Stochastic models — homework problems

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The number of dots $\bullet$ is the value of an exercise. Recall that 15 points are due on April 3, another 15 by April 30 Thu 2 pm (in my Department Office pigeonhole). If you have seriously tried to solve some problem, but got stuck, I will be happy to help. Also, if your final solution to a problem has some mistake but has some potential to work, then I will give it back and you can try and correct the mistake.

▷ **Exercise 1.** Let $D_n := \mathrm{dist}(X_n, X_0)$ be the distance of SRW from the starting point on a graph.

(a) $\bullet\bullet$ Using the Central Limit Theorem, prove that $\mathbf{E}[\,D_n\,] \asymp \sqrt{n}$ on any $\mathbb{Z}^d$.

(b) $\bullet\bullet$ Use the notion of stochastic domination to compare D_n on the d-regular tree $\mathbb{T}_d$ with a biased random walk on $\mathbb{Z}$, then prove carefully from Azuma-Hoeffding that the return probability $p_n(o,o)$ on $\mathbb{T}_d$ decays exponentially in n.

(c) $\bullet$ Using the exponential decay in the previous part, prove that $\mathbf{E}[\,D_n\,] \sim \frac{d-2}{d} n$, as $n \to \infty$.

▷ **Exercise 2.**

(a) $\bullet\bullet$ Recall (and give a reference), or prove using the reflection principle, that if $\{X_k\}_{k\geq 0}$ is SRW on $\mathbb{Z}$, and $M_n = \max_{k\leq n} X_k$, then

$$2\,\mathbf{P}[\,X_n \geq t\,] \geq \mathbf{P}[\,M_n \geq t\,]\,.$$

Using this and Exercise 1 (a), show that the expected number of vertices visited by $\{X_k\}$ by time n is

$$\mathbf{E}|\{X_0, X_1, \dots, X_n\}| \asymp \sqrt{n}\,.$$

(b) $\bullet\bullet\bullet\bullet\bullet$ For SRW on $\mathbb{Z}^2$, show that the expected number of vertices visited by time n is

$$\mathbf{E}|\{X_0, X_1, \dots, X_n\}| \asymp n/\log n\,.$$

(c) $\bullet\bullet\bullet\bullet$ Prove that, for SRW on any transient transitive graph,

$$\lim_{n\to\infty} \frac{\mathbf{E}|\{X_0, X_1, \dots, X_n\}|}{n} = \mathbf{P}[\,X_k \neq X_0,\ k = 1, 2, \dots\,] > 0\,.$$

▷ **Exercise 3.** $\bullet\bullet$ Prove that for Green's function of simple random walk on a connected graph, for any vertices x, y, a, b and any real $z > 0$,

$$G(x, y|z) < \infty \ \Leftrightarrow\ G(a, b|z) < \infty\,.$$

Therefore, by Pringsheim's theorem, we have that the radius of convergence is independent of x, y.

▷ **Exercise 4.** $\bullet\bