Formulas on hyperbolic volume*

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Abstract

This paper collects some important formulas on hyperbolic volume. To determine concrete values of volume function of polyhedra is a very hard question requiring the knowledge of various methods. Our goal is to give (in Section 3.3, Theorem 1) a new non-elementary integral on the volume of the orthoscheme (to obtain it without the Lobachevsky-Schläfli differential formula), using edge-lengthes as the only parameters.

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1 Introduction

In the first section we give certain formulas to some important coordinate systems and models, respectively.

Then we collect the classical results on the three-dimensional hyperbolic volume of J.Bolyai and N.I.Lobachevsky. The most famous volume-integral (depending on the dihedral angles of the orthosceme) discovered by N.I.Lobachevsky, is known and investigated worldwide, however it is not well-known that for this volume J.Bolyai also gave two integrals. He used as parameters both of the measure of the dihedral angles and the edges, respectively. We observed that there is no volume-formula by edge-lengthes as parameters. So as an application of our general formulas we compute such an integral (in Section 3.3, Theorem 1). We will use to this calculation the system of hyperbolic orthogonal coordinates.

Finally, we give a collection of some new interesting volume formulas of bodies discovered by contemporary mathematics showing that this old and hard problem is evergreen.

1.1 Notation

$\mathbb{R}^n$, $\mathbb{E}^n$, $\mathbb{H}^n$: The vector space of the real n-tuples, the n-dimensional Euclidean space and the n-dimensional hyperbolic space, respectively.

x_i: The i^{th} coordinate axis, and coordinate value with respect to the Cartesian coordinate system of $\mathbb{E}^n$ or the analogous hypercycle (hypersphere) coordinate system $\mathcal{H}$ of $\mathbb{H}^n$.

*Dedicated to the memory of János Bolyai on the 150^{th} anniversary of his death.

2 General formulas

In hyperbolic geometry, we have a good chance to get a concrete value of the volume function if we can transform our problem into either a suitable coordinate system or an adequate model of the space, respectively. In this section, we give volume-integrals with respect to some important system of coordinates. In our computation we also use the distance parameter k (used by J.Bolyai expressing the curvature $K = \frac{-1}{k^2}$ of the hyperbolic space in the modern terminology).

2.1 Coordinate system based on paracycles (horocycles)

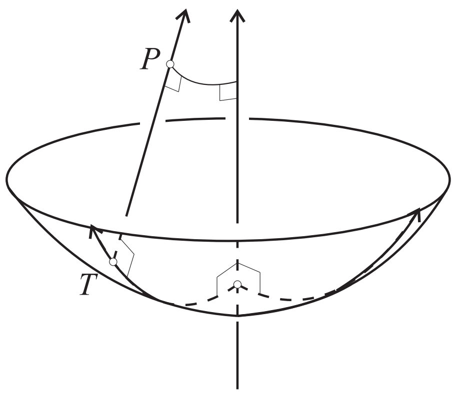

Figure 1: Coordinate system based on paracycles (paraspheres) of $\mathbb{H}^n$.

Consider in $\mathbb{H}^n$ a parasphere of dimension $n-1$ and its bundle of rays of parallel lines. Let ξ_n be the last coordinate axis, one of these rays, the origin will be the intersection of this line with the parasphere. The further $(n-1)$-"axes" are pairwise orthogonal paracycles. The coordinates of P in this system are $(\xi_1, \xi_2, \cdots, \xi_n)^T$, where the last coordinate is the distance of P and the parasphere, while the further coordinates are the coordinates of the orthogonal projection T with respect to the Cartesian coordinate system in $\mathbb{E}^{n-1}$ given by the above mentioned paracycles.

We can correspond to P a point p in $\mathbb{R}^n$ by the following Cartesian coordinates (T refers to transposed column matrix written into row one):

$$(x_1, x_2, \cdots, x_n)^T = \left(e^{-\frac{\xi_n}{k}}\xi_1, e^{-\frac{\xi_n}{k}}\xi_2, \cdots, e^{-\frac{\xi_n}{k}}\xi_{n-1}, \xi_n\right)^T.$$

By definition let the volume of a "Jordan measurable" set D in $\mathbb{H}^n$ be

$$v(D) := v_n \int\limits_{D^\star} \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where $D^\star$ in $\mathbb{R}^n$ is the image of domain D in $\mathbb{H}^n$ by the above mapping and v_n is a constant which we will choose later. Our first volume formula is:

$$v(D) = v_n \int\limits_{D} e^{-(n-1)\frac{\xi_n}{k}} \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n,$$

depending on the coordinates of the points of D, in the given paracycle system. Let now the domain $D = [0, a_1] \times \cdots \times [0, a_{n-1}]$ be a parasphere sector of parallel segments of length a_n based on a coordinate brick of the corresponding parasphere. Then we get by successive integration

$$v(D) = v_n \int\limits_0^{a_1} \cdots \left[\int\limits_0^{a_n} e^{-(n-1)\frac{\xi_n}{k}} \mathrm{d}\xi_n\right] \cdots \mathrm{d}\xi_1 = \frac{kv_n}{n-1}\left[-e^{-(n-1)\frac{a_n}{k}} + e^0\right] \prod_{i=1}^{n-1} a_i =$$

$$= \frac{kv_n}{n-1}[1 - e^{-(n-1)\frac{a_n}{k}}]\prod_{i=1}^{n-1} a_i.$$

If a_n tends to infinity and $a_i = 1$ for every $i = 1, \cdots, (n-1)$, then the volume is equal to $\frac{kv_n}{n-1}$. Note that J.Bolyai and N.I.Lobachevski used the value $v_n = 1$ only for $n = 2, 3$ so in their calculations the volume is independent of the dimension but depends on the constant k which determine the curvature of the space. To follow them we will determine the constant v_n such that for every fixed k the measure of a thin layer divided by its height tends to the measure of the limit figure of lower dimension. Now the limit:

$$\lim_{a_n \to 0} \frac{v(D)}{a_n} = \frac{kv_n}{n-1} \lim_{n\to\infty} \frac{[1 - e^{-(n-1)\frac{a_n}{k}}]}{a_n} \prod_{i=1}^{n-1} a_i = v_n \prod_{i=1}^{n-1} a_i,$$

would be equal to $v_{n-1} \prod\limits_{i=1}^{n-1} a_i$ showing that $1 = v_1 = v_2 = \ldots = v_n = \ldots$.

Thus $v_n = 1$ as indicated earlier. On the other hand if for a fixed n the number k tends to infinity the volume of a body tends to the euclidean volume of the corresponding euclidean body. In every dimension n we also have a k for which the corresponding hyperbolic n-space contains a natural body with unit volume, if k equal to $n-1$ then the volume of the paraspheric sector based on a unit cube of volume 1 is also 1.

Finally, with respect to paracycle coordinate system our volume function by definition will be

$$v(D) = \int_D e^{-(n-1)\frac{\xi_n}{k}} \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n.$$

2.2 Volume in the Poincare half-space model

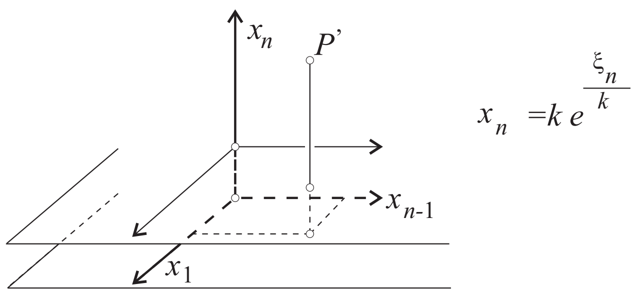

Figure 2: Coordinate system in the half-space model.

In the Poincare half-space model we consider a "Cartesian" coordinate system via Fig. 2. The first $n-1$ axes lie in the bounding hyperplane at infinity, the last one is perpendicular to it. If the hyperbolic coordinates of a point P (with respect to a paracycle coordinate system) is $(\xi_1, \xi_2, \cdots, \xi_n)^T$, correspond to P the point P' with coordinates:

$$(x_1, x_2, \cdots, x_n)^T = \left(\xi_1, \xi_2, \cdots, \xi_{n-1}, ke^{\frac{\xi_n}{k}}\right)^T.$$

The Jacobian determinant of this substitution is $\frac{k}{x_n}$ showing that

$$v(D) = k^n \int_{D'} \frac{1}{x_n^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where D' means the image of D via the mapping $\xi \to \mathbf{x}$.

2.3 Hyperbolic orthogonal coordinate system

Put an orthogonal system $\mathcal{H}$ of axes to the paracycle coordinate system, such that the new half-axes $x_1, \cdots, x_{n-1}$ are tangent half-lines at the origin to the former paracycles. (We can see the situation in Fig.3.) To determine the new coordinates of the point P we project P orthogonally to the hyperplane spanned by the axes $x_1, x_2, \cdots, x_{n-2}, x_n$. The projection will be P_{n-1}. Then we project orthogonally P_{n-1} onto the $(n-2)$-space spanned by axes $x_1, x_2, \cdots, x_{n-3}, x_n$. The new point is P_{n-2}. Now the $(n-1)^{th}$ coordinate is the distance of P and P_{n-1}, the $(n-2)^{th}$ one is the distance of P_{n-1} and P_{n-2} and so on .... In the last step we get the n^{th} coordinate which is the distance of the point P_1 (in axis x_n) from the origin O. Since the connection between the

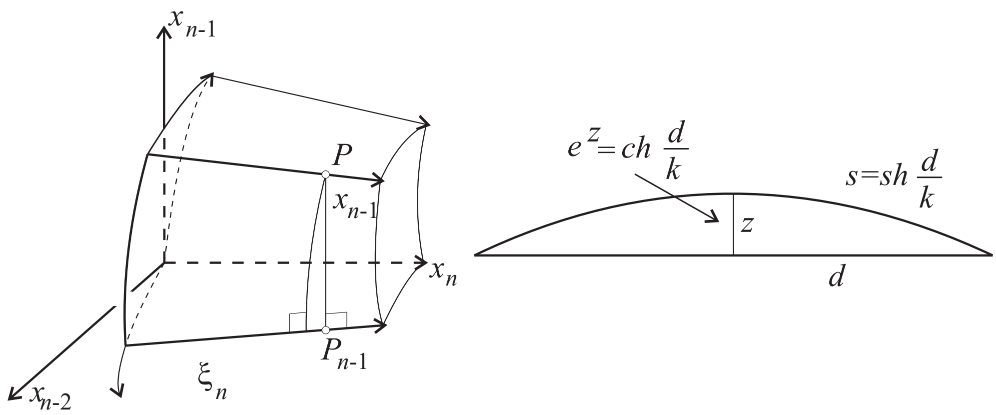

Figure 3: Coordinate system based on orthogonal axes ($n = 3$).

distance $2d$ of two points of a paracycle and the length of the connecting paracycle arc $2s$ is

$$s = k \sinh \frac{d}{k}.$$

Thus (Fig.3), the distance z of the respective halving points can be calculated as:

$$z = k \ln \cosh \frac{d}{k}.$$

Now a non-trivial but elementary calculation shows (using also the hyperbolic Pythagorean theorem) that the connection between the coordinates with respect to the two systems of coordinates is:

$$\begin{aligned}
\xi_{n-1} &= e^{\frac{\xi_n}{k}} k \sinh \frac{x_{n-1}}{k} \\
\xi_{n-2} &= e^{\frac{\xi_n}{k} + \ln \cosh \frac{x_{n-1}}{k}} k \sinh \frac{x_{n-2}}{k} \\
&\vdots \\
\xi_1 &= e^{\frac{\xi_n}{k} + \ln \cosh \frac{x_{n-1}}{k} + \cdots + \ln \cosh \frac{x_2}{k}} k \sinh \frac{x_1}{k} \\
x_n &= \xi_n + k \ln \cosh \frac{x_{n-1}}{k} + \cdots + k \ln \cosh \frac{x_2}{k} + k \ln \cosh \frac{x_1}{k}.
\end{aligned}$$

From this we get a new connection, corresponding point $(\xi_1, \xi_2, \cdots, \xi_n)^T \in \mathbb{H}^n$ to point $(u_1, \cdots, u_n)^T \in \mathbb{R}^n$ as in our first calculation in 2.1. The corresponding system of equations is:

$$
\begin{aligned}
u_1 &= k\cosh\frac{x_2}{k}\cdots\cosh\frac{x_{n-1}}{k}\sinh\frac{x_1}{k}\\
&\vdots\\
u_{n-1} &= k\sinh\frac{x_{n-1}}{k}\\
u_n &= x_n - k\ln\cosh\frac{x_1}{k} - \cdots - k\ln\cosh\frac{x_{n-1}}{k}
\end{aligned}
$$

The Jacobian determinant of this transformation is

$$\left(\cosh\frac{x_1}{k}\right)\left(\cosh\frac{x_2}{k}\right)^2\cdots\left(\cosh\frac{x_{n-1}}{k}\right)^{n-1},$$

and we get our third formula on the volume:

$$v(D) = \int\limits_D \left(\cosh\frac{x_{n-1}}{k}\right)^{n-1}\cdots\left(\cosh\frac{x_2}{k}\right)^2\left(\cosh\frac{x_1}{k}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

Here we have applied hyperbolic orthogonal coordinates.

