Random Disease on the Square Grid

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Abstract. We introduce some generalizations of a nice combinatorial problem, the central notion of which is the so-called *Disease Process.* Let us color independently each square of an $n \times n$ chessboard black with a probability $p(n)$, this is a random initial configuration of our process. Then we have a deterministic painting or expansion rule, and the question is the behaviour of the disease process determined by this rule of spreading. In particular, how large $p(n)$ must be for painting the whole chessboard black? The main result of this paper is the almost exact determination of the threshold function in the fundamental case of this *Random Disease Problem.* Further investigations are involved about the general randomized and deterministic cases.

1. Introduction and main results

The following nice exercise might be well-known:

Each square of an 8×8 chessboard can have one of two states: "clean" or "weedy". We have some originally weedy squares, and each square of the board can change its state in time according to the following expansion rule: a weedy square remains weedy forever, and if previous day a square was clean then it stays clean if and only if at most one of its neighbors was weedy. (Adjacency among the squares is defined by having a common edge.) What is the minimum number of weedy squares one needs to make the whole chessboard weedy? Unfortunately, we do not know the origin of this simple combinatorial puzzle.

Our goal is to consider some generalizations of this elementary problem.

We call an $n \times n$ chessboard with each of its squares with a specific state a *configuration.* In the later parts of the paper we will refer to weedy squares as "black squares" and to clean squares as "white squares"; so a configuration is just a colored board. We start with an *initial configuration* and a *painting rule* that tells us which white squares become black next day. The expansion rule in the original problem was the *2-neighbor rule*: a white square becomes black iff it has at least 2 black neighbors.

The initial configuration and the rule define a finite or infinite sequence of configurations. We call this sequence a *disease process*. The initial configuration is called *contagious* or *successful* if the corresponding sequence has the all-weedy board as a member.

Now let $G(n)$ be the minimal number of weedy (or black) squares in a contagious configuration. The solution of the initial exercise is the following:

Fact. [Folklore] $G(n) = n$.

Proof. If we paint the squares of a diagonal black, it will be a contagious configuration, so $G(n) \leq n$. For a lower bound we can use the so-called *invariant method*: a suitable invariant is the perimeter of the black part of the board, which can never increase if we use our 2-neighbor painting rule. If we succeed in painting the whole chessboard black, we will have a perimeter $4n$, so we need at least n black squares at the beginning. ∎

One can easily generalize the problem to the k-dimensional $n \times \ldots \times n$ chessboard with an l-neighbor painting rule, where $1 \leq l \leq 2k$. This case was considered in [P], published in Hungarian. These results and the main ideas used are summarized in *Section 6, Appendix.*

Another way to modify the original problem is considering a random initial configuration and investigate the random process we obtain. A natural problem is as follows:

Let us color each square black independently with probability $p = p(n)$. We say that the configuration we obtain in this way is *p-random*. "Being contagious" is a monotone set property if the configurations are considered as ordered sets of black squares, and it is clear that the probability

$$Q(p, n) = \Pr[\text{a } p\text{-random initial configuration is contagious}]$$

is a strictly monotone increasing function of p. So it makes sense to define the critical probability $P(n)$ as $Q(P(n), n) = 1/2$, and the classical result of [BT] says that this $P(n)$ is a real threshold function: if $p(n)/P(n) \to 0$ then $Q(p(n), n) \to 0$, and if $p(n)/P(n) \to \infty$ then $Q(p(n), n) \to 1$. We call the problem of determining $P(n)$ and other properties of $Q(p, n)$ the *Random Disease Problem.*

Of course one can easily extend this problem to the k-dimensional board with an l-neighbor expansion rule. The corresponding threshold function is denoted by $P_{k,l}(n)$, so $P(n) = P_{2,2}(n)$.

