follows that $\sin(\beta_1 - \beta_2) = 0$ and then $\sin\beta_1 = \sin\beta_2 = 0$. So (1.14) has the form

$$m_0(\lambda_n) = m_0^*(\lambda_n);$$

in other words the λ_n are common eigenvalues of q^* and q. In this case $m_0^* \equiv m_0$ (,i.e., $q^* = q$) follows if and only if $e(\Lambda)$ is closed in $L_p(a-\pi, \pi-a)$; this is stated in Theorem 1.1. The proofs of Theorems 1.5 and 1.6 are complete. □

4. Applications

This section is devoted to demonstrate how the formerly known theorems listed in Section 1 can be deduced from the new results. At the end of this section we provide the proof of Proposition 1.3.

Consider an arbitrary sequence $\{\mu_n : n \geq 1\}$ of different complex numbers satisfying

$$|\mu_n| \to \infty. \tag{4.1}$$

Define the counting function

$$n(r) = \sum_{|\mu_n| \leq r} 1 \quad \text{for} \quad r > 0 \tag{4.2}$$

and the function

$$N(r) = \int_1^r \frac{n(t)}{t}\, dt. \tag{4.3}$$

Recall the following classical closedness test of Levinson:

Theorem 4.1. [15], *see also Young* [21]. *Let* $0 \leq a < \pi$, $1 \leq p < \infty$, $1/p + 1/p' = 1$. *If*

$$\limsup_{r\to\infty}(N(r) - 2(1 - a/\pi)r + 1/p' \ln r) > -\infty \tag{4.4}$$

then the system $\{e^{i\mu_n x} : n \geq 1\}$ *is closed in* $L_p(a-\pi, \pi-a)$.

Remark. The original form of Theorem 4.1 in [21] refers to $1 < p$ and to the case $a = 0$ because it is formulated as a completeness test in $L_{p'}$ and this is equivalent to closedness in L_p only if $p > 1$. However the proof given in [21] works also for $p = 1$ and it can be transformed into the form (4.4).

Remark. We can easily extend Theorem 4.1 to those cases where there are values μ_n taken with multiplicities $1 < m_n < \infty$; in this case (4.2) is accordingly modified and the exponential system contains the members $e^{i\mu_n x}, xe^{i\mu_n x}, \ldots, x^{m_n-1}e^{i\mu_n x}$.

In applying Theorem 4.1 we need the following estimates for the N-function corresponding to a complete spectrum.

Lemma 4.2. *Denote by N_σ the N-function for the set $\{\pm 2\sqrt{\lambda_n} : \lambda_n \in \sigma(q,\alpha,\beta)\}$; if $\lambda_n = 0$ then the value 0 has multiplicity 2 in this set. Then, as $r \to +\infty$,*

$$\sin\alpha = \sin\beta = 0 \Rightarrow N_\sigma(r) = r - \ln r + \mathbf{O}(1), \tag{4.5}$$

$$\begin{aligned} &\sin\alpha = 0 \neq \sin\beta \text{ or } \sin\alpha \neq 0 = \sin\beta \\ &\Rightarrow N_\sigma(r) = r + \mathbf{O}(1), \end{aligned} \tag{4.6}$$

$$\sin\alpha \neq 0,\ \sin\beta \neq 0 \Rightarrow N_\sigma(r) = r + \ln r + \mathbf{O}(1). \tag{4.7}$$

Proof. Consider first the Dirichlet case (4.5). Recall the well-known eigenvalue asymptotics

$$\sqrt{\lambda_n} = n + \mathbf{O}(\frac{1}{n}), \quad n \geq 1 \tag{4.8}$$

see ,e.g., [17]. Let $n^{(1)}$ and $N^{(1)}$ be the corresponding functions if we substitute the values $\pm 2\sqrt{\lambda_n}$ by $\pm 2n$. From (4.8) it follows that

$$N_\sigma(r) - N^{(1)}(r) = \mathbf{O}(1). \tag{4.9}$$

We can count $N^{(1)}(r)$ for $2k \leq r \leq 2k+2$ as follows

$$\begin{aligned} N^{(1)}(r) &= \int_1^r \frac{n^{(1)}(t)}{t}\,dt = \sum_{i=2}^{k}(2i-2)[\ln(2i) - \ln(2i-2)] + \\ &+ 2k[\ln r - \ln(2k)] = 2k\ln r - 2\sum_{i=1}^{k}\ln(2i) = \\ &= 2k\ln r - 2k\ln 2 - 2\ln(k!) = 2k(\ln r - \ln 2) - 2k(\ln k - 1) - \\ &- \ln k + \mathbf{O}(1) = 2k\ln(\frac{r}{2k}) + 2k - \ln k + \mathbf{O}(1) = \\ &= 2k - \ln k + \mathbf{O}(1) = r - \ln r + \mathbf{O}(1). \end{aligned} \tag{4.10}$$

From (4.9) we get $N_\sigma(r) = r - \ln r + \mathbf{O}(1)$ as asserted.

In the second case (4.6) we start with the asymptotics

$$\sqrt{\lambda_n} = n + 1/2 + \mathbf{O}(\frac{1}{n+1}), \quad n \geq 0. \tag{4.11}$$

Define the functions $n^{(2)}$ and $N^{(2)}$ by the main term of (4.11). Taking into account $n^{(2)}(t) = n^{(1)}(t+1)$, $n^{(1)}(t) = t + \mathbf{O}(1)$ we obtain

$$N^{(2)}(r) = \int_1^r \frac{n^{(1)}(t+1)}{t}\,dt = \int_1^r \frac{n^{(1)}(t+1)}{t+1}\,dt + \int_1^r \frac{n^{(1)}(t+1)}{t(t+1)}\,dt$$
$$= N^{(1)}(r+1) + \int_1^r \frac{t+\mathbf{O}(1)}{t(t+1)}\,dt + \mathbf{O}(1) = r + \mathbf{O}(1).$$