2.4 Coordinate system based on hyperbolic spherical coordinates

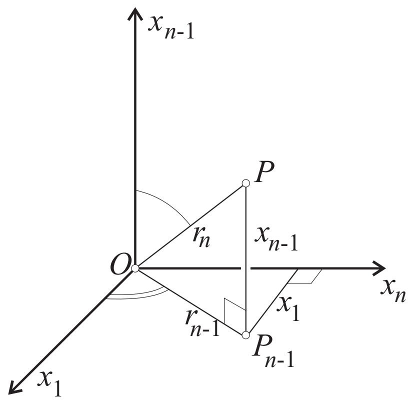

Figure 4: Spherical polar coordinates in $\mathbb{H}^n$ $(n = 3)$.

From the hyperbolic orthogonal coordinates $x_1, x_2, \cdots, x_n$ we get the spherical coordinates (ϕ_i and r_n) of a point P. Let the distance of $P = P_n$ from the origin is r_n and denote by ϕ_i the angle between the i^{th} coordinate axis and the segment OP_{i+1} for $i = n-1, n-2, \cdots 1$. (Here P_{n-1} is the orthogonal projection of P into the coordinate subspace of the axes x_1, x_{n-2}, x_n, and so on...) We have

$$\sinh\frac{x_{n-1}}{k} = \sinh\frac{r_n}{k}\cos\phi_{n-1}$$

by the hyperbolic Sin theorem. For general i, we get that

$$\cosh\frac{x_{n-1}}{k}\cdots\cosh\frac{x_{n-i+1}}{k}\sinh\frac{x_{n-i}}{k} = \sinh\frac{r_n}{k}\sin\phi_{n-1}\cdots\sin\phi_{n-i+1}\cos\phi_{n-i},$$

and by the Pythagorean theorem we can get a last equation:

$$\cosh\frac{x_{n-1}}{k}\cdots\cosh\frac{x_2}{k}\sinh\frac{x_1}{k} = \sinh\frac{r_n}{k}\sin\phi_{n-1}\cdots\sin\phi_2\cos\phi_1.$$

Straightforward computation shows that the volume of a sector domain D:

$$v(D) = k^{n-1}\int\limits_D\left(\sinh\frac{r_n}{k}\right)^{n-1}\sin^{n-2}\phi_{n-1}\cdots\sin\phi_2\mathrm{d}\phi_1\cdots\mathrm{d}\phi_{n-1}\mathrm{d}r_n.$$

2.5 Volume in the projective model

At the origin of the projective (Beltrami-Cayley-Klein) ball model we consider a Cartesian system of coordinates. Regarding the factor k we assume that the radius of the sphere is k. The considered system is an orthogonal coordinate system both of the embedding Euclidean and the modelled hyperbolic space.

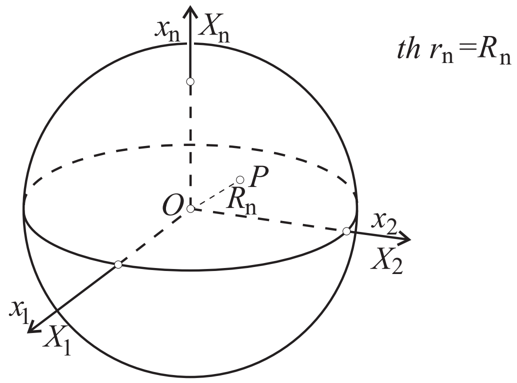

Figure 5: Coordinates in the Beltrami-Cayley-Klein model

We connect the hyperbolic spherical coordinates and the euclidean spherical ones by the system of equations:

$$r_n = k \tanh^{-1} \frac{R_n}{k}, \qquad \phi_i = \theta_i, \qquad i = 1, \cdots, n-1.$$

Since the Jacobian determinant is

$$\frac{1}{1 - \left(\frac{R_n}{k}\right)^2},$$

and

$$\left(\sinh \frac{r_n}{k}\right)^{n-1} = \left(\sinh\left(\frac{1}{2}\ln\frac{1+\frac{R_n}{k}}{1-\frac{R_n}{k}}\right)\right)^{n-1} = \left(\frac{\frac{R_n}{k}}{\sqrt{1-\left(\frac{R_n}{k}\right)^2}}\right)^{n-1},$$

the volume is:

$$v(D) = \int_D \frac{R_n^{n-1}}{\sqrt{1-\left(\frac{R_n}{k}\right)^2}^{n+1}} \sin^{n-2}\theta_{n-1}\cdots\sin\theta_2 \mathrm{d}\theta_1\cdots\mathrm{d}\theta_{n-1}\mathrm{d}R_n.$$

Transforming it into the usual Cartesian coordinates $(X_1, \cdots, X_n)$ the new formula is:

$$v(D) = \int_D \frac{1}{\left(1 - \left(\sum_{i=1}^{n}\left(\frac{X_i}{k}\right)^2\right)\right)^{\frac{n+1}{2}}} \mathrm{d}X_1\cdots\mathrm{d}X_{n-1}\mathrm{d}X_n.$$

For comparison with equivalent formulas see also [11].

3 The three dimensional case

3.1 Formulas of J. Bolyai

In this section k is a constant giving the curvature of the hyperbolic space and the value of our constant v_n is 1. The following formulas can be found in [1], [2] and [14]. Some of them can be easily determined using the results of the previous section. An important exception is the volume of the orthoscheme. We will give a new formula for it in section 3.3.