The main result of this paper is giving almost exact bounds on $P(n)$, namely we prove that

Theorem 1. *If $\epsilon > 0$ is arbitrary and n is large enough then*

$$\frac{1}{200e^2 \ln n} < P(n) < \frac{(\log^* n)^{1+\epsilon}}{\ln n}, \tag{1}$$

where $\log^* n$ *is a rather slowly growing function, i.e. it denotes the minimum number* k *such that for the sequence* $a_1 = 2$ *and* $a_{i+1} = 2^{a_i}$ *the inequality* $a_k \geq n$ *holds.*

Our notions can be easily extended for the infinite board. Configurations and different expansion rules are the same in this case. A configuration is *contagious* iff every square becomes black at some point of the disease process, and *strongly contagious* iff the sequence of the configurations contains the all-black plane. The p-random initial configuration is the same as above: each square is painted black with probability p, independently from each other. We can speak about a $p(x, y)$-random configuration, as well: we fix a coordinate system with axes x and y ($x, y \in \mathbf{Z}$), and the square with coordinates (x, y) is chosen to be black with probability $p(x, y)$.

The fact that $P(n) \to 0$ as $n \to \infty$ means that our disease depends not only on the local properties of a configuration, on a bounded neighborhood of the squares. So an easy corollary of our main theorem can be formulated as follows:

Theorem 2.

(a) *Let* $\widehat{P}(n)$ *be arbitrary with* $\widehat{P}(n)/P(n) \to \infty$. *Then the* $p(x, y)$*-random initial configuration is contagious with probability* 1, *where* $p(x, y) = \widehat{P}(\|(x, y)\|)$ *and* $\|(x, y)\| = \max\{|x|, |y|\}$. *Thus the* p*-random configuration is almost surely contagious for any* $p > 0$ *fixed.*

(b) *Starting with a p-random initial configuration (*$p > 0$ *fixed) the time* $t(p)$ *needed for the complete painting of the plane is almost surely infinite, i.e. the probability that a p-random configuration is strongly contagious is 0.*

Lots of other questions can be asked about our Disease Process, some of them, together with a conjecture generalizing *Theorem 1*, we discuss in *Section 5*.

During the paper 'w.h.p. (with high probability)' will mean that 'with a probability tending to 1'. We also remark that we may and will assume, whenever this is needed, that n is sufficiently large.

2. Proof of the Lower Bound on $P(n)$

In the disease process a black square remains black forever. Hence in the case of a finite board our sequence of the configurations will be constant after a certain time. We call this configuration the *final configuration.* What can be the final configuration? Obviously, the black squares of the final configuration can be partitioned into groups, such that the squares in each group form a rectangle, and the rectangles of different groups are *far* from each other in the sense that no square is neighbored with two rectangles. In particular, two rectangles will be far if they can be separated by a strip consisting of two neighboring columns or rows of the board. We call this type of strips *width 2 strips.*

In some sense the reverse is also true: if the initial black squares can be covered by rectangles that are pairwise separated by width 2 strips then during the disease process all the black squares stay in the covering rectangles. This can be easily proved by induction on the time of the process.

So for the lower bound we will define a $p(n)$ such that we will be able to show that with high probability the black squares can be covered by rectangles, in such a way that any two rectangles can be separated from each other with width 2 strips.

Let us divide the $n \times n$ chessboard into smaller rectangular pieces, *subboards*, of size $(L \text{ or } L+1) \times (L \text{ or } L+1)$, where $L = \lfloor c_1 \ln n \rfloor$, c_1 will be determined later. We will have $\lfloor n/L \rfloor^2$ many subboards.

First we show that with an appropriate choice for $p = p(n)$ the p-random initial configuration contains only $L/10$ many black squares in each subboard with high probability.

Lemma 1. *Let $p = c_2(\ln n)^{-1}$, $L = \lfloor c_1 \ln n \rfloor$. Then with a probability tending to 1 there will be no subboard with more than $L/10$ black initial squares, if $c_1 = 20$ and $c_2 = 1/(200e^2)$.*

Proof. Easy to give an upper bound on the probability of that an $L \times L$ chessboard contains more than $L/10$ black squares of a p-random configuration:

$$\binom{L^2}{L/10} p^{L/10}.$$

Thus the probability that there is no subboard with more than $L/10$ initial black squares is at most:

$$\left(\frac{n}{L}\right)^2 \binom{L^2}{L/10} p^{L/10} < 2\frac{n^2}{L^{2.5}}(10eLp)^{L/10}, \tag{2}$$

where we used *Stirling's formula.*

Our choice for c_1 and c_2 is nearly optimal for getting $\overline{\lim}_{n\to\infty} n^2(10eLp)^{L/10} < \infty$, so by (2) we have obtained w.h.p. that there is no subboard of size $L \times L$ with more than $L/10$ initial black squares.

A similar calculation handles the cases of subboards of size $L \times (L+1)$, $(L+1) \times L$ and $(L+1) \times (L+1)$. ∎

We take two subboards, B_1 and B_2, sharing a common vertical side. Let B be the rectangle we obtain by gluing together the two subboards: $B = B_1 \cup B_2$. We define the following property of the initial configuration:

$\mathcal{P}(B) =$ "there are two horizontal width 2 strips in $B_1 \cup B_2$ which contain only white squares, one in the upper half of B and one in the lower half of B"

If $B_1 \cup B_2$ has property $\mathcal{P}$ we fix the two strips which proves this and call them *lower and upper channels.*

One can easily define an analogous $\mathcal{P}^*$ property for subboards, B_1 and B_2, sharing a horizontal side:

$$\mathcal{P}^*(B) = \text{"there are two vertical width 2 strips in } B_1 \cup B_2 \text{ which contain only white squares, one in the left half of } B \text{ and one in the right half of } B\text{"}$$

If $B_1 \cup B_2$ has property $\mathcal{P}^*$ we fix the two strips which proves this and call them *left and right channels*.

We will prove that with the choice of *Lemma 1* for p the initial p-random configuration will have the property $\mathcal{P}(B)$ or $\mathcal{P}^*(B)$ for any pair B of two neighboring subboards (depending whether their common side is vertical or horizontal), i.e. it has property $\widehat{\mathcal{P}}$.

Lemma 2. *If in a configuration C each subboard contains at most $L/10$ initial black squares, then C has property $\widehat{\mathcal{P}}$.*

Proof. By symmetry it is enough to prove $\mathcal{P}(B)$. There are at least $(9/10)L$ white rows in each subboard, hence in each half of a subboard there are at least $(4/10)L - 1$ white rows. So if we have two neighboring subboards, then in each half of them there will exist at least $(3/10)L - 2$ common white rows, thus 2 of them will be neighboring, if L is large enough. ∎

From now on we assume that our initial configuration has property $\widehat{\mathcal{P}}$. We are going to prove that the configuration is not contagious by providing a cover of the initial black squares by rectangles as promised.

Let B be an "inner subboard". We define two partitions of B into five rectangles with a border between them. B has four neighboring subboards: the upper, the right, the lower and the left one. The left neighbor with B together have property $\mathcal{P}$, hence we have an upper and a lower channel. We will call these the left-upper and left-lower channels. Similarly, we can define right-upper, right-lower, upper-left, upper-right, lower-left and lower-right channels. Using four of these channels we can get a desired partition, called leftist, and using the other four channels we get the other partition, called rightist. The construction of a leftist partition of B is shown on the self-explanatory *Figure 1*, the borders between the five rectangles are the four dark channels.

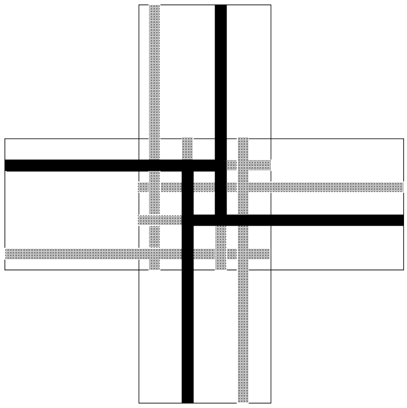

Figure 1.

Now we combine the leftist and rightist partitions of the subboards alternating in a chessboard manner. The result is a suitable cover: rectangles and all-white borders providing a width 2 strip for any two rectangles to separate them (see *Figure 2*).

