Stochastic Models at CEU — First HW problem set

Gábor Pete
http://www.math.bme.hu/~gabor

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Solve 6 of the 14 problems below by March 7. Since we do not have a class that week, it is OK to hand them in later. You can ask me for help if you get stuck with something.

Let us start with a problem on stochastic domination:

▷ **Exercise 1.** Consider the graph G with 6 vertices and 7 edges that looks like a figure 8 on a digital display. Consider the uniform measure on the 15 spanning trees of G, denoted by UST, and the uniform measure on the 7 connected subgraphs with 6 edges (one more than a spanning tree), denoted by $\mathsf{UST}+1$. Find an explicit monotone coupling between the two measures (i.e., with $\mathsf{UST} \subset \mathsf{UST}+1$).

Open problem. Is there such a monotone coupling for every finite graph?

The most basic problem on phase transitions:

▷ **Exercise 2.** Prove the Bollobás-Thomason **threshold** theorem: for any sequence monotone events $\mathcal{A} = \mathcal{A}_n$ and any ϵ there is $C_\epsilon < \infty$ such that $\left|p^{\mathcal{A}}_{1-\epsilon}(n) - p^{\mathcal{A}}_{\epsilon}(n)\right| < C_\epsilon \left(p^{\mathcal{A}}_{\epsilon}(n) \wedge (1 - p^{\mathcal{A}}_{1-\epsilon}(n))\right)$. (Hint: take many independent copies of low density to get success with good probability at a larger density.)

Four problems on the Galton-Watson phase transition:

▷ **Exercise 3.** Let Z_n be the size of the nth generation in a GW tree with offspring distribution ξ.

(a) Assuming that $\mu > 1$ and $\mathbf{E}[\,\xi^2\,] < \infty$, first show that $\mathbf{E}\big[\,Z_n^2\,\big] \le C(\mathbf{E}Z_n)^2$. (Hint: use the conditional variance formula $\mathbf{D}^2[Z_n] = \mathbf{E}\big[\,\mathbf{D}^2[Z_n \mid Z_{n-1}]\big] + \mathbf{D}^2\big[\mathbf{E}[\,Z_n \mid Z_{n-1}\,]\big]$.) Then, using the Second Moment Method, deduce that the GW process survives with positive probability.

(b) Extend the result on survival to the cases $\mathbf{E}\xi = \infty$ or $\mathbf{D}\xi = \infty$ by a truncation $\xi \mathbf{1}_{\xi<K}$ for K large enough.

▷ **Exercise 4.** Let $(X_i)_{i\ge0}$ be a random walk on $\mathbb{Z}$, with i.i.d. increments ξ_i that have zero mean and an exponential tail: there exist $K \in \mathbb{N}$ and $0 < q < 1$ such that $\mathbf{P}[\,\xi \ge k+1\,] \le q\,\mathbf{P}[\,\xi \ge k\,]$ for all $k \ge K$. Starting from $X_0 = \ell \in \{1, 2, \dots, k-1\}$, let τ_0 be the first time the walk is at most 0, and let τ_k be the first time the walk is at least k. For any $0 < X_0 = \ell < k$, show that $\mathbf{P}_\ell[\,\tau_k < \tau_0\,] \asymp \ell/k$. (Hint: first prove that $X_{\tau_k} - k$, conditioned on $\tau_k < \tau_0$, has an exponential tail, independently of k.)

▷ **Exercise 5.** For the GW tree with offspring distribution Poisson$(1+\epsilon)$, show that the survival probability is asymptotically 2ϵ, as $\epsilon \to 0$.

▷ **Exercise 6.** Using the exploration Markov chain for GW trees and a Doob transform, show that if we condition the GW tree with offspring distribution Poisson(λ) on extinction, where $\lambda > 1$, then we get a GW tree with offspring distribution Poisson(μ) with $\mu < 1$, where $\lambda e^{-\lambda} = \mu e^{-\mu}$.

We used exponential concentration bounds for the Erdős-Rényi giant cluster phase transition at two places, where the usual Azuma-Hoeffding for bounded MG-differences does not exactly apply. The next exercise fills in these gaps:

▷ **Exercise 7.** Using the exponential Markov inequality as for Azuma-Hoeffding, together with the moment generating function $m_X(t) = \mathbf{E}\big[\, e^{tX} \,\big]$, prove the following two exponential concentration inequalities:

(a) If $S_n = X_1 + \cdots + X_n$ is a sum of i.i.d. variables with $\mathbf{E}X_i = \mu$ and $m_X(t_0) < \infty$ for some $t_0 > 0$, then, for any $\delta > 0$ there exist $c_\delta > 0$ and $C_\delta < \infty$ (which also depend on the distribution of X_i) such that

$$\mathbf{P}\big[\, |S_n/n - \mu| > \delta \,\big] < C_\delta \, e^{-c_\delta n},$$

for any n. (Hint: use that $\frac{d}{dt} \log m_X(t)\big|_{t=0} = 0$, while $\frac{d}{dt} \delta t\big|_{t=0} > 0$.)

(b) For any $\delta > 0$ there exist $c_\delta > 0$ and $C_\delta < \infty$ such that

$$\mathbf{P}\big[\, |\mathsf{Poi}(\lambda) - \lambda| > \delta\lambda \,\big] < C_\delta \, e^{-c_\delta \lambda},$$

for any $\lambda > 0$. (Hint: we know what the exponential generating function of $\mathsf{Poi}(\lambda)$ is.)

A simple example to practice influences, algorithmic revealment, and noise sensitivity:

▷ **Exercise 8.** Consider the Tribes_n function on $n = k2^k$ bits.