Equidistant body: Volume of the body determined by a disk of area p and the equal segments orthogonal to it with length q, respectively:

$$v = \frac{1}{8}pk\left(e^{\frac{2q}{k}} - e^{-\frac{2q}{k}}\right) + \frac{1}{2}pq = \frac{1}{4}pk\sinh\frac{2q}{k} + \frac{1}{2}pq.$$

Paraspherical sector: Volume of the sector of parallel half-lines intersecting orthogonally a paraspherical basic domain of area p:

$$\frac{1}{2}pk.$$

Ball: Volume of the ball of radius x:

$$\frac{1}{2}\pi k^3(e^{\frac{2x}{k}} - e^{\frac{-2x}{k}}) - 2\pi k^2 x = \pi k^3 \sinh\frac{2x}{k} - 2\pi k^2 x.$$

Barrel: Volume of the set of those points of $\mathbb{H}^3$, which distances from a fixed segment AB of length p is not greater then q:

$$\frac{1}{4}\pi k^2 p(e^{\frac{q}{k}} - e^{-\frac{q}{k}})^2 = \pi k^2 p \sinh^2\frac{q}{k}.$$

Orthoscheme: Volume of a special tetrahedron. Two edges a and b are orthogonal to each other and a third one c (skew to a) is orthogonal to the plane of a and b.

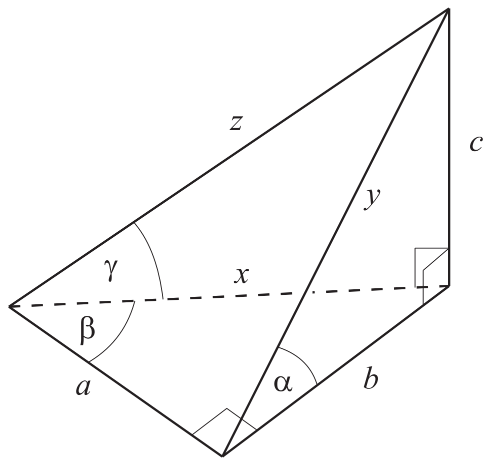

Figure 6: Orthoscheme

The dihedral angle at a is α, the angle opposite to b of the triangle with edges a and b is β and the angle opposite to the edge c in the triangle with edges c and z is γ, respectively (see Fig.6). J.Bolyai gave two formulas (with $k = 1$ in the following):

$$v = \frac{\tan\gamma}{2\tan\beta}\int_0^c \frac{u\sinh u}{\left(\frac{\cosh^2 u}{\cos^2\alpha} - 1\right)\sqrt{\frac{\cosh^2 u}{\cos^2\gamma} - 1}}\mathrm{d}u,$$

and

$$v=\frac{1}{2}\int_0^{\alpha}\left(-a+\frac{\sinh a\cos\phi}{2\sqrt{\tanh^2 b+\sinh^2 a\cos^2\phi}\ln\frac{\cosh a\cos\phi+\sqrt{\tanh^2 b+\sinh^2 a\cos^2\phi}}{\cosh a\cos\phi-\sqrt{\tanh^2 b+\sinh^2 a\cos^2\phi}}}\right)\mathrm{d}\phi$$

Asymptotic orthoscheme: Volume of the orthoscheme with ideal vertex (which is the common endpoint of the edges a, x and z) is:

$$v=\frac{\sin 2\alpha}{4}\int_0^{c}\frac{u}{\cosh^2 u-\cos^2\alpha}\mathrm{d}u$$

and

$$v=\frac{1}{2}\int_0^{\alpha}\ln\frac{\cos\phi}{\sqrt{\cos^2\phi-\tanh^2 b}}\mathrm{d}\phi.$$

Circular cone: Volume of a cone with a basic circle of radius b and with half-angle β at its apex is

$$v=\pi\int_0^{b}\frac{\sinh^2 y}{\cosh y\sqrt{\frac{\cosh^2 y}{\cos^2\beta}-1}}\mathrm{d}y.$$

Asymptotic circular cone: The apex B of a circular cone tends to the ideal point on its axis of rotation. Then the volume tends to

$$v=\pi\ln\cosh b.$$

3.2 Formulas of N.I.Lobachevsky

The formulas of this subsection can be found in [13] or [10].

Barrel-wedge: Barrel-wedge is a sector of a barrel intersected from it by two meridian-plane through its axis of rotation. Let T be the area of a meridian-intersection and p be the length of its parallel circular arcs. Then we have

$$v=\frac{1}{2}pT.$$

Orthoscheme: Let the essential (non-rectangular) dihedral angles of an orthoscheme be α, β and γ. They are admitted to the edges a, z and c, respectively. (See in Fig.6 with other β and γ). Introduce the parameter δ by the equalities:

$$\tanh\delta:=\tanh a\tan\alpha=\tanh c\tan\gamma,$$

and the Milnor's form of the Lobachevsky-function (see in [8]):

$$\Lambda(x)=-\int_0^{x}\ln|2\sin\xi|\mathrm{d}\xi,$$

respectively. Then the volume v of the orthoscheme is (k=1)

$$\frac{1}{4}\left[\Lambda(\alpha+\delta)-\Lambda(\alpha-\delta)-\Lambda\left(\frac{\pi}{2}-\beta+\delta\right)+\right.$$

$$\left.+\Lambda\left(\frac{\pi}{2}-\beta-\delta\right)+\Lambda(\gamma+\delta)-\Lambda(\gamma-\delta)+2\Lambda\left(\frac{\pi}{2}-\delta\right)\right].$$

A consequence of this formula (can be seen also in [8]) is the following one:

Since the opposite dihedral angles of the ideal tetrahedron are equal to each other so $A+B+C=\pi$, then its volume v is equal to

$$v=\Lambda(A)+\Lambda(B)+\Lambda(C),$$

where $\Lambda(x)=-\int\limits_0^x \ln|2\sin\zeta|\mathrm{d}\zeta$.

3.3 Once more again on the volume of orthoscheme

As an application of our general formulas we determine the volume of the orthosceme as the function of its edge-lengthes a, b and c. We note that there are formulas to transform the dihedral angles into the edge-lengthes. By the notation of the previous section these are (they are just for derivation of the former Lobachevsky-Schläfli volume differential, see [13] or [10]):

$$a=\frac{1}{2}\ln\frac{\sin(\alpha+\delta)}{\sin(\alpha-\delta)},\quad c=\frac{1}{2}\ln\frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)},\quad z=\frac{1}{2}\ln\frac{\sin(\frac{\pi}{2}-\beta+\delta)}{\sin(\frac{\pi}{2}-\beta-\delta)}.$$

3.3.1 The 3-dimensional case

The following lemma plays an important role in the n-dimensional investigation and immediately can be seen in the 3-dimensional one.