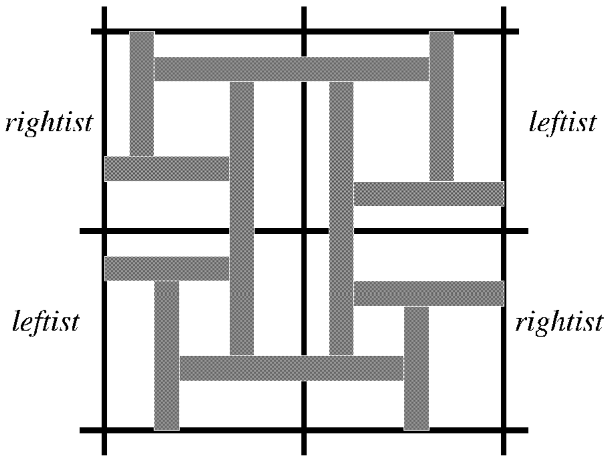

Figure 2.

Side and corner subboards can be covered naturally by rectangles between the channels constructed in partitioning the inner subboards that are the neighbors of our outer ones.

Summarizing our construction:

Lemma 3. *A configuration with property $\widehat{\mathcal{P}}$ is not contagious.* ∎

Now the combination of *Lemma 1,2,3* gives us the lower bound on $P(n)$. ∎

3. Proofs of the Upper Bound and the Whole Plane case

For the upper bound first of all we observe that in the initial configuration there are quite long horizontal runs of pure black squares in each row of the chessboard. The proof that the initial configuration is contagious with high probability then can be based on the existence of these initial runs. The method we use is a clever and efficient modification of the one in a previous version of this paper; this improvement is due to *N. Alon*, [A].

Throughout this section we will use that

$$(1-a_n)^{b_n} \sim \exp(-a_n b_n)$$

if $a_n < 1$ and $b_n \to \infty$ such that $a_n^2 b_n \to 0$. We need this simple analytic fact in almost every estimation of our probabilities, and we will not refer to it if the computations are clear.

Split the first row into, say, $n^{0.99}$ pairwise disjoint segments of equal size. (From now on we can allow not to bother with the integer parts, this simplification causes no problem.) In a $p(n)$-random initial configuration the probability that such a segment starts with a run of $f(n)$ black squares is $p(n)^{f(n)}$. If we plug in $p(n) \geq \frac{1}{\ln n}$ and $f(n) = \frac{\ln n}{100 \ln \ln n}$ we get that this probability is at least $n^{-1/100}$. Now we can easily estimate the probability that a bunch of $n^{0.04}$ consequetive segments does not contain a run of $f(n)$ black squares: this is less than $\exp(-n^{0.02})$. We can make $n^{0.95}$ pairwise disjoint bunches, and so the probability that there exists such a bunch without a long horizontal black run is less than $1-\exp\left(-e^{-n^{0.02}} n^{0.95})\right) \to 0$. Thus we have proved the following

Many Black-runs Lemma. *For $p(n) \geq \frac{1}{\ln n}$ the initial configuration contains at least $n^{0.95}$ horizontal black runs of length $\frac{\ln n}{100 \ln \ln n}$ each, where each such run starts from the leftmost point of one of our disjoint segments of length $n^{0.01}$, and the probability that this does not hold is exponentially small.* ∎

Remark 1. More sophisticated methods give that the length of the longest black run in a row of length n is $\mu(n) \sim \frac{\ln n}{\ln 1/p(n)}$ w.h.p. For $p(n) = 1/2$ it was stated first by *P. Erdős* and *A. Rényi* in [ErRn] from a little bit different point of view, and more precise results can be found in [ErRv].

Remark 2. Actually, we need only "almost pure" black-runs instead of the pure ones, namely we can allow single white squares between the black ones, as they will change into black by the next day. But this relaxation does not help much, the expected length of the longest run would increase only by a constant factor, and in the proof of the Upper Bound this improvement means nothing.

