Stochastic Models — First HW problem set

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The number of dots $^\bullet$ is the value of an exercise. **Hand in solutions for 12 points by March 11 Thursday 23:59.** If you have seriously tried to solve some problem, but got stuck, I will be happy to help. Also, if your final solution to a problem has some mistake but has some potential to work, then I will give it back and you can try and correct the mistake.

▷ **Exercise 1.**$^\bullet$ Show that if $G(V,E)$ is a connected graph, and simple random walk started at some $o \in V$ visits o infinitely often almost surely, then the walk started at any $x \in V$ visits any given $y \in V$ infinitely often, almost surely. Consequently, recurrence is a property solely of the graph. In a transient graph, the walk visits any given finite set only finitely many times.

▷ **Exercise 2.**$^{\bullet\bullet}$ Give an example of an iid random walk on $\mathbb{Z}$ with symmetric jump distribution that is transient. (Hint: simple random walk on $\mathbb{Z}^3$ is transient.)

▷ **Exercise 3.**$^{\bullet\bullet}$ Prove that for Green's function of simple random walk on a connected graph, $G(a,b|z) := \sum_{n\geq 0} p_n(a,b)\, z^n$, for any vertices x,y,a,b and any real $z>0$,

$$G(x,y|z) < \infty \;\Leftrightarrow\; G(a,b|z) < \infty\,.$$

Therefore, by Pringsheim's theorem, we have that the radius of convergence is independent of x,y.

▷ **Exercise 4.**$^\bullet$ Prove Pringsheim's theorem. (Hint: $\big|\sum_n a_n z^n\big| \leq \sum_n a_n |z|^n$.)

▷ **Exercise 5.** Let $D_n := \mathrm{dist}(X_n, X_0)$ be the distance of SRW from the starting point.

(a)$^{\bullet\bullet}$ Using the Central Limit Theorem, prove that $\mathbf{E}[\,D_n\,] \asymp \sqrt{n}$ on any $\mathbb{Z}^d$.

(b)$^{\bullet\bullet}$ Using the transience of the SRW on $\mathbb{T}_k$, $k\geq 3$, show that $D_n/n \to \frac{k-2}{k}$ almost surely, and $\mathbf{E}[\,D_n\,] \sim \frac{k-2}{k} n$, as $n\to\infty$.

▷ **Exercise 6.**$^{\bullet\bullet}$ Let $\mathbb{T}_{k,\ell}$ be the tree where, if $v_n \in \mathbb{T}_{k,\ell}$ is a vertex at distance n from the root, then

$$\deg v_n = \begin{cases} k & \text{if } n \text{ is even} \\ \ell & \text{if } n \text{ is odd}\,. \end{cases}$$

Show the almost sure limiting speed $\lim_{n\to\infty} d(X_0,X_n)/n$ exists, and compute its value.

▷ **Exercise 7.**$^{\bullet\bullet\bullet}$ Compute the spectral radius $\rho(\mathbb{T}_{k,\ell})$ for the previous tree.

▷ **Exercise 8.**$^\bullet$ Take a connected infinite transitive graph G of degree d. Let Γ_n be the number of self-avoiding paths (no repeated vertices) of length n starting from a fixed vertex. Show that $\gamma := \lim_{n\to\infty} \Gamma_n^{1/n}$ exists, and is at most $d-1$. This is called the connective constant of the graph G.

▷ **Exercise 9.**$^\bullet$ In First Passage Percolation on a graph $G(V,E)$, we assign iid nonnegative random weights ω_e to the edges $e\in E$, then study the resulting random metric $\mathrm{dist}_\omega(\cdot,\cdot)$ on $V\times V$, where the length of each edge is not 1, but its weight. Let the graph be $\mathbb{Z}^2$, and let the weight distribution be $\mathbf{P}[\,\omega_e = a\,] = 1 - \mathbf{P}[\,\omega_e = b\,] = p$, with some fixed $0<a<b<\infty$ and $p\in(0,1)$. Let $L_n := \mathbf{E}\big[\,\mathrm{dist}_\omega\big((0,0),(n,n)\big)\,\big]$. Show that $\lim_n L_n/n$ exists and is positive and finite.

▷ **Exercise 10.**$^{\bullet\bullet\bullet}$ A simple version of the Tetris game (with no player): on the discrete cycle of length K, unit squares with sticky corners are falling from the sky, at places $[i, i+1]$ chosen uniformly at random ($i = 0, 1, \dots, K-1$, mod K). Let R_t be the size of the roof after t squares have fallen: those squares of the current configuration that could have been the last to fall. Show that $\lim_{t\to\infty} \mathbf{E} R_t = K/3$.

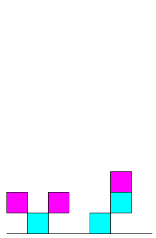

Figure 1: Sorry, this picture is on the segment, not on the cycle.