With $N_\sigma(r) = N^{(2)}(r) + \mathbf{O}(1)$ this implies (4.6). Finally in case (4.7) we argue similarly starting from

$$\sqrt{\lambda_n} = n + \mathbf{O}(\frac{1}{n+1}), \quad n \geq 0 \tag{4.12}$$

and using $n^{(3)}(t) = n^{(1)}(t+2)$. The proof is complete. □

Remark. Let $d > 0$ and denote by N_d the N-function corresponding to the set $\{\pm 2nd : n \geq 1\}$. Then we have

$$N_d(r) = \frac{r}{d} - \ln r + \mathbf{O}(1). \tag{4.13}$$

Indeed, $n_d(t) = n^{(1)}(t/d)$ implies

$$N_d(r) = \int_1^r \frac{n^{(1)}(\frac{t}{d})}{t}\,dt = \int_{\frac{1}{d}}^{\frac{r}{d}} \frac{n^{(1)}(t)}{t}\,dt =$$
$$= N(\frac{r}{d}) + \mathbf{O}(1) = \frac{r}{d} - \ln r + \mathbf{O}(1).$$

Proof of Theorem C

Remark first that $\sigma(q,\alpha_1,\beta) \cap \sigma(q,\alpha_2,\beta) = \emptyset$. If $\sin\beta = 0$, we have two spectra of type (4.6) or one of type (4.6) and the other of type (4.5). Thus

$$N_{\Lambda_0}(r) \geq 2r - \ln r + \mathbf{O}(1)$$

if N_{Λ_0} is the N-function corresponding to the values $\pm 2\sqrt{\lambda}$ where λ runs over the eigenvalues from the two spectra. Adjoining the pair $\pm 2\mu$ we get

$$N_\Lambda(r) \geq 2r + \ln r + \mathbf{O}(1).$$

By Theorem 4.1 this implies that $e(\Lambda)$ is closed in $L_1(-\pi,\pi)$ and then the potential is determined by the two spectra. Now, if $\sin\beta \neq 0$, we have two spectra of type (4.7) or one of type (4.7) and one of type (4.6). Hence

$$N_{\Lambda_0}(r) \geq 2r + \ln r + \mathbf{O}(1)$$

so $e_0(\Lambda)$ is closed in $L_1(-\pi,\pi)$, thus the potential is again determined. □

Proof of Theorem D
This follows from the estimates:

$$\sin\beta \neq 0 \Rightarrow N_{\Lambda_0} = N_\sigma \geq r + \mathbf{O}(1),$$

$$\sin\beta = 0 \Rightarrow N_\Lambda = N_\sigma + 2\ln r + \mathbf{O}(1) \geq r + \ln r + \mathbf{O}(1).$$

□

Proof of Theorem E
Denote by $n_S(t)$ the n-function corresponding to the set $\{\pm 2\sqrt{\lambda_n} : \lambda_n \in S\}$. Then for large t

$$n_S(t) = 2\#\{\lambda_n \in S : \lambda_n \leq t^2/4\} \geq 4(1-a/\pi)\#\{\lambda_n \in \sigma : \lambda_n \leq t^2/4\} + \\ + 2a/\pi - 1 = 2(1-a/\pi)n_\sigma(t) + 2a/\pi - 1.$$

Hence

$$\begin{aligned} \sin\beta \neq 0 &\Rightarrow N_{\Lambda_0} \geq 2(1-a/\pi)r + (2a/\pi - 1)\ln r + \mathbf{O}(1), \\ \sin\beta = 0 &\Rightarrow N_\Lambda \geq 2(1-a/\pi)(r - \ln r) + 2\ln r \\ &\qquad + (2a/\pi - 1)\ln r + \mathbf{O}(1) = \\ &\qquad = 2(1-a/\pi)r + 2a/\pi \ln r + (2a/\pi - 1)\ln r + \mathbf{O}(1) \end{aligned}$$

which verifies Theorem E even after deleting $a/\pi - 1/2$ from the density condition on S. □

Proof of Theorem F

$$\begin{aligned} \sin\beta \neq 0 \Rightarrow N_{\Lambda_0} &\geq 2/3(N_{\sigma_1} + N_{\sigma_2} + N_{\sigma_3}) + \mathbf{O}(1) \geq \\ &\geq 2/3(3r + 2\ln r) + \mathbf{O}(1), \\ \sin\beta = 0 \Rightarrow N_\Lambda &\geq 2/3(N_{\sigma_1} + N_{\sigma_2} + N_{\sigma_3}) + 2\ln r + \mathbf{O}(1) \geq \\ &\geq 2/3(3r - \ln r) + 2\ln r + \mathbf{O}(1). \end{aligned}$$

□

Proof of Theorem H
By Theorem 1.5 it is enough to verify that (1.15) implies the closedness of $e_0(\Lambda)$ in $L_1(-\pi,\pi)$. Consider the N-function for the set $\{\pm 2\sqrt{\lambda_n} : n \geq 0\}$; we have to check that

$$N(r) - 2r \not\to -\infty \quad (r \to +\infty). \tag{4.14}$$

We shift the values $\lambda_n < \frac{n^2}{4}$ into $\frac{n^2}{4}$. This will diminish $N(r)$ and it is enough to prove (4.14) for the diminished N. But we can also shift the values $\lambda_n > \frac{n^2}{4}$ into $\frac{n^2}{4}$ since this will grow N by a bounded quantity. Indeed, the growth is at most

$$\sum_{n<r} 2\int_n^{2\sqrt{\lambda_n}} \frac{dt}{t} = \sum_{n<r} \ln \frac{4\lambda_n}{n^2} = \sum_{n<r} \ln\left(1 + \frac{4(\lambda_n - n^2/4)}{n^2}\right) = \mathbf{O}(1)$$

by (1.15). For the shifted system $\{\lambda_n = n^2/4 : n \geq 0\}$ we have $N_0(r) = 2r + \ln r + \mathbf{O}(1)$ hence $N(r) \geq 2r + \ln r + \mathbf{O}(1)$ and (4.14) follows. □