If we pick a horizontal black run of length $f(n)$, i.e. a black block of size $1 \times f(n)$, we can see in the neighboring row that the block "below" our black one will change into black in at most $f(n)$ days even if only one square is initially black in it. So a neighboring block becomes black with probability $q = q(n)$, where

$$1 - q(n) = (1 - p(n))^{f(n)}\,. \tag{3}$$

If this event with probability q does happen, then the same thing can be repeated for the $2 \times f(n)$ black block we have just obtained, and so on; we stop when we find a pure white block. So we get a run of black blocks with a random length Z, where

$$\mathrm{E}[Z] = 1q(1-q) + 2q^2(1-q) + \ldots + (n-1)q^{n-1}(1-q) > \frac{1}{2}\frac{q}{(1-q)},$$

and for $g(n) \leq n$ we have

$$\Pr[Z \geq g(n)] \sim q(n)^{g(n)}. \tag{4}$$

As two special cases of (4) we can state the following two lemmas. For a detailed verification one should use our analytic fact again and estimations like $1/(2k) \leq 1 - \exp(-1/k)$ if $k > 1$.

Lemma 4. *Suppose we have a black horizontal run of length $l = \frac{\ln n}{k}$ (with $k > 1$), and suppose each square is now becoming black, randomly and independently, with probability $p(n) \geq \frac{1}{\ln n}$. Then the probability that the process described above creates a black vertical run of length $\frac{\delta \ln n}{\ln(2k)}$ is at least $n^{-\delta}$.* ∎

Lemma 5. *Suppose we have a black horizontal run of length $l = k \ln n$ (with $o(\ln n) = k > 1$), and suppose each square is now becoming black, randomly and independently, with probability $p(n) \geq \frac{1}{\ln n}$. Then the probability that the process creates a black vertical run of length $\frac{e^k \delta \ln n}{2}$ is at least $n^{-\delta}$.* ∎

Needless to say, the assertions of both lemmas hold if we replace vertical by horizontal and vice versa. In this case the longer horizontal runs are created to the right of the existing black blocks (see *Figure 3*). This method of enlarging the black blocks also yields that we will be able to iterate these lemmas such that the realizations of the iteration steps will be mutually independent of each other.

Figure 3.

Now we are ready to describe why a $p(n)$-random initial configuration is contagious if $p(n) = \frac{(\log^* n)^{1+\epsilon}}{\ln n}$ and n is large enough. At the beginning by the *Many Black-runs Lemma* in the first row we have $n^{0.95}$ pairwise disjoint horizontal black runs of length $\frac{\ln n}{100 \ln \ln n}$ each w.h.p., and these blocks are rather far away from each other. Note that here and sometimes later, as well, we use only $p(n) \geq \frac{1}{\ln n}$. We do this only for the sake of simplicity, and the usage of the stronger condition would not produce a better result. It also simplifies matters to note that in the last few iterations we will do, it will be convenient to have each square become black again with probability $1/\ln n$, so as to obtain independence. Clearly, if every square first becomes black with probability $\frac{(\log^* n)^{1+\epsilon}}{\ln n}$ and then (if it is white) becomes black with probability $1/\ln n$ then this still corresponds to having each square becoming black with probability $(1+o(1))\frac{(\log^* n)^{1+\epsilon}}{\ln n}$.

Define, now, for $i \geq 1$,

$$\delta_i = \frac{\nu}{i^{1+\epsilon/3}}, \qquad k_1 = 100 \ln\ln n,$$

$$k_{i+1} = \ln(2k_i)/\delta_i = \frac{1}{\nu}\, i^{1+\epsilon/3} \ln(2k_i),$$

where $\nu > 0$ is small enough to have $\sum_{i=1}^{\infty} \delta_i = C(\nu, \epsilon) = C < \frac{1}{100}$. If we consider one of our long black runs then by applying *Lemma 4* $\log^* n + O(1)$ times repeatedly we conclude that with a probability larger than $\prod_{i=1}^{\infty} n^{-\delta_i} = n^{-C} > n^{-0.01}$ our iteration results a horizontal black run of length greater than

$$\frac{\ln n}{(\log^* n)^{1+2\epsilon/3}}.$$

This claim can be easily verified by the following argument. If $\ln \frac{k_i}{2} > \frac{1}{\nu}\, i^{1+\epsilon/3}$ for all $1 \leq i \leq \log^* n$ then $k_{i+1} < \ln^2 k_i$, and by induction we have $k_{(\log^* n)} < c \log^* n$. Otherwise there exists a $j \leq \log^* n$ with $\ln \frac{k_j}{2} < \frac{1}{\nu} j^{1+\epsilon/3}$ and so $k_{j+1} < (\log^* n)^{1+2\epsilon/3}$, supposing that n is large enough.

