Stochastic Models — Third HW problem set

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The number of dots $^\bullet$ is the value of an exercise. **Hand in solutions for 12 points by May 17 Monday 11:59 am.** If you have seriously tried to solve some problem, but got stuck, I will be happy to help. Also, if your final solution to a problem has some mistake but has some potential to work, then I will give it back and you can try and correct the mistake.

▷ **Exercise 1.**$^\bullet$ Consider the configuration model for a random d-regular multi-graph on n vertices, with nd even. (Given by a uniform random perfect matching on the nd half-edges). Show that if we condition this random graph to have no multiple edges and no self-loops, then we get the uniform distribution on d-regular simple graphs on n vertices.

Let G_n be a sequence of finite graphs. Pick a uniform random root ρ_n from $V(G_n)$, and take the ball $B_{G_n,\rho_n}(r)$ around it in the graph metric, with some fixed radius $r \in \mathbb{Z}_+$. We get a distribution $\mu_{n,r}$ on finite rooted graphs. We say that the sequence $\{G_n\}$ converges in the **Benjamini-Schramm sense** (also called **local weak convergence**) to a random rooted graph (G, ρ), if, for every r, the distributions $\mu_{n,r}$ converge weakly as $n \to \infty$ to the distribution of $B_{G,\rho}(r)$. The simplest case is when the limit is a transitive infinite graph G: the measures $\mu_{n,r}$ converge to the Dirac measure on a single graph, the r-ball of G. The following exercise generalizes the examples of boxes in $\mathbb{Z}^d$ and balls in the d-regular tree $\mathbb{T}_d$ that we saw on class:

▷ **Exercise 2.**$^{\bullet\bullet}$ Show that a transitive graph G has a sequence G_n of *subgraphs* converging to it in the Benjamini-Schramm (local weak) sense iff it is amenable.

▷ **Exercise 3.** We considered the following simple model for a random d-regular bipartite (multi-)graph: take d independent uniform random permutations $\pi_i : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$, then take all the edges $\big\{(j, n + \pi(j)) : j \in \{1, \dots, n\},\ i \in \{1, \dots, d\}\big\}$.

(a)$^{\bullet\bullet}$ Show that for any $k \geq 2$, the number of k-cycles is tight in n. (We saw this for $k = 2$.)

(b)$^\bullet$ Conclude from part (a) that this random graph converges to the d-regular tree $\mathbb{T}_d$ in the local weak sense. (Here the randomness for the measure $\mu_{n,r}$ comes from two sources: we take a random root ρ_n in the random graph G_n, and want to show convergence in this joint probability space.)

▷ **Exercise 4.**$^{\bullet\bullet}$ Consider a GW process with offspring distribution ξ, having $\mathbf{E}\xi = \mu > 1$ and $\mathbf{E}[\,\xi^2\,] < \infty$. Let Z_n be the size of the nth level, with $Z_0 = 1$, the root. Show that $\mathbf{E}\big[\,Z_n^2\,\big] \leq C(\mathbf{E}Z_n)^2$, with a uniform $C < \infty$. (*Hint:* use the conditional variance formula $\mathbf{D}^2[Z_n] = \mathbf{E}\big[\,\mathbf{D}^2[Z_n \mid Z_{n-1}]\,\big] + \mathbf{D}^2\big[\mathbf{E}[\,Z_n \mid Z_{n-1}\,]\big]$.)

▷ **Exercise 5.**$^{\bullet\bullet}$ Give an example of a random sequence $(M_n)_{n=0}^\infty$ such that $\mathbf{E}[\,M_{n+1} \mid M_n\,] = M_n$ for all $n \geq 0$, but which is not a martingale (in its natural filtration).

▷ **Exercise 6.**$^{\bullet\bullet}$ Let $G(V, E)$ be any bounded degree infinite graph, and $S_n \nearrow V$ an exhaustion by finite connected subsets. Consider Bernoulli(p) bond percolation on G. Is it always true that, for $p > p_c(G)$, we have

$$\lim_{n\to\infty} \mathbf{P}_p[\,\text{largest cluster for percolation inside } S_n \text{ is the subset of an infinite cluster}\,] = 1\,?$$

An **invariant bond (or site) percolation** on a finite or infinite transitive graph G is a random subset of $E(G)$ (or $V(G)$) whose distribution is invariant under any graph-automorphism. The connected components are called clusters, and the cluster of a vertex o is denoted by $\mathscr{C}_o$.

▷ **Exercise 7.**$^{\bullet\bullet}$ Assume you know about a sequence of invariant percolation measures $\mathbf{P}_n$ on finite transitive graphs G_n on n vertices that the cluster size distribution of a fixed vertex is tight: for any $\epsilon > 0$ there is $K < \infty$ such that, $\mathbf{P}_n\big[\,|\mathscr{C}_o| > K\,\big] < \epsilon$ for all n large enough, for any $o \in V(G_n)$.

Show that the largest cluster on G_n has size $o(n)$ with probability tending to 1. That is, for any $\delta, \gamma > 0$, if n is large enough, then $\mathbf{P}_n\big[\,\exists\, x : |\mathscr{C}_x| > \delta n\,\big] < \gamma$.

(*Hint:* consider the expected number of vertices x with $|\mathscr{C}_x| > \delta n$.)

▷ **Exercise 8.**$^{\bullet}$ Define a random perfect matching ω on the 3-regular tree $\mathbb{T}_3$ in the following way. Fix a vertex $o \in V(\mathbb{T}_3)$. Take uniformly at random one of the three edges incident to o into ω. Denote the other two edges by (o, x_1) and (o, x_2). For each x_i, take uniformly at random one of the two other edges (not incident to o) into ω, independently from the previous choices. And so on, take edges further and further from o into ω, by choosing independently always one of the two possible new edges incident to a vertex. This construction is not invariant at all. Show that nevertheless the distribution of the resulting ω is invariant.