Lemma 1 *We have two k-dimensional hyperbolic subspaces H_k and H'_k, respectively for which they intersection has dimension $k-1$. Assume that the points $P\in H_k$, $P'\in H'_k$ and $P''\in H_k\cap H'_k$ hold the relations $PP'\perp H'_k$ and $P'P''\perp H_k\cap H'_k$, respectively. Then the angle*

$$\alpha=\tan^{-1}\frac{\tanh(PP')}{\sinh P'P''},$$

is independent from the position of P in H_k.

Let us we now follow another way for computation, we will determine the integral (in 2.3)

$$v(D)=\int\limits_D(\cosh z)^2(\cosh y)\mathrm{d}z\mathrm{d}y\mathrm{d}x=\int\limits_0^a\left[\int\limits_0^{\phi(x)}\left(\int\limits_0^{\psi(x,y)}(\cosh z)^2(\cosh y)\mathrm{d}z\right)\mathrm{d}y\right]\mathrm{d}x$$

getting it from hyperbolic orthogonal coordinates $\mathcal{H}$ using the parameter value $k=1$. The functions $\phi(x)$ and $\psi(x,y)$ can be determined as follows. Consider the orthoscheme in Fig.7. In the rectangular triangle $\triangle_{OP_2P_1}$ we know that the tangent of the angle $P_2OP_1\angle$ is:

$$\tan P_2OP_1\angle=\frac{\tanh b}{\sinh a}=\frac{\tanh\Phi(x)}{\sinh x}.$$

Hence

$$\tanh\Phi(x)=\frac{\tanh b}{\sinh a}\sinh x,$$

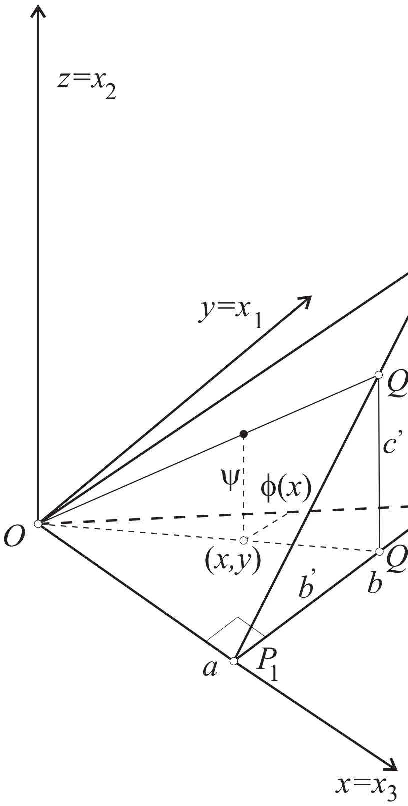

Figure 7: Orthoscheme and orthogonal coordinates

and

$$0 \leq y \leq \phi(x) = \tanh^{-1}\left(\frac{\tanh b}{\sinh a}\sinh x\right) =: \lambda.$$

Consider now the triangle $\triangle_{P_1P_2P}$. The line $O(x,y,0)$ intersects that point Q for which $|P_1Q| = b'$, and let denote the point of the segment PP_1 above Q be Q'. Thus we get the equality

$$\tanh c' = \frac{\tanh c}{\sinh b}\sinh b'.$$

Take into consideration again the equality

$$\tanh b' = \frac{\tanh y}{\sinh x}\sinh a,$$

and apply the hyperbolic Pythagorean theorem. From the triangle $\triangle_{OQQ'}$ we get

$$\tanh\Psi(x,y) = \tanh c'\left[\frac{\sinh\left(\cosh^{-1}(\cosh x\cosh y)\right)}{\sinh\left(\cosh^{-1}(\cosh a\cosh b')\right)}\right] =$$

$$= \frac{\tanh c}{\sinh b}\sinh b'\left[\frac{\sqrt{\cosh^2 x\cosh^2 y - 1}}{\sqrt{\cosh^2 a\cosh^2 b' - 1}}\right] =$$

$$= \frac{\tanh c}{\sinh b}\sinh b'\left[\frac{\sqrt{\sinh^2 y + \sinh^2 x\cosh^2 y}}{\sqrt{\sinh^2 b' + \sinh^2 a\cosh^2 b'}}\right] = \frac{\tanh c}{\sinh b}\sinh y\frac{\sqrt{1+\sinh^2 x\coth^2 y}}{\sqrt{1+\sinh^2 a\coth^2 b'}} =$$

$$= \frac{\tanh c}{\sinh b}\sinh y,$$

since

$$\tan QOP_2\angle = \frac{\tanh b'}{\sinh a} = \frac{\tanh y}{\sinh x}.$$

Hence the assumption

$$0 \leq z \leq \psi(x,y) = \tanh^{-1}\left(\frac{\tanh c}{\sinh b}\sinh y\right) =: \mu$$

holds if we fix the first two variables, but $\Psi(x,y)$ does not depend on x, as it can be expected for and exploited later. Thus the desired volume is:

$$v = \int_0^a \int_0^\lambda \int_0^\mu (\cosh z)^2 (\cosh y) \mathrm{d}z \mathrm{d}y \mathrm{d}x =$$

$$= \int_0^a \int_0^\lambda \frac{1}{2}\left[z + \frac{1}{2}(\sinh 2z)\right]_0^\mu (\cosh y)\mathrm{d}y\mathrm{d}x.$$

For $\Phi(x)$ and $\Psi(x,y)$ we apply the identities $\tanh\rho = \frac{\sinh\rho}{\cosh\rho} = \frac{e^{2\rho}-1}{e^{2\rho}+1}$, i.e. $\rho = \frac{1}{2}\ln\frac{1+\tanh\rho}{1-\tanh\rho}$. We get $\mu = \frac{1}{2}\ln\frac{\sinh b + \tanh c \sinh y}{\sinh b - \tanh c \sinh y}$, and $\lambda = \frac{1}{2}\ln\frac{\sinh a + \tanh b \sinh x}{\sinh a - \tanh b \sinh x}$. Hence

$$v = \frac{1}{4}\left\{\int_0^a \left(\left(\int_0^\lambda \ln\frac{\sinh b + \tanh c \sinh y}{\sinh b - \tanh c \sinh y}\cosh y \mathrm{d}y + \right.\right.\right.$$

$$\left.\left.\left. + \int_0^\lambda \sinh\left(\ln\frac{\sinh b + \tanh c \sinh y}{\sinh b - \tanh c \sinh y}\right)\cosh y\mathrm{d}y\right)\mathrm{d}x\right\}.$$