We have $n^{0.95}$ samples of the random iteration process described above, so with very high probability we will have in the resulting configuration at least, say, $n^{0.9}$ pairwise disjoint black horizontal runs of length at least $\frac{\ln n}{(\log^* n)^{1+2\epsilon/3}}$ each.

Given these runs, let each square become black with probability $\frac{(\log^* n)^{1+\epsilon}}{\ln n}$. Then we get w.h.p. at least $n^{0.8}$ black pairwise disjoint vertical runs of length, say, $100 \ln n$ each, these last computations are routine.

Define

$$k_1 = 100, \qquad \delta_i = \frac{\nu}{i^{1+\epsilon/3}},$$

$$k_{i+1} = \delta_i e^{k_i}/2.$$

By repeatedly applying *Lemma 5* we can now conclude that after some $\Theta(\log^* n)$ additional iterations we get w.h.p. many (and hence at least one) horizontal black run of length at least $(\ln n)^{3/2}$, and it is easy to see that this implies, after two more additional iterations, that the whole grid becomes black with a probability tending to 1. (Note that until the last two iterations we deal with runs of length less than $n^{0.01}$, so the whole processes of the iterations for our disjoint starting runs are mutually independent of each other.) This completes the proof of the Upper Bound and *Theorem 1*. ∎

Proof of Theorem 2. **(a)** Fix $\widehat{P}(n)$ as in the statement, and let us first suppose that it is a monotone decreasing function of n, just as $P(n)$ was. Let Ω be the probability space of the $\widehat{P}(\|(x,y)\|)$-random configurations of the infinite square grid, and let S_n be the $2n \times 2n$ square with vertices $(-n,-n),(-n,n),(n,-n),(n,n)$. Define A to be the event that not every square of the plane can be painted black with an initial configuration of Ω, and let A_n be the event that there is a square in S_n remaining white forever. It is clear that $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ and $\cup_{n=1}^{\infty} A_n = A$,

$$\Pr[A] = \lim_{n\to\infty} \Pr[A_n]. \tag{5}$$

Now define the event B_n as follows: for all $n \in \mathbf{N}$ consider the subalgebra of Ω generated by S_n, i.e. let Ω_n be the probability space of the $\widehat{P}(\|(x,y)\|)$-random configurations of S_n, and let B_n be the event that the painting is not complete on S_n, forgetting about the influence of the other part of the plane. Because of losing the possible positive influence of the black squares of the plane outside the square S_n we have $\Pr_{\Omega}[A_n] \le \Pr_{\Omega_n}[B_n]$. But inside S_n we have $\|(x,y)\| \le n$, so

$$\widehat{P}(\|(x,y)\|) \ge \widehat{P}(n) \gg P(n),$$

since $\widehat{P}(n)$ is monotone decreasing. Thus $\Pr_{\Omega_n}[B_n] \to 0$ as $n \to \infty$, which proves our statement because of (5), as $\Pr[A] = 0$.

If $\widehat{P}(n)$ is not monotone decreasing then define $\overline{P}(n) = \min_{1\le k\le n} \widehat{P}(n)$. Now $\widehat{P}(n) \ge \overline{P}(n)$, $\overline{P}(n)$ is already monotone decreasing, but still $\overline{P}(n)/P(n) \to \infty$, so we can repeat the argument above, and we are done.

The second part of the statement *(a)* follows from our Upper Bound on $P(n)$: looking at the probability field of the p-random configurations we have $p > \widehat{P}(n)$ for $n > n_p$, so $\Pr_{\Omega_n}[B_n] \to 0$ again.