In order to check Theorem B we need a stability result of Riesz bases. By definition, a Riesz basis is an isomorphic image of an orthonormal basis of a Hilbert space. A famous result of Kadec [14] says that if λ_n are arbitrary real numbers with $|\lambda_n - n| \leq L < 1/4$ for all $n \in \mathbf{Z}$ then the system $\{e^{i\lambda_n x} : n \in \mathbf{Z}\}$ forms a Riesz basis in $L_2(-\pi, \pi)$. It has been previously known that the constant $1/4$ is best possible here; see ,e.g., Young [21]. Later on, S. A. Avdonin [2] realized that it is not necessary to impose the bound $L < 1/4$ for every individual shift $|\lambda_n - n|$; instead, it is enough to take this bound only for the average shifts in the following sense:

Theorem 4.3. [2] *Suppose that the shifts $\delta_n \in \mathbf{C}$ are bounded and the shifted exponents $\lambda_n = n + \delta_n$ are separated, i.e., $\inf_{n\neq m} |\lambda_n - \lambda_m| > 0$. If the average Kadec condition*

$$\limsup_{R\to\infty} \sup_{x\in\mathbf{R}} \frac{1}{R}\Big| \sum_{x<n+\Re\delta_n<x+R} \delta_n\Big| < \frac{1}{4} \tag{4.15}$$

holds then the shifted system $\{e^{i\lambda_n x} : n \in \mathbf{Z}\}$ forms a Riesz basis in $L_2(-\pi, \pi)$.

Proof of Theorem B. (in case $\sin\beta = 0$).
The sufficiency of two spectra is proved in Theorem C, we investigate the necessity. For the eigenvalues $\lambda_n^{(1)}$ of $\sigma(q, 0, 0)$ and $\lambda_n^{(2)}$ of $\sigma(q, \alpha_2, 0)$ we have

$$\sqrt{\lambda_n^{(1)}} = n + \mathbf{o}(1)\ (n \geq 1), \quad \sqrt{\lambda_n^{(2)}} = n + 1/2 + \mathbf{o}(1)\ (n \geq 0). \tag{4.16}$$

So the set of all values $\pm 2\sqrt{\lambda}$ is a $\mathbf{o}(1)$-perturbation of $\mathbf{Z} \setminus \{0\}$. Since the eigenvalues are different, this means that the shifted exponents are separated and (4.15) holds with $\limsup = 0$. Consequently $e_0(\Lambda)$ is a Riesz basis of codimension 1. Hence $e(\Lambda)$ is complete in L_2 and after deleting an arbitrary eigenvalue it becomes incomplete (of codimension 1). In other words, after the deletion it is not closed in L_2, thus it is neither closed in L_1. By Theorem 1.1 this proves Theorem B if $\sin\beta = 0$. □

Remark. The case $\sin\beta \neq 0$ can not be dealt with in this general framework. Roughly speaking, we have "half an eigenvalue" deficiency and excess in $e_0(\Lambda)$ and $e(\Lambda)$, respectively. This prevents us from applying Theorems 1.2 and 1.4. It would be possible to give ad hoc modifications, based on the special structure of the set of eigenvalues in order to cover this special case; we do not give the details.

Our last topic in this section is the proof of Proposition 1.3. We need the following elementary

Lemma 4.4. *In the domain $|w| > 1$ the function*

$$f(w) = w\sin\frac{\pi}{2}w + \frac{1}{w}\sin\frac{\pi}{2w}$$

has only real zeros.

Proof. Since $f(-w) = f(w)$, we can suppose $\Re w \geq 0$. From the well-known formula

$$|\sin(a+ib)| = \sqrt{\sin^2 a + \sinh^2 b} \tag{4.17}$$

we can easily check that

$$|\sin\frac{\pi}{2}w| > |\sin\frac{\pi}{2w}| \quad \text{if } |w| > 1,\ 0 \leq \Re w \leq 1, \tag{4.18}$$

hence $f(w)$ has no zeros in this domain. Indeed, if $w = x+iy$, $0 \leq x \leq 1$, $x^2+y^2 > 1$ then $\sin^2\frac{\pi}{2}x \geq \sin^2\frac{\pi}{2}\frac{x}{x^2+y^2}$, $\sinh^2\frac{\pi}{2}y \geq \sinh^2\frac{\pi}{2}\frac{y}{x^2+y^2}$ and equality cannot occur in both cases. Now consider the case $x = 1+\varepsilon$, $\varepsilon > 0$ being appropriately small. From

$$\sin\frac{\pi}{2}(1+\varepsilon) = 1 - \mathbf{O}(\varepsilon^2)$$

and (4.17) we get

$$|\sin\frac{\pi}{2}w| > |\sin\frac{\pi}{2w}| - \mathbf{O}(\varepsilon^2) \quad \text{if } x = 1+\varepsilon.$$

Consequently

$$|w\sin\frac{\pi}{2}w| > |\frac{1}{w}\sin\frac{\pi}{2w}| \quad \text{if } x = 1+\varepsilon. \tag{4.19}$$

Indeed, this is trivial if $|y|$ is large enough, and for other values y $|\sin\frac{\pi}{2w}|$ is not very small, so

$$\begin{aligned}|w\sin\frac{\pi}{2}w| &> (1+\varepsilon)\Big(|\sin\frac{\pi}{2w}| - \mathbf{O}(\varepsilon^2)\Big) > \\ &> |\frac{1}{w}\sin\frac{\pi}{2w}| + 2\varepsilon|\sin\frac{\pi}{2w}| - \mathbf{O}(\varepsilon^2) > |\frac{1}{w}\sin\frac{\pi}{2w}|.\end{aligned}$$

Hence f has no zeros on the line $x = 1 + \varepsilon$. We can simply check by (4.17) that