Here are two quite canonical invariant random spanning tree models on finite graphs:

▷ **Exercise 9.** On any finite graph $G(V, E)$, assign iid random edge weights $\xi = (\xi_e)_{e\in E}$ to the edges, from an atomless non-negative valued distribution. Consider the spanning tree of G that minimizes the sum of the edge weights — this is the Minimal Spanning Tree MST_ξ.

(a)$^{\bullet}$ Show that one can construct this tree by removing from every cycle of G the edge with the largest label.

(b)$^{\bullet}$ Conclude that the distribution of MST_ξ does not depend on the distribution of the ξ_e's. Hence we can denote this random tree just by MST, the Minimal Spanning Tree of the graph.

(c)$^{\bullet\bullet}$ Consider the uniform distribution on all the spanning trees of G — this is the Uniform Spanning Tree UST. Give a finite graph on which $\mathsf{MST} \neq \mathsf{UST}$ with positive probability.

An invariant percolation on an infinite transitive graph is called **ergodic** if every automorphism-invariant event (e.g., $\{\exists\,\infty$ cluster$\}$, or $\{$every cluster has a vertex with degree 1$\}$) has probability 0 or 1.

▷ **Exercise 10.**$^{\bullet}$ Show that an invariant percolation measure μ on an infinite transitive graph is ergodic iff it is not a non-trivial convex combination $p\mu_1 + (1-p)\mu_2$ of two different invariant percolation measures.

▷ **Exercise 11.**$^{\bullet\bullet}$ Give an ergodic invariant site percolation on some transitive infinite graph such that if we take the union of the open sites in two independent copies, the resulting percolation is not ergodic.

An invariant percolation $\mathbf{P}$ on an infinite transitive graph G is called **mixing** if, for any two events A, B, and $\epsilon > 0$, there is a finite set $K \subset \mathrm{Aut}(G)$ such that if $\gamma \notin K$, then $\big|\mathbf{P}[\,A \cap \gamma(B)\,] - \mathbf{P}[\,A\,]\mathbf{P}[\,B\,]\big| < \epsilon$.

▷ **Exercise 12.**

(a)$^{\bullet}$ Show that Bernoulli percolation on any infinite graph is mixing.

(b)$^{\bullet}$ Show that mixing implies ergodicity.

▷ **Exercise 13.**$^{\bullet\bullet\bullet}$ Consider the following site percolation on $\mathbb{Z}$: let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be fixed, $U \sim \mathsf{Unif}[0,1]$ random, and let $\omega_n := 1$ if $U + n\alpha \pmod 1$ is in $[0, 1/2)$, and $\omega_n := 0$ otherwise. Show that this is $\mathbb{Z}$-invariant, ergodic, but not mixing.

▷ **Exercise 14.**$^{\bullet}$ For Bernoulli bond percolation on any connected infinite graph G, any $o \in V(G)$, define

$$p_{T,o} := \inf\big\{p : \mathbf{E}_p[\,|\mathscr{C}_o|\,] = \infty\big\}.$$

T is for the honour of Temperley. Show that this does not depend on o.

$\triangleright$ **Exercise 15.** Consider Bernoulli bond percolation on the canopy tree Λ (the Benjamini-Schramm limit of the balls $B_n(o)$ in the 3-regular tree $\mathbb{T}_3$).

(a)$^\bullet$ Show that $p_c(\Lambda) = 1$.

(b)$^{\bullet\bullet}$ Find $p_T(\Lambda)$, with the definition of the previous exercise.

$\triangleright$ **Exercise 16.**$^\bullet$ Prove using subadditivity that, for Bernoulli percolation on any transitive graph, $\sigma(p) := \lim_{n\to\infty} \frac{-1}{n} \log \mathbf{P}_p[\, o \longleftrightarrow \partial B_n(o)\,]$ exists, and is in $[0,\infty)$.

The **Fortuin-Kasteleyn random cluster measure** $\mathsf{FK}(p,q)$ on a finite graph G, with $p \in [0,1]$ and $q > 0$, is the invariant bond percolation model given by, for any $\omega \subset E(G)$,

$$\mathbf{P}_{\mathsf{FK}(p,q)}[\omega] := \frac{p^{|\omega|}\,(1-p)^{|E\setminus\omega|}\,q^{k(\omega)}}{Z_{\mathsf{FK}(p,q)}} \qquad \text{with} \quad Z^{\pi}_{\mathsf{FK}(p,q)} := \sum_{\omega\subseteq E} p^{|\omega|}\,(1-p)^{|E\setminus\omega|}\,q^{k(\omega)}\,,$$

where $k(\omega)$ is the number of clusters of ω.

$\triangleright$ **Exercise 17.**$^{\bullet\bullet}$ Consider $\mathsf{FK}(p,q)$ on the $n\times n$ two-dimensional lattice torus $(\mathbb{Z}/n\mathbb{Z})^2$. Given a configuration ω, the dual configuration ω^* is defined on the dual torus: the dual vertices are the primal faces, and two are connected by a dual edge iff the edge between the primal faces is not present in ω. Show that for $p = p_{\text{self-dual}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$, the dual configuration ω^* has the same distribution as ω.

$\triangleright$ **Exercise 18.**$^\bullet$ For any finite tree, show that $\mathsf{FK}(p,q)$ is just Bernoulli bond percolation at some density $\tilde{p}(q)$, which you should identify.

$\triangleright$ **Exercise 19.**$^\bullet$ For any finite graph, show carefully that $\lim_{p\to 0+}\lim_{q\to 0+} \mathsf{FK}(p,q) = \mathsf{UST}$.