To determine the second integral, we apply $\sinh u = \frac{e^u - e^{-u}}{2}$. Now

$$\int_0^\lambda \sinh\left(\ln\frac{\sinh b + \tanh c \sinh y}{\sinh b - \tanh c \sinh y}\right)\cosh y\mathrm{d}y =$$

$$= \frac{1}{2}\int_0^\lambda \left(\frac{\sinh b + \tanh c \sinh y}{\sinh b - \tanh c \sinh y} - \frac{\sinh b - \tanh c \sinh y}{\sinh b + \tanh c \sinh y}\right)\cosh y\mathrm{d}y =$$

$$= 2\int_0^\lambda \frac{\sinh y \cosh y}{\frac{\sinh b}{\tanh c} - \frac{\tanh c}{\sinh b}\sinh^2 y}\mathrm{d}y = 2\int_0^\lambda \frac{\sinh 2y}{2\frac{\sinh b}{\tanh c} - \frac{\tanh c}{\sinh b}\cosh 2y + \frac{\tanh c}{\sinh b}}\mathrm{d}y =$$

$$= -\frac{\sinh b}{\tanh c}\left[\ln\left(2\frac{\sinh b}{\tanh c} - \frac{\tanh c}{\sinh b}\cosh 2y + \frac{\tanh c}{\sinh b}\right)\right]_0^\lambda =$$

$$= -\frac{\sinh b}{\tanh c}\ln\left(2\frac{\sinh b}{\tanh c} - \frac{\tanh c}{\sinh b}\cosh 2\lambda + \frac{\tanh c}{\sinh b}\right) + \frac{\sinh b}{\tanh c}\ln\left(2\frac{\sinh b}{\tanh c}\right).$$

From the above expression of $\lambda = \Phi(x)$ we can calculate $\cosh 2\lambda$ and get:

$$\cosh 2\lambda = \frac{1}{2}\left(\frac{\sinh a + \tanh b \sinh x}{\sinh a - \tanh b \sinh x} + \frac{\sinh a - \tanh b \sinh x}{\sinh a + \tanh b \sinh x}\right).$$

Thus the second integral (denoted by II) is:

$$\mathrm{II} := -\frac{\sinh b}{\tanh c}\ln\left(1-\frac{\tanh^2 c\sinh^2 x}{\cosh^2 b(\sinh^2 a-\tanh^2 b\sinh^2 x)}\right).$$

The first integral to v can be integrated by parts as follows:

$$\int\limits_0^\lambda \ln\left(\frac{\sinh b+\tanh c\sinh y}{\sinh b-\tanh c\sinh y}\right)\cosh y\mathrm{d}y = \left\{\left[\ln\left(\frac{\sinh b+\tanh c\sinh y}{\sinh b-\tanh c\sinh y}\right)\sinh y\right]_0^\lambda - \right.$$

$$\left.-\int\limits_0^\lambda \frac{\tanh c\cosh y[(\sinh b-\tanh c\sinh y)+(\sinh b+\tanh c\sinh y)]}{\sinh^2 b-\tanh^2 c\sinh^2 y}\sinh y\mathrm{d}y\right\} =$$

$$=\left\{\sinh\lambda\ln\frac{\sinh b+\tanh c\sinh\lambda}{\sinh b-\tanh c\sinh\lambda}-\int\limits_0^\lambda\frac{2\tanh c\sinh b\cosh y\sinh y}{\sinh^2 b-\tanh^2 c\cosh^2 y+\tanh^2 c}\mathrm{d}y\right\}=$$

$$=\left\{\sinh\lambda\ln\frac{\sinh b+\tanh c\sinh\lambda}{\sinh b-\tanh c\sinh\lambda}+\frac{\sinh b}{\tanh c}\left[\ln(\sinh^2 b-\tanh^2 c\cosh^2 y+\tanh^2 c)\right]_0^\lambda\right\}=$$

$$=\left\{\sinh\lambda\ln\frac{\sinh b+\tanh c\sinh\lambda}{\sinh b-\tanh c\sinh\lambda}+\frac{\sinh b}{\tanh c}\left(\ln(\sinh^2 b-\tanh^2 c\sinh^2\lambda)-\ln(\sinh^2 b)\right)\right\}.$$

Since

$$\sinh^2\lambda=\frac{\tanh^2 b\sinh^2 x}{\sinh^2 a-\tanh^2 b\sinh^2 x},$$

the first integral is:

$$\left\{\sinh\lambda\ln\frac{\sinh b+\tanh c\sinh\lambda}{\sinh b-\tanh c\sinh\lambda}+\frac{\sinh b}{\tanh c}\ln\left(1-\frac{\tanh^2 c\sinh^2 x}{\cosh^2 b(\sinh^2 a-\tanh^2 b\sinh^2 x)}\right)\right\}=$$

$$=\sinh\lambda\ln\frac{\sinh b+\tanh c\sinh\lambda}{\sinh b-\tanh c\sinh\lambda}-\mathrm{II}.$$

The sum of the two parts is:

$$\sinh\lambda\ln\frac{\sinh b+\tanh c\sinh\lambda}{\sinh b-\tanh c\sinh\lambda}.$$

A change of integration variable will have some benefits $x\mapsto\lambda$, $a\mapsto b$, $\mathrm{d}x=\frac{\mathrm{d}x}{\mathrm{d}\lambda}\mathrm{d}\lambda$. From $\lambda=\tanh^{-1}\left(\frac{\tanh b}{\sinh a}\sinh x\right)$ follows

$$x=\sinh^{-1}\left(\frac{\tanh\lambda\sinh a}{\sinh b}\right)=\ln\frac{\tanh\lambda\sinh a+\sqrt{\tanh^2\lambda\sinh^2 a+\tanh^2 b}}{\tanh b}$$

and we get in a straightforward way

$$v=\frac{1}{4}\int\limits_0^b\frac{\tanh\lambda\sinh a}{\sqrt{\tanh^2 b\cosh^2\lambda+\sinh^2 a\sinh^2\lambda}}\ln\left(\frac{\sinh b+\tanh c\sinh\lambda}{\sinh b-\tanh c\sinh\lambda}\right)\mathrm{d}\lambda,$$

proving our main theorem as follows:

Theorem 1 *Let the edges of an orthoscheme be a, b, c, respectively, where $a \perp b$ and $(a,b) \perp c$. If $k = 1$ then its volume is:*

$$v = \frac{1}{4}\int_0^b \frac{\tanh\lambda \sinh a}{\sqrt{\tanh^2 b \cosh^2\lambda + \sinh^2 a \sinh^2\lambda}} \ln\left(\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda}\right) \mathrm{d}\lambda.$$