(b) Note that if we have a pure white $k \times k$ square in the initial configuration then it takes at least $k-1$ days to paint it black from the outside. Now divide the plane into subboards of size $k \times k$. Each of them is pure white in the initial configuration with a *positive* probability $(1-p)^{k^2}$. So we can find a pure white one in the whole plane with probability 1. If the exceptional event (namely, there is no initial pure white $k \times k$ subboard) is denoted by C_k, then $\Pr[C_k] = 0$ and

$$\begin{aligned}\Pr[t(p)=\infty] &\ge \Pr[\forall k \ \exists \text{ an initial white } k \times k \text{ square in the plane}] \\ &= 1 - \Pr[\cup_{k=1}^{\infty} C_k] \ge 1 - \sum_{k=1}^{\infty} \Pr[C_k] = 1,\end{aligned}$$

what proves our second statement. ∎

4. The general $P_{k,l}(n)$ case

1. For a successful disease process the exact thing we need is to have at least one initial black cube, so $(1 - P_{k,1}(n))^{n^k} = \frac{1}{2}$, that is

$$P_{k,1}(n) \sim \frac{\ln 2}{n^k}. \tag{6}$$

2. By the procedure of the Upper Bound we can obtain an all-black 2-dimensional $n \times n$ face of our board, and then can continue enlarging the black blocks as long as we will, and, moreover, the method of the Lower Bound can also be generalized, so it is not difficult to prove that

$$\Omega\left(\frac{1}{(\ln n)^{k-1}}\right) < P_{k,2}(n) < \frac{(\log^*)^{1+\epsilon}}{\ln n}. \tag{7}$$

3. In the problem of $P_{k,k+1}(n)$ the complete painting cannot be done if there exists a $2 \times \ldots \times 2$ white cube in the initial configuration. Thus dividing the board into $(n/2)^k$ subboards containing 2^k cubes each, one can easily see that

$$P_{k,k+1}(n) \geq 1 - \frac{O(1)}{n^{k/2^k}}. \tag{8}$$

4. The complete painting is equivalent with the lack of two initial white cubes side-by-side and the total lack of white cubes on the border, so an easy computation gives

$$P_{k,2k}(n) \geq 1 - \frac{O(1)}{n^{k-1}}. \tag{9}$$

5. The simple methods above can be easily converted to the whole space case, so based on (7) and (8):

$$P_{k,l}(\infty) = \begin{cases} 0, & \text{if } l \leq 2 \\ 1, & \text{if } l \geq k+1, \\ ?, & \text{if } 3 \leq l \leq k, \end{cases} \tag{10}$$

i.e. the $p_{k,l}$-random initial configuration of the infinite grid is a.s. contagious for $l \leq 2$ with any $p_{k,l} > 0$ fixed, and it is a.s. contagious for $l \geq k+1$ only with $P_{k,l}(\infty) = 1$. Our conjecture on the middle cases is described after *Question 2* in the next section.

5. Some open problems

As shown in the previous section, $P_{k,1}(n)$ is very small and $P_{k,2k}(n)$ is very large. Thus a crucial question is the following: what is the maximal $f(k)$ and minimal $g(k)$ for which $P_{k,f(k)}(n) \to 0$ and $P_{k,g(k)}(n) \to 1$? In the deterministic version we have

$$\lim_{n\to\infty} \frac{G_{k,k}(n)}{n^k} = 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{G_{k,k+1}(n)}{n^k} \geq \frac{1}{k+1},$$

so we can feel that the case $l = k$ has a special role. (See the *Appendix*; the exact value of the second limit is a special case of the *Conjecture 2* stated there.) Actually, we think that the method of the lower bound in *Theorem 1* can be generalized, and gives a sharp result, which, together with the result $P_{k,k+1}(n) \to 1$ ($n \to \infty$) in *Section 4*, would imply that $k = f(k) < g(k) = k + 1$.

Conjecture 1. *For $l \le k$*

$$P_{k,l}(n) = \frac{1}{(\ln n)^{k-l+1-o(1)}}. \tag{11}$$

Of course, this conjecture is indeed a generalization of *Theorem 1*, because the bounds of (1) can be interpreted as $P_{2,2}(n) = (\ln n)^{-1+o(1)}$. Moreover, it seems to be plausible to conjecture that $\Theta\left((\ln n)^{-k+l-1}\right)$ is the truth. In particular, we can ask the following:

Question 1. *What is the exact order of $P_{k,l}(n)$? Is it true that $P_{k,l}(n) \le P_{k+1,l}(n)$?*

The appearance of the threshold $P(n)$ in the whole plane case is very natural, but is *Theorem 2 (a)* really sharp?