(4.20)
$$|w \sin \frac{\pi}{2} w| > |\frac{1}{w} \sin \frac{\pi}{2w}| \quad \text{if } x = 2k+1\,(k = 1, 2, \dots) \text{ and } y \in \mathbf{R}$$

and that for large $R > 0$

$$|w \sin \frac{\pi}{2} w| > |\frac{1}{w} \sin \frac{\pi}{2w}| \quad \text{if } |y| \geq R. \tag{4.21}$$

This means by Rouché's theorem that $f(w)$ has exactly one zero in each of the rectangles $[1+\varepsilon, 3] \times [-R, R]$ and $[2k+1, 2k+3] \times [-R, R]$ for $k \geq 1$ and no other zeros exist. These zeros must be real since $f(\overline{w}) = \overline{f(w)}$ and this completes the proof. $\square$

Proof of Proposition 1.3
On the interval $[\pi/2, \pi]$ we have $v(x, z) = \cos(\pi - x)\sqrt{z-1}$. Thus $v(\frac{\pi}{2}, z) = \cos\frac{\pi}{2}\sqrt{z-1}$, $v'(\frac{\pi}{2}, z) = \sqrt{z-1}\sin\frac{\pi}{2}\sqrt{z-1}$ and then in $[0, \pi/2]$

$$v(x,z) = \cos(\frac{\pi}{2} - x)\sqrt{z} \cdot \cos\frac{\pi}{2}\sqrt{z-1} - \\ - \frac{\sin(\frac{\pi}{2} - x)\sqrt{z}}{\sqrt{z}} \sqrt{z-1} \sin\frac{\pi}{2}\sqrt{z-1}.$$

Finally we get that

(4.22)
$$v(0,z) = \cos\frac{\pi}{2}\sqrt{z} \cdot \cos\frac{\pi}{2}\sqrt{z-1} - \frac{\sqrt{z-1}}{\sqrt{z}} \sin\frac{\pi}{2}\sqrt{z} \cdot \sin\frac{\pi}{2}\sqrt{z-1},$$

(4.23)
$$v'(0,z) = \sqrt{z}\sin\frac{\pi}{2}\sqrt{z} \cdot \cos\frac{\pi}{2}\sqrt{z-1} + \sqrt{z-1}\cos\frac{\pi}{2}\sqrt{z} \cdot \sin\frac{\pi}{2}\sqrt{z-1}.$$

Similarly

(4.24)
$$v^*(0,z) = \cos\frac{\pi}{2}\sqrt{z} \cdot \cos\frac{\pi}{2}\sqrt{z-1} - \frac{\sqrt{z}}{\sqrt{z-1}} \sin\frac{\pi}{2}\sqrt{z} \cdot \sin\frac{\pi}{2}\sqrt{z-1},$$

(4.25)
$$v^{*\prime}(0,z) = v'(0,z).$$

Consider the function

(4.26)

$$\begin{aligned} F(z) &= v'(0,z)v^*(0,z) - v(0,z)v^{*\prime}(0,z) = \\ &= v'(0,z)(v^*(0,z)-v(0,z)) = -v'(0,z)\frac{\sin\frac{\pi}{2}\sqrt{z}}{\sqrt{z}}\cdot\frac{\sin\frac{\pi}{2}\sqrt{z-1}}{\sqrt{z-1}}. \end{aligned}$$

Its real zeros are precisely the common eigenvalues of q and q^*. In order to find the zeros of $v'(0,z)$, consider the decomposition

(4.27)

$$\begin{aligned} v'(0,z) &= \sqrt{z}\sin\frac{\pi}{2}(\sqrt{z}+\sqrt{z-1}) - \\ &- (\sqrt{z}-\sqrt{z-1})\cos\frac{\pi}{2}\sqrt{z}\sin\frac{\pi}{2}\sqrt{z-1} = \\ &= \sqrt{z}\sin\pi\sqrt{z} + \Big[\sqrt{z}(\sin\pi\sqrt{z} - \sin\frac{\pi}{2}(\sqrt{z}+\sqrt{z-1})) - \\ &- (\sqrt{z}-\sqrt{z-1})\cos\frac{\pi}{2}\sqrt{z}\sin\frac{\pi}{2}\sqrt{z-1}\Big] = \\ &\overset{def}{=} g(z) + [h(z)]. \end{aligned}$$

From

$$\begin{aligned} &\sin\pi\sqrt{z} - \sin\frac{\pi}{2}(\sqrt{z}+\sqrt{z-1}) = \\ &= 2\sin\frac{\pi}{2}\frac{\sqrt{z}-\sqrt{z-1}}{2}\cdot\cos\frac{\pi}{2}\frac{3\sqrt{z}+\sqrt{z-1}}{2} \end{aligned}$$

we infer

$$\begin{aligned} &\sqrt{z}(\sin\pi\sqrt{z} - \sin\frac{\pi}{2}(\sqrt{z}+\sqrt{z-1})) = \\ &= \mathbf{O}\left(\sqrt{z}\frac{1}{\sqrt{z}}e^{\frac{\pi}{4}(3|\Im\sqrt{z}|+|\Im\sqrt{z-1}|)}\right) = \mathbf{O}\left(e^{\pi|\Im\sqrt{z}|}\right) \end{aligned}$$

and then

$$h(z) = \mathbf{O}\left(e^{\pi|\Im\sqrt{z}|}\right), \quad |z|\to\infty. \tag{4.28}$$

The (simple) zeros of the function $g(z)$ are $z = k^2,\ k = 0,1,\ldots$. We know that

$$|g(z)| \geq c\sqrt{|z|}e^{\pi|\Im\sqrt{z}|} \quad \text{if} \quad |z| = (N+1/2)^2,\ n\in\mathbf{N}$$

with $c > 0$ independent of z and N. Comparing this estimate with (4.28) we get from Rouché's theorem that $v'(0,z)$ has precisely $N+1$

zeros in the disk $|z| < (N + 1/2)^2$ and (again by Rouché's theorem) that the zeros satisfy

$$\sqrt{\lambda_n^{(1)}} = n + \mathbf{O}(\frac{1}{n+1}) \quad (n \geq 0,\ n \to \infty). \tag{4.29}$$