Corollary: This formula can be simplified in the case of asymptotic orthoschemes. If the edge-length a tends to infinity, the function $\frac{\tanh\lambda \sinh a}{\sqrt{\tanh^2 b \cosh^2\lambda + \sinh^2 a \sinh^2\lambda}}$ tends to $\frac{1}{\cosh\lambda}$ showing that the volume of the orthosceme with one ideal vertex is

$$v = \frac{1}{4}\int_0^b \frac{1}{\cosh\lambda} \ln\left(\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda}\right) \mathrm{d}\lambda.$$

If the length of the edge c also grows to infinity, then this formula simplifies to:

$$v = \frac{1}{4}\int_0^b \frac{1}{\cosh\lambda} \ln\left(\frac{\sinh b + \sinh\lambda}{\sinh b - \sinh\lambda}\right) \mathrm{d}\lambda,$$

which is the volume of an orthosceme with two ideal vertices. If now we reflect this one in the face containing the edges b and c then we get a tetrahedron with three ideal vertices. If then we reflect the previous tetrahedron in the face containing the edges b and a we get another one with four ideal vertices. The volume of the last one is

$$v = \int_0^b \frac{1}{\cosh\lambda} \ln\left(\frac{\sinh b + \sinh\lambda}{\sinh b - \sinh\lambda}\right) \mathrm{d}\lambda.$$

This tetrahedron has two edges (a and c) which are skew and orthogonal to each other (its common normal transversal is b). Since the reflection in the line of b is a symmetry of this ideal tetrahedron, we can see that there are two types of its dihedral angles, two opposite (at the edges a and c) are equal to each other, (say A); and the other four ones are also equal to each other (say B). Then we have $A + 2B = \pi$, and its volume by Milnor's formula is equal to

$$v' = \Lambda(\pi - 2B) + 2\Lambda(B) = \Lambda(2B) + 2\Lambda(B) = 4\Lambda(B) + 2\Lambda\left(B + \frac{\pi}{2}\right).$$

(We have exploited that the Lobachevsky function is odd, of period π, and satisfies the identity $\Lambda(2B) = 2\Lambda(B) + 2\Lambda(B + \frac{\pi}{2})$.) Then we get the following connection between the two integrals:

$$0 = \int_0^b \frac{1}{\cosh\lambda} \ln\left(\frac{\sinh b + \sinh\lambda}{\sinh b - \sinh\lambda}\right) \mathrm{d}\lambda + 2\int_0^{B+\frac{\pi}{2}} \ln|2\sin\xi| \mathrm{d}\xi + 4\int_0^B \ln|2\sin\xi| \mathrm{d}\xi.$$

Remark: If we substitute into this formula the first-order terms of the Taylor series of the functions in the integrand, respectively, we get

$$v = \frac{1}{4}\int_0^b \frac{\tanh\lambda \sinh a}{\sqrt{\tanh^2 b \cosh^2\lambda + \sinh^2 a \sinh^2\lambda}} \ln\left(\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda}\right) \mathrm{d}\lambda =$$

$$= \frac{1}{2}\int_0^b \frac{\lambda a}{\sqrt{b^2 + a^2\lambda^2}} \frac{c\lambda}{b} \mathrm{d}\lambda = \frac{ac}{2b^2}\int_0^b \frac{\lambda^2}{\sqrt{1}} \mathrm{d}\lambda = \frac{abc}{6}.$$

This shows that our formula gives back the euclidean one for small values in a limit procedure.

3.3.2 The case of dimension two

The following calculation shows, of course, that the area of a rectangular triangle also can be obtained with our method.

$$\int_0^a \int_0^{\phi(x)} (\cosh y)\mathrm{d}y\mathrm{d}x = \int_0^a \frac{\frac{\tanh b}{\sinh a}\sinh x}{\sqrt{1-\left(\frac{\tanh b}{\sinh a}\sinh x\right)^2}}\mathrm{d}x =$$

$$= \frac{1}{2}\frac{\sinh a}{\tanh b}\int_0^{(\tanh b)^2} \frac{1}{\sqrt{1+\left(\left(\frac{\sinh a}{\tanh b}\right)^2-1\right)t-\left(\frac{\sinh a}{\tanh b}\right)^2 t^2}}\mathrm{d}t$$

At this point we can use the following result on antiderivative: if $y = ax^2 + bx + c$ and $a < 0$ then $\int \frac{1}{\sqrt{y}}\mathrm{d}t = \frac{1}{\sqrt{-a}}\sin^{-1}\frac{-y'}{\sqrt{b^2-4ac}}$. Thus the desired area is:

$$\frac{1}{2}\left[\sin^{-1}\frac{(2\sinh^2 a)t-\sinh^2 a+\tanh^2 b}{\sinh^2 a+\tanh^2 b}\right]_0^{\tanh^2 b} =$$

$$= \frac{1}{2}\left[\sin^{-1}(2\sin^2\beta\sinh^2 a-\cos 2\beta)-\sin^{-1}(-\cos 2\beta)\right].$$

Applying now the equalities $\sin\beta = \frac{\sinh b}{\sinh c}$, $\sin\alpha = \frac{\sinh a}{\sinh c}$ we get that $\frac{-\sinh^2 a+\tanh^2 b}{\sinh^2 a+\tanh^2 b} = -\cos(2\beta)$, $\frac{(2\sinh^2 a)\tanh^2 b}{\sinh^2 a+\tanh^2 b} = 2\sin^2\beta\sinh^2 a = 2\frac{\sinh^2 b}{\sinh^2 c}\sinh^2 a$ and $1 = \frac{\sinh^2 b\sinh^2 a}{\sinh^2 c} + \frac{\sinh^2 b+\sinh^2 a}{\sinh^2 c}$. Now our formula simplifies to the form

$$\frac{1}{2}\left[\sin^{-1}(2-2\frac{\sinh^2 b+\sinh^2 a}{\sinh^2 c}+\sin^2\beta-\cos^2\beta)+(\frac{\pi}{2}-2\beta)\right] = \pi-\left(\alpha+\beta+\frac{\pi}{2}\right),$$

as we stated.