Question 2. *Is there a function $\widehat{P}_{k,l}(n)$ with $\widehat{P}_{k,l}(n)/P_{k,l}(n) \to 0$ for some k, l for which the $\widehat{P}_{k,l}(\|\underline{x}\|)$-random initial configuration of the infinite k-dimensional square grid is almost surely contagious?*

The similar question about the existence of an a.s. contagious $P_{k,l}(\infty)$-random configuration with a fixed $P_{k,l}(\infty) < \lim_{n\to\infty} P_{k,l}(n)$ would be answered negatively if *Conjecture 1* held, because it would imply $P_{k,k}(\infty) = 0$, and in *Section 4* we proved $P_{k,k+1}(\infty) = 1$.

And finally, according to *Theorem 2 (b)*, two questions about the time needed for a complete painting:

Question 3. *If we pick an individual square of the plane, what is the expected time of its getting black, if the initial configuration is $p(x, y)$-random? Can anything be said about the time needed in the finite chessboard problems?*

6. Appendix — deterministic results from [P]

As we cited some results of the paper on the deterministic version, and it is available only in Hungarian, perhaps it is worth saying few words about it.

Let $G_{k,l}(n)$ denote the minimum number of the initial black squares needed for the complete painting of the k-dim chessboard, if we follow the l-neighbor painting rule. Generalizing the method of our starting exercise we get a relevant result for $l \ge k$:

Perimeter Lemma. *$G_{k,l}(n) \ge \frac{l-k}{l} n^k + \frac{k}{l} n^{k-1}$. Further, $G_{k,k}(n) = n^{k-1}$.* ∎

With a simple geometric trick one can get a lower bound even for the cases $l < k$:

Projection Lemma. *$G_{k,l}(n) \geq G_{k-1,l}(n)$.* $\blacksquare$

An upper bound comes from a recursive painting technique:

Recursive Lemma. *$G_{k,l}(n) \leq f_k(k)G_{k,l}(n-2) + f_k(k-1)G_{k-1,l-1}(n-2) + f_k(k-2)G_{k-2,l-2}(n-2) + \ldots + f_k(0)G_{0,l-k}(n-2)$, where $f_k(m)$ is the number of m-faces of the k-dimensional cube.* $\blacksquare$

Combining these results we have

Main Deterministic Theorem. *For fixed k, l we have*

$$G_{k,l}(n) = \begin{cases} \Theta(n^{l-1}), & \text{if } 1 \leq l \leq k \\ \Theta(n^k), & \text{if } k+1 \leq l \leq 2k \end{cases} \tag{12}$$

$\blacksquare$

For the first few special cases we have exact asymptotics, as well; some of them are trivial, others need some tricky ideas:

Disease on the Square.
(a) $G_{2,1}(n) = 1$
(b) $G_{2,2}(n) = n$
(c) $G_{2,3}(n) \sim \frac{1}{3}n^2$
(d) $G_{2,4}(n) \sim \frac{1}{2}n^2$ $\blacksquare$

Disease on the Cube.
(a) $G_{3,1}(n) = 1$
(b) $G_{3,2}(n) = \lfloor \frac{3}{2}n \rfloor$
(c) $G_{3,3}(n) = n^2$
(d) $G_{3,4}(n) \sim \frac{1}{4}n^3$
(e) $\frac{2}{5}n^3 \leq G_{3,5}(n) \leq \frac{3}{7}n^3$
(f) $G_{3,6}(n) \sim \frac{1}{2}n^3$ $\blacksquare$

Finally, based on our results above and other ideas, we stated the following conjecture:

Conjecture 2. *For $k+1 \leq l \leq 2k$ the Perimeter Lemma is sharp, i.e. $G_{k,l}(n) = \frac{l-k}{l}n^k + O(n^{k-1})$.*

The case $l < k$ seems to be hopeless at this time.

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