We have to check that these zeros are real. Apply trigonometric identities to obtain

$$\begin{aligned}
v'(0,z) = \sqrt{z}&\frac{\sin\frac{\pi}{2}(\sqrt{z}+\sqrt{z-1}) + \sin\frac{\pi}{2}(\sqrt{z}-\sqrt{z-1})}{2} + \\
&+ \sqrt{z-1}\frac{\sin\frac{\pi}{2}(\sqrt{z}+\sqrt{z-1}) - \sin\frac{\pi}{2}(\sqrt{z}-\sqrt{z-1})}{2} = \\
&= \frac{\sqrt{z}+\sqrt{z-1}}{2}\sin\frac{\pi}{2}(\sqrt{z}+\sqrt{z-1}) + \\
&+ \frac{\sqrt{z}-\sqrt{z-1}}{2}\sin\frac{\pi}{2}(\sqrt{z}-\sqrt{z-1}) = \\
&= 1/2 f(w), \quad w = \sqrt{z}+\sqrt{z-1};
\end{aligned}$$

here we used $(\sqrt{z}+\sqrt{z-1})(\sqrt{z}-\sqrt{z-1}) = 1$. By appropriately defining the square roots we can suppose $|w| = |\sqrt{z}+\sqrt{z-1}| \geq 1$. If $|w| = 1$, then $\overline{\sqrt{z}+\sqrt{z-1}} = \overline{w} = \frac{1}{w} = \sqrt{z}-\sqrt{z-1}$, hence $w + \overline{w} = 2\sqrt{z}$ is real, $w - \overline{w} = 2\sqrt{z-1}$ is purely imaginary. This means that $0 \leq z \leq 1$. If $|w| > 1$ and $f(w) = 0$ then by Lemma 4.4 the root $w = \sqrt{z}+\sqrt{z-1}$ is real. Then $\frac{1}{w} = \sqrt{z}-\sqrt{z-1}$ and hence $\sqrt{z}$ and $\sqrt{z-1}$ are also real ,i.e., $z \geq 1$. This shows indeed that $v'(0,z)$ has only real zeros. The other two factors in (4.26) have the zeros $\lambda_n^{(2)}$, $\lambda_n^{(3)}$ satisfying

$$\sqrt{\lambda_n^{(2)}} = 2n \quad (n \geq 1), \tag{4.30}$$

$$\sqrt{\lambda_n^{(3)}} = \sqrt{4n^2+1} = 2n + \mathbf{O}(\frac{1}{n}) \quad (n \geq 1). \tag{4.31}$$

So for the N-function of the sets of values $\pm 2\sqrt{\lambda_n}$ we get by (4.13) that

$$N^{(1)}(r) = r + \ln r + \mathbf{O}(1), \quad N^{(2)}(r),\, N^{(3)}(r) = \frac{r}{2} - \ln r + \mathbf{O}(1).$$

If $\mu \neq \pm\sqrt{\lambda_n^{(j)}}$, then the N-function of all values $\pm 2\sqrt{\lambda_n^{(j)}}$ and 2μ satisfies

$$N(r) = 2r + \mathbf{O}(1)$$

which means by the Levinson test that the system

$$e_1(\Lambda) = \{e^{2i\mu x}, e^{\pm 2i\sqrt{\lambda}x} : F(\lambda) = 0\}$$

is closed in $L_p(-\pi,\pi)$. On the other hand $e_0(\Lambda)$ cannot be closed by Theorem 1.2, so it has deficiency 1 as asserted. $\square$

5. Technical background

In this last part of the paper we give the auxiliary results used in the above proofs. More precisely we provide integral representations for the products of eigenfunctions and a connection between the closedness of cosine and exponential systems. The first result is a refinement of the known representation (5.1) below, see ,e.g., Marchenko [18].

Lemma 5.1. *Let $1 \le p \le \infty$, $0 < d < \infty$, $q \in L_p(-d,d)$ and consider the solution $e(x,\lambda)$ of the initial value problem*

$$-y'' + qy = \lambda^2 y \text{ on } (-d,d), \quad y(0) = 1, \quad y'(0) = i\lambda.$$

It has a representation of the form

$$e(x,\lambda) = e^{i\lambda x} + \int_{-x}^{x} K_1(x,t) e^{i\lambda t} dt \tag{5.1}$$

with a continuous kernel $K_1(x,t)$. If we have two potentials q^*, $q \in L_p(-d,d)$ with norm $\le D$ then

$$|K_1(x,t,q)| \le c(D), \tag{5.2}$$

$$|K_1(x,t,q^*) - K_1(x,t,q)| \le c(D)\|q^* - q\|_p \tag{5.3}$$

with a constant $c(D) = c(D,p,d)$ independent of q, q^*, x and t.

Proof. Define $H(\alpha,\beta) = K_1(\alpha+\beta,\alpha-\beta)$ for α, $\beta \ge 0$. Introduce the notations

$$\sigma(u) = \int_0^u |q|, \quad \varrho(u,v) = \int_0^u \int_0^v |q(\alpha+\beta)|\, d\beta\, d\alpha.$$

It is shown in Marchenko [18] that

$$H(u,v) = 1/2 \int_0^u q + \int_0^u \int_0^v q(\alpha+\beta) H(\alpha,\beta)\, d\beta\, d\alpha \tag{5.4}$$

and

$$|H(u,v)| \le 1/2\sigma(u) e^{\varrho(u,v)}. \tag{5.5}$$

From $\sigma(u) \le c(D)$, $\varrho(u,v) \le c(D)$ we get $|H(u,v)| \le c(D)$ which is (5.2). To show (5.3) consider the decomposition

$$H^*(u,v) - H(u,v) = 1/2 \int_0^u (q^* - q) + \tag{5.6}$$
$$+ \int_0^u \int_0^v (q^*(\alpha+\beta) - q(\alpha+\beta)) H^*(\alpha,\beta)\, d\beta\, d\alpha +$$
$$+ \int_0^u \int_0^v q(\alpha+\beta)(H^*(\alpha,\beta) - H(\alpha,\beta))\, d\beta\, d\alpha.$$