3.3.3 The case of dimension n

Let now our orthosceme is the convex hull of the vertices $0 = A_0, A_1, \dots, A_n$ placed into a hyperbolic orthogonal coordinate system as we did it in the three dimensional case. More precisely, the coordinates of the vertices are $(0,\dots,0)^T$, $(0,\dots,0,a_n)^T$, $(a_1,0,\dots,0,a_n)^T$, $(a_1,a_2,0\dots,0,a_n)^T$ $\dots$ $(a_1,a_2,\dots,a_n)^T$, respectively. Introduce the function giving the upper boundary of the successive integrals. These are $\phi_0 = a_n$, $\phi_1 : x_n \mapsto x_1$ for a point $(x_1,0,\dots,0,x_n)^T$ of the edge $\operatorname{conv}(OA_2)$, $\phi_2 : (x_n,x_1) \mapsto x_2$ on the points $(x_1,x_2,0\dots,x_n)^T$ of the triangle $\operatorname{conv}(OA_2A_3)$. In general $\phi_k : (x_1,\dots,x_{k-1},0,\dots,0,x_n) \mapsto x_k$ if the corresponding point $(x_1,x_2\dots x_{k-1},x_k,0,\dots,0,x_n)^T$ is on the $k-1$-face $\operatorname{conv}(OA_2A_3\cdots A_{k+1})$ and so on... From Lemma 1 we get that

$$\frac{\tanh\phi_{k+1}(x_n,x_1,\dots,x_k)}{\sinh x_k} = \frac{\tanh a_{k+1}}{\sinh a_k},$$

implying that

$$\phi_{k+1}(x_n,x_1,\dots,x_k) = \tanh^{-1}\left(\frac{\tanh a_{k+1}}{\sinh a_k}\sinh x_k\right).$$

In the case of $k = 1$, the volume can be determined by the following n-times integral: $v(O) =$

$$\int_0^{a_n}\int_0^{\tanh^{-1}\left(\frac{\tanh a_1}{\sinh a_n}\sinh x_n\right)}\cdots\int_0^{\tanh^{-1}\left(\frac{\tanh a_{n-1}}{\sinh a_{n-2}}\sinh x_{n-1}\right)}$$

$$(\cosh^{n-1}x_{n-1})(\cosh^{n-2}x_{n-2})\cdots(\cosh x_1)\mathrm{d}x_{n-1}\cdots\mathrm{d}x_1\mathrm{d}x_n.$$

3.4 Further results on concrete volumes

Here we mention some important results from the last few decades, which are related to hyperbolic volumes. Considering them we can see immediately that our formula has a relative simple structure. In this section the capital latin letters mean the measures of the dihedral angles at the edges denoted by the corresponding small ones.

Theorem 2 (R.Kellerhals [6]) *Let R be a Lambert cube with essential angles w_k at 3 skew edges, $0 \leq w_k \leq \frac{\pi}{2}, k = 0, 1, 2$. Then the volume $V(R)$ of R is given by*

$$V(R) = \frac{1}{4}\left\{\sum_{0}^{2}(\Lambda(w_i + \theta) - \Lambda(w_i - \theta)) - \Lambda(2\theta) + 2\Lambda\left(\frac{\pi}{2} - \theta\right)\right\}$$

with

$$0 < \theta = \tan^{-1}\frac{\sqrt{\cosh^2 V_1 - \sin^2 w_0 \sin^2 w_2}}{\cos w_0 \cos w_2} \leq \frac{\pi}{2}.$$

Theorem 3 (Y. Mohanty [9]) *Let O be an ideal symmetric octahedron with all vertices on the infinity. Then $C = \pi - A$, $D = \pi - B$, $F = \pi - E$ and the volume of O is:*

$$v = 2\left(\Lambda\left(\frac{\pi + A + B + E}{2}\right) + \Lambda\left(\frac{\pi - A - B + E}{2}\right) + \Lambda\left(\frac{\pi + A - B - E}{2}\right) + \right.$$
$$\left. + \Lambda\left(\frac{\pi - A + B - E}{2}\right)\right).$$

Theorem 4 (D. Derevin and A.Mednykh [5]) *The volume of the hyperbolic tetrahedron $T = T$ (A,B,C,D,E,F) is equal to*

$$Vol(T) = \frac{-1}{4}\int_{z_1}^{z_2} \log \frac{\cos\frac{A+B+C+z}{2}\cos\frac{A+E+F+z}{2}\cos\frac{B+D+F+z}{2}\cos\frac{C+D+E+z}{2}}{\sin\frac{A+B+D+E+z}{2}\sin\frac{A+C+D+F+z}{2}\sin\frac{B+C+E+F+z}{2}\sin\frac{z}{2}}\mathrm{dz},$$

where z_1 and z_2 are the roots of the integrand, given by

$$z_1 = \tan^{-1}\frac{k_2}{k_1} - \tan^{-1}\frac{k_4}{k_3}\ ,\ z_2 = \tan^{-1}\frac{k_2}{k_1} + \tan^{-1}\frac{k_4}{k_3},$$

with

$$k_1 = -(\cos S + \cos(A + D) + \cos(B + E) + \cos(C + F) + \cos(D + E + F) +$$
$$\cos(D + B + C) + + \cos(A + E + C) + \cos(A + B + F)),$$
$$k_2 = \sin S + \sin(A + D) + \sin(B + E) + \sin(C + F) + \sin(D + E + F) +$$
$$\sin(D + B + C) + \sin(A + E + C) + \sin(A + B + F),$$
$$k_3 = 2(\sin A \sin D + \sin B \sin E + \sin C \sin F),$$
$$k_4 = \sqrt{k_1^2 + k_2^2 - k_3^2},$$

and $S = A + B + C + D + E + F$.

The above theorem implies the theorem of Murakami and Yano:

Theorem 5 (J.Murakami, M.Yano [12]) *The volume of the simplex T is*

$$v = \frac{1}{2}\Im\left(U(z_1, T) - U(z_2, T)\right),$$

where $\Im$ means the imaginary part and

$$U(z,T) = \frac{1}{2}(l(z) + l(A+B+D+E+z) + l(A+C+D+F+z) + l(B+C+E++F+z) -$$

$$-l(\pi+A+B+C+z) - l(\pi+A+E+F+z) - l(\pi+B+D+F+z) - l(\pi+C+D+E+z),$$

and $l(z) = \mathrm{Li}_2(e^{iz})$ by the Dilogarithm function

$$\mathrm{Li}_2(z) = -\int_0^x \frac{\log(1-t)}{t}\mathrm{dt}.$$

Finally, I mention the theorem of Y.Cho and H. Kim describing the volume of a tetrahedron using Lobachevsky function, too. This is also a very complicated formula, the reader can find it in [4].

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