Remark. If there are two types of squares, particles and antiparticles that annihilate each other when falling on exactly on top of each other, this process is a SRW on a group, and the size of the roof has to do with the speed of the SRW. Here, for $K \geq 4$, the expected limiting size of the roof is already less than $0.32893K$, but this is far from trivial. What's the situation for $K = 3$?

▷ **Exercise 11.** Recall (or look it up in Durrett's book) that the reflection principle implies the following: if $\{X_k\}_{k\geq 0}$ is SRW on $\mathbb{Z}$, and $M_n = \max_{k\leq n} X_k$, then

$$2\,\mathbf{P}[\,X_n \geq t\,] \geq \mathbf{P}[\,M_n \geq t\,]\,.$$

Consider now SRW on the lamplighter group $\oplus_{\mathbb{Z}}\mathbb{Z}_2 \rtimes \mathbb{Z}$, with the lazy generators Left, Right, Switch, Nothing, each with probability 1/4 (but the exact probabilities will not matter).

(a)$^{\bullet\bullet}$ Prove that the return probability is at least $p_n(o,o) \geq \exp(-c\sqrt{n})$, for some absolute constant $c > 0$. (Note that the subexponential decay corresponds to the graph being amenable.)

(b)$^{\bullet\bullet\bullet\bullet}$ Find a smarter version of this strategy and prove $p_n(o,o) \geq \exp(-cn^{1/3})$. This is actually the sharp exponent (but that you don't have to give any upper bound).

▷ **Exercise 12.** Recall that a bounded degree infinite graph satisfies the isoperimetric inequality IP_d if $|\partial S| > c|S|^{\frac{d-1}{d}}$ for every finite $S \subset V(G)$. In particular, IP_∞ means non-amenable.

(a)$^{\bullet\bullet\bullet}$ Show that a bounded degree tree without leaves is amenable iff there is no bound on the length of "hanging chains", i.e., chains of vertices with degree 2. (Consequently, for trees, $IP_{1+\epsilon}$ implies IP_∞.)

(b)$^{\bullet\bullet}$ Give an example of a bounded degree tree of exponential volume growth that satisfies no $IP_{1+\epsilon}$ and is recurrent for the simple random walk on it.

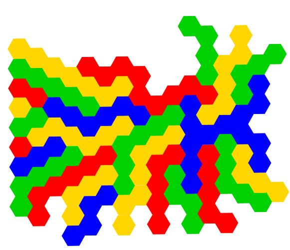

Figure 2: Trying to create at least 7 neighbours for each country.

▷ **Exercise 13.**$^{\bullet\bullet\bullet\bullet}$ Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.

▷ **Exercise 14.**

(a)•••• Show that a bounded degree graph $G(V,E)$ is nonamenable if and only if it has a wobbling paradoxical decomposition: two injective maps $\alpha, \beta : V \longrightarrow V$ such that $\alpha(V) \sqcup \beta(V) = V$ is a disjoint union, and both maps are at a bounded distance from the identity, or wobbling: $\sup_{x \in V} d(x, \alpha(x)) < \infty$. (Hint: State and use the locally finite infinite bipartite graph version of the Hall marriage theorem, called the Hall-Rado theorem.)

(b)• Deduce from part (a) that any bounded degree nonamenable graph has a Ponzi pyramid scheme (bounded transactions over the edges, but uniformly positive gain per vertex).

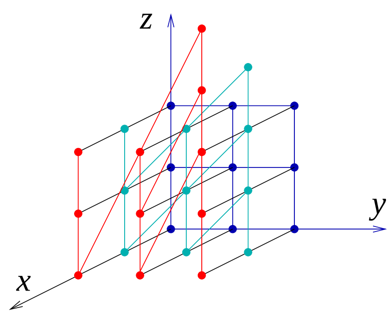

Figure 3: The Cayley graph of the Heisenberg group with generators X, Y, Z.

The **3-dimensional discrete Heisenberg group** is the matrix group

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

If we denote by X, Y, Z the matrices given by the three permutations of the entries $1, 0, 0$ for x, y, z, then $H_3(\mathbb{Z})$ is given by the presentation $\big\langle X, Y, Z \,\big|\, [X,Z] = 1, [Y,Z] = 1, [X,Y] = Z \big\rangle$, where $[a,b] = aba^{-1}b^{-1}$.

▷ **Exercise 15.** We say that a bounded degree graph $G(V,E)$ has d-dimensional volume growth if there exist $0 < c < C < \infty$ such that $cr^d < |B_r(o)| < Cr^d$ for any $o \in V$ and every large enough $r > r^*(o)$.

(a)• Show that if a group has a finitely generated Cayley graph with d-dimensional volume growth, then all its Cayley graphs have d-dimensional volume growth.

(b)•• Show that the discrete Heisenberg group has 4-dimensional volume growth.