It implies

$$|H^*(u,v) - H(u,v)| \le c\|q^* - q\|_p + \tag{5.7}$$
$$+c(D) \int_0^u \int_0^v |q^*(\alpha+\beta) - q(\alpha+\beta)|\, d\beta\, d\alpha +$$
$$+ \int_0^u \int_0^v |q(\alpha+\beta)||H^*(\alpha,\beta) - H(\alpha,\beta)|\, d\beta\, d\alpha \le$$
$$\le c(D)\|q^* - q\|_p + \int_0^u \int_0^v |q(\alpha+\beta)||H^*(\alpha,\beta) - H(\alpha,\beta)|\, d\beta\, d\alpha.$$

Recall the following inequality of Wendroff (see ,e.g., in [3]): Let $c \ge 0$, $u(s,r) \ge 0$, $v(s,r) \ge 0$, u continuous, v locally integrable in the domain $r, s \ge 0$. Now if

$$u(x,y) \le c + \int_0^x \int_0^y v(r,s) u(r,s)\, ds\, dr, \quad x, y \ge 0 \tag{5.8}$$

then

$$u(x,y) \le c e^{\int_0^x \int_0^y v(r,s)\, ds\, dr}, \quad x, y \ge 0. \tag{5.9}$$

Applying this to (5.7) gives

$$|H^*(u,v) - H(u,v)| \le c(D)\|q^* - q\|_p e^{\int_0^u \int_0^v |q(\alpha+\beta)|\, d\beta\, d\alpha} \le \tag{5.10}$$
$$\le c(D)\|q^* - q\|_p$$

which is equivalent to (5.3). □

Our next topic is an integral representation for $v(x,\lambda)v^*(x,\lambda)$:

Lemma 5.2. *Let $\beta = 0$ in (1.10) and $\mu \in \mathbf{C}$, $1 \le p$, q^*, $q \in L_p(0,\pi)$. Then for $z \in \mathbf{C}$*

$$1 - 2(z^2 - \mu^2) v(\pi - x, z^2) v^*(\pi - x, z^2) = \tag{5.11}$$
$$= \cos 2zx + \int_0^{2x} \cos z\tau\, M(x,\tau,\mu^2)\, d\tau$$

where the kernel function $M(x,\tau,\mu^2)$ is linear in μ^2, continuous in (x,τ) and independent of z. Further if $q^{**}\in L_p$ and $\|q\|_p$, $\|q^*\|_p$, $\|q^{**}\|_p\le D$ then

$$|M(x,\tau,\mu^2,q,q^*)|\le c(D,\mu,p), \tag{5.12}$$

$$|M(x,\tau,\mu^2,q,q^{**})-M(x,\tau,\mu^2,q,q^*)|\le c(D,\mu,p)\|q^{**}-q^*\|_p. \tag{5.13}$$

Proof. It can be checked from (5.1) that there exists a continuous kernel $K(x,t)$, $0\le t\le x$ satisfying $K(x,0)=0$, the analogues of (5.2), (5.3) and

$$v(\pi-x,z^2)=\frac{\sin zx}{z}+\int_0^x K(x,t)\frac{\sin zt}{z}\,dt. \tag{5.14}$$

Indeed, define K_1 for the potential $q(\pi-x)$, then $K(x,t)=K_1(x,t)-K_1(x,-t)$ satisfies (5.14). Consequently

$$\begin{aligned}(5.15)\quad & 1-2z^2v(\pi-x,z^2)v^*(\pi-x,z^2)=\\ &=1-2\sin^2 zx-2\int_0^x K(x,t)\sin zx\sin zt\,dt-\\ &-2\int_0^x K^*(x,t)\sin zx\sin zt\,dt-\\ &-2\int_0^x\int_0^x K(x,t)K^*(x,u)\sin zt\sin zu\,du\,dt=\\ &=\cos 2zx-\int_0^x K(x,t)[\cos z(x-t)-\cos z(x+t)]\,dt-\\ &-\int_0^x K^*(x,t)[\cos z(x-t)-\cos z(x+t)]\,dt-\\ &-\int_0^x\int_0^x K(x,t)K^*(x,u)[\cos z(t-u)-\cos z(t+u)]\,du\,dt=\\ &=\cos 2zx-I_1-I_1^*-I_2.\end{aligned}$$

We have to check that I_1, I_1^* and I_2 have integral representations like in the right side of (5.11) with continuous kernels satisfying (5.12) and (5.13). In I_1

$$\int_0^x K(x,t)\cos z(x-t)\,dt=\int_0^x K(x,x-\tau)\cos z\tau\,d\tau,$$
$$\int_0^x K(x,t)\cos z(x+t)\,dt=\int_x^{2x} K(x,\tau-x)\cos z\tau\,d\tau$$

,i.e.,

$$I_1 = \int_0^{2x} \cos z\tau M_1(x,\tau)\,d\tau, \tag{5.16}$$

$$M_1(x,\tau) = \begin{cases} K(x,x-\tau) & \text{if } 0 \le \tau \le x \\ -K(x,\tau-x) & \text{if } x \le \tau \le 2x. \end{cases}$$