(a) Find the total influence of the input bits.

(b) Find a $o(1)$-revealment algorithm that computes this function.

(c) Prove that this function is noise sensitive (directly, without invoking any noise-sensitivity theorems): for any $\epsilon > 0$ fixed, if ω^ϵ denotes the configuration where every bit in ω is resampled independently with probability ϵ, then $\mathrm{Corr}(\mathsf{Tribes}_n(\omega), \mathsf{Tribes}_n(\omega^\epsilon)) \to 0$ as $n \to \infty$.

Two exercises on random walks and isoperimetry:

▷ **Exercise 9.** Why it is hard to construct large expanders:

(a) If $G' \longrightarrow G$ is a covering map of infinite graphs, then the spectral radii satisfy $\rho(G') \le \rho(G)$, i.e., the larger graph is more non-amenable. In particular, if G is an infinite k-regular graph, then $\rho(G) \ge \rho(\mathbb{T}_k) = \frac{2\sqrt{k-1}}{k}$.

(b) If $G' \longrightarrow G$ is a covering map of finite graphs, then $\lambda_2(G') \ge \lambda_2(G)$, i.e., the larger graph is a worse expander.

▷ **Exercise 10.** Recall (or look it up in Durrett's book) that the reflection principle implies the following: if $\{X_k\}_{k \ge 0}$ is SRW on $\mathbb{Z}$, and $M_n = \max_{k \le n} X_k$, then

$$2\, \mathbf{P}[\, X_n \ge t \,] \ge \mathbf{P}[\, M_n \ge t \,] \,.$$

Consider now SRW on the lamplighter group $\oplus_{\mathbb{Z}} \mathbb{Z}_2 \rtimes \mathbb{Z}$, with the lazy generators Left, Right, Switch, Nothing, each with probability $1/4$ (but the exact probabilities will not matter).

(a) Prove that the return probability is at least $p_n(o,o) \ge \exp(-c\sqrt{n})$, for some absolute constant $c > 0$. (Note that the subexponential decay corresponds to the graph being amenable.)

(b) Find a smarter version of this strategy and prove $p_n(o,o) \ge \exp(-cn^{1/3})$, which is actually sharp.

Now some exercises with almost no probability content, only coarse geometry and graph theory.

▷ **Exercise 11.** Recall that a bounded degree infinite graph satisfies the isoperimetric inequality IP_d if $|\partial S| > c|S|^{\frac{d-1}{d}}$ for every finite $S \subset V(G)$. In particular, IP_∞ means non-amenable.

(a) Show that a bounded degree tree without leaves is amenable iff there is no bound on the length of "hanging chains", i.e., chains of vertices with degree 2. (Consequently, for trees, $IP_{1+\epsilon}$ implies IP_∞.)

(b) Give an example of a bounded degree tree of exponential volume growth that satisfies no $IP_{1+\epsilon}$, recurrent for the simple random walk on it, and has $p_c = 1$ for percolation.

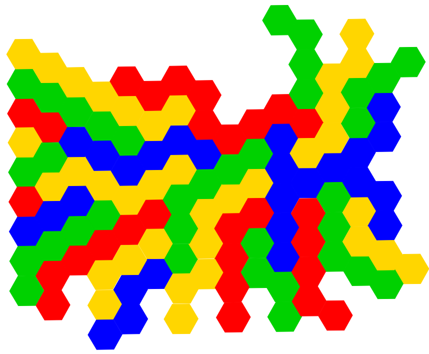

Figure 1: Trying to create at least 7 neighbours for each country.

▷ **Exercise 12.** Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.

▷ **Exercise 13.** Show that a bounded degree graph $G(V, E)$ is nonamenable if and only if it has a wobbling paradoxical decomposition: two injective maps $\alpha, \beta : V \longrightarrow V$ such that $\alpha(V) \sqcup \beta(V) = V$ is a disjoint union, and both maps are at a bounded distance from the identity, or wobbling: $\sup_{x \in V} d(x, \alpha(x)) < \infty$. (Hint: State and use the locally finite infinite bipartite graph version of the Hall marriage theorem, called the Hall-Rado theorem.)

Remark. The previous exercise can be used to give a simple nice proof that groups with Følner-non-amenable Cayley graphs are also von Neumann non-amenable: they do not have group-translation-invariant finitely-additive probability measures defined on all their subsets.

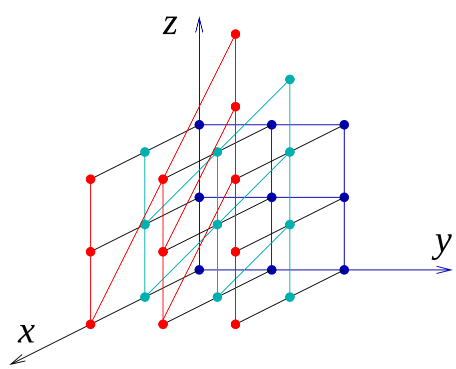

Figure 2: The Cayley graph of the Heisenberg group with generators X, Y, Z.

The **3-dimensional discrete Heisenberg group** is the matrix group

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

If we denote by X, Y, Z the matrices given by the three permutations of the entries $1, 0, 0$ for x, y, z, then $H_3(\mathbb{Z})$ is given by the presentation $\langle X, Y, Z \,|\, [X, Z] = 1, [Y, Z] = 1, [X, Y] = Z \rangle$.

▷ **Exercise 14.** Show that the discrete Heisenberg group has 4-dimensional volume growth.