The kernel M_1 is continuous since $K(x,0)=0$ and the analogues of (5.12), (5.13) are also satisfied ((5.13) is trivial). In I_1^* we argue similarly. In I_2 we change the order of integrations:

$$\begin{aligned} &\int_0^x \int_0^x K(x,t)K^*(x,u)\cos z(t-u)\,du\,dt = \\ &= \int_0^x \int_{t-x}^t K(x,t)K^*(x,t-\tau)\cos z\tau\,d\tau\,dt = \\ &= \int_0^x \left(\int_\tau^x K(x,t)K^*(x,t-\tau)\,dt\right)\cos z\tau\,d\tau + \\ &+ \int_0^x \left(\int_0^{x-\tau} K(x,t)K^*(x,t+\tau)\,dt\right)\cos z\tau\,d\tau \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &\int_0^x \int_0^x K(x,t)K^*(x,u)\cos z(t+u)\,du\,dt = \\ &= \int_0^x \int_t^{x+t} K(x,t)K^*(x,\tau-t)\cos z\tau\,d\tau\,dt = \\ &= \int_0^x \left(\int_0^\tau K(x,t)K^*(x,\tau-t)\,dt\right)\cos z\tau\,d\tau + \\ &+ \int_x^{2x} \left(\int_{\tau-x}^x K(x,t)K^*(x,\tau-t)\,dt\right)\cos z\tau\,d\tau. \end{aligned} \tag{5.18}$$

Consequently

$$I_2 = \int_0^{2x} \cos z\tau M_2(x,\tau)\,d\tau \tag{5.19}$$

with the kernel

(5.20)

$$M_2(x,\tau) = \begin{cases} \int_\tau^x K(x,t)K^*(x,t-\tau)\,dt + \\ + \int_0^{x-\tau} K(x,t)K^*(x,t+\tau)\,dt - \\ - \int_0^\tau K(x,t)K^*(x,\tau-t)\,dt & \text{if } 0 \le \tau \le x, \\ - \int_{\tau-x}^x K(x,t)K^*(x,\tau-t)\,dt & \text{if } x \le \tau \le 2x \end{cases}$$

continuous also at $\tau = x$ by definition. Here (5.12) and (5.13) also follow from Lemma 5.1. In order to complete the proof of (5.19) we have to find an integral representation of $2\mu^2 v(\pi - x, z^2)v^*(\pi - x, z^2)$. We apply the identity

$$\int_0^t (t-\tau)\cos z\tau\, d\tau = \frac{1-\cos zt}{z^2} \tag{5.21}$$

and

$$\begin{aligned}
&v(\pi - x, z^2)v^*(\pi - x, z^2) = \frac{1-\cos 2zx}{2z^2}+\\
&+\int_0^x K(x,t)\frac{\cos z(x-t)-\cos z(x+t)}{2z^2}\,dt+\\
&+\int_0^x K^*(x,t)\frac{\cos z(x-t)-\cos z(x+t)}{2z^2}\,dt+\\
&+\int_0^x\int_0^x K(x,t)K^*(x,u)\frac{\cos z(t-u)-\cos z(t+u)}{2z^2}\,du\,dt\\
&=\frac{1}{2}I_3 + I_4 + I_4^* + I_5.
\end{aligned}$$

Now

$$I_3 = \int_0^{2x} (2x-\tau)\cos z\tau\, d\tau$$

and in I_4

$$\begin{aligned}
&\int_0^x K(x,t)\frac{1-\cos z(x+t)}{z^2}\,dt=\\
&=\int_0^x K(x,t)\int_0^{x+t}(x+t-\tau)\cos z\tau\,d\tau\,dt=\\
&=\int_0^{2x}\cos z\tau\left(\int_{\max(0,\tau-x)}^x K(x,t)(x+t-\tau)\,dt\right)d\tau,\\
&\int_0^x K(x,t)\frac{1-\cos z(x-t)}{z^2}\,dt=\\
&=\int_0^x \cos z\tau\left(\int_0^{x-\tau}K(x,t)(x-t-\tau)\,dt\right)d\tau.
\end{aligned}$$

The kernel arising here is zero at $\tau = x$, so it can be continuously extended to $x \le \tau \le 2x$. This proves an appropriate integral representation for I_4. The case of I_4^* is similar. Finally in I_5 we get by twofold interchange of integrations

$$\int_0^x \int_0^x K(x,t)K^*(x,u)\frac{1-\cos z(t+u)}{z^2}\,du\,dt =$$
$$= \int_0^{2x} \left(\int_{\max(\tau-x,0)}^x K(x,t) \int_{\max(\tau-t,0)}^x K^*(x,u)(t+u-\tau)\,du\,dt \right) \cdot$$
$$\cdot \cos z\tau\,d\tau,$$
$$\int_0^x \int_0^x K(x,t)K^*(x,u)\frac{1-\cos z(t-u)}{z^2}\,du\,dt =$$
$$= \int_0^x \cos z\tau \left(\int_\tau^x K(x,t) \int_0^{t-\tau} K^*(x,u)(t-u-\tau)\,du\,dt \right) d\tau +$$
$$+ \int_0^x \cos z\tau \left(\int_0^{x-\tau} K(x,t) \int_{t+\tau}^x K^*(x,u)(u-t-\tau)\,du\,dt \right) d\tau.$$

Since the last two kernels can be continuously extended by zero to the domain $x \le \tau \le 2x$ and the analogue of (5.12), (5.13) is again a trivial corollary of (5.2) and (5.3), the proof of Lemma 5.3 is complete. $\square$

A similar statement holds for $\sin\beta \neq 0$:

Lemma 5.3. *Let* $\sin\beta \neq 0$, $1 \le p$ *and* q, $q^* \in L_p(0,\pi)$, *then for* $z \in \mathbf{C}$

$$v(\pi-x,z^2)v^*(\pi-x,z^2) - 1/2\sin^2\beta = \tag{5.22}$$
$$= 1/2\sin^2\beta\cos 2zx + \int_0^{2x} L(x,t)\cos zt\,dt$$

with a kernel $L(x,t)$ *continuous in* (x,t). *Further if* $q^{**} \in L_p$ *and* $\|q\|_p$, $\|q^*\|_p$, $\|q^{**}\|_p \le D$ *then*

$$|L(x,t,q,q^*)| \le c(D,p), \tag{5.23}$$

$$|L(x,t,q,q^{**}) - L(x,t,q,q^*)| \le c(D,p)\|q^{**}-q^*\|_p. \tag{5.24}$$

Proof. From Lemma 5.1 we know that

$$v(\pi-x,z^2) = \sin\beta\cos zx + \int_0^x N(x,t)\cos zt\,dt \tag{5.25}$$

with a continuous kernel N satisfying the analogue of (5.2), (5.3). Indeed, if we define the kernel K_1 for the potential $q(\pi-x)$ then

$$N(x,t) = \sin\beta\frac{K_1(x,t)+K_1(x,-t)}{2} + \cos\beta\frac{K_1(x,t)-K_1(x,-t)}{2}.$$

Now

$$
\begin{aligned}
&v(\pi - x, z^2)v^*(\pi - x, z^2) - 1/2\sin^2\beta = \\
&= 1/2\sin^2\beta\cos 2zx + I_1\sin\beta + I_1^*\sin\beta + 2I_2, \\
&I_1 = \int_0^x N(x,t)\left(\cos z(x-t) + \cos z(x+t)\right)\,dt, \\
&I_1^* = \int_0^x N^*(x,t)\left(\cos z(x-t) + \cos z(x+t)\right)\,dt, \\
&I_2 = \int_0^x \int_0^x N(x,t)N^*(x,u)\left(\cos z(t-u) + \cos z(t+u)\right)\,du\,dt.
\end{aligned}
$$

As above we can check that

$$
I_1 = \int_0^{2x} \cos z\tau L_1(x,\tau)\,d\tau,\ L_1(x,\tau) = \begin{cases} N(x, x-\tau) & \text{if } 0 \le \tau \le x, \\ N(x, \tau - x) & \text{if } x \le \tau \le 2x; \end{cases}
$$

$$
I_2 = \int_0^{2x} \cos z\tau L_2(x,\tau)\,d\tau + \int_0^x \cos z\tau L_3(x,\tau)\,d\tau
$$

with

$$
\begin{aligned}
L_2(x,\tau) &= \int_{\max(\tau - x, 0)}^{\min(\tau, x)} N(x,t)N^*(x,\tau - t)\,dt, \\
L_3(x,\tau) &= \int_0^{x-\tau} N(x,t)N^*(x,t+\tau)\,dt + \int_\tau^x N(x,t)N^*(x,t-\tau)\,dt.
\end{aligned}
$$

Since L_3 can be continuously extended by zero to the domain $x \le \tau \le 2x$ and (5.12), (5.13) follows from Lemma 5.1, the proof is complete. □

After obvious modifications we can also prove

Lemma. 5.3' *Let* $\sin\beta_1 \ne 0$, $\sin\beta_2 \ne 0$ *and* q, $q^* \in L_1(0,\pi)$; *then for* $z \in \mathbf{C}$

$$
\begin{aligned}
&v(\pi - x, z^2, \beta_1)v^*(\pi - x, z^2, \beta_2) - 1/2\sin\beta_1\sin\beta_2 = \\
&\qquad = 1/2\sin^2\beta_1\sin\beta_2\cos 2zx + \int_0^{2x} \cos zt L(x,t,q,q^*,\beta_1,\beta_2)\,dt
\end{aligned}
\tag{5.26}
$$

with a kernel $L(x,t)$ *continuous in* (x,t).

Our final auxiliary result is a connection between the closedness of exponential systems and that of cosine systems.

Lemma 5.4. *Let* $z_n, n \ge 1$ *be arbitrary different complex numbers and let* $d > 0$, $1 \le p \le \infty$. *The system* $\{\cos z_n x : n \ge 1\}$ *is closed in* $L_p(0,d)$ *if and only if the system* $\{e^{\pm i z_n x} : n \ge 1\}$ *is closed in* $L_p(-d,d)$. *In case* $z_n = 0$ *we take* 1 *and* x *instead of* $e^{\pm i z_n x}$.

Proof. The only if part: if the cosine system is not closed in $L_p(0,d)$, then there exists $0 \neq h \in L_p(0,d)$ with

$$\int_0^d h(x)\cos z_n x\,dx = 0, \quad n \geq 1. \tag{5.27}$$

Define $h(-x) = h(x)$; then (5.27) implies

$$0 = \int_{-d}^d h(x)\cos z_n x\,dx = \int_{-d}^d h(x)e^{\pm i z_n x}\,dx$$

and in case $z_n = 0$ we also have $\int_{-d}^d h(x)x\,dx = 0$. Consequently $\{e^{\pm i z_n x} : n \geq 1\}$ is not closed in $L_p(-d,d)$.

The if part: if the exponential system is not closed then there exists a function $0 \neq h \in L_p(-d,d)$ with

$$0 = \int_{-d}^d h(x)e^{\pm i z_n x}dx, \quad n \geq 1. \tag{5.28}$$

Then

$$\begin{aligned} 0 &= \int_{-d}^d h(-x)e^{\pm i z_n x}dx, \\ 0 &= \int_{-d}^d (h(x)+h(-x))e^{\pm i z_n x}dx = 2\int_0^d (h(x)+h(-x))\cos z_n x\,dx \end{aligned}$$

and this proves that the cosine system is not closed unless h is odd. Now if h is odd, we get from (5.28) that

$$0 = \int_0^d h(x)\sin z_n x\,dx, \quad n \geq 1.$$

Integrating by parts gives

$$0 = z_n \int_0^d \cos z_n x \left(\int_x^d h\right)\,dx.$$

In other words, $0 \neq \int_x^d h \in L_p(0,d)$ is orthogonal to all functions $\cos z_n x$, $z_n \neq 0$. If $z_n = 0$, then

$$\begin{aligned} 0 = \int_{-d}^d xh(x)\,dx = 2\int_0^d xh(x)\,dx = 2\int_0^d \left(\int_x^d h\right)\,dx = \\ = 2\int_0^d \cos z_n x \left(\int_x^d h\right)\,dx \end{aligned}$$

so the cosine system is not closed in $L_p(0,d)$ and this was to be proved. $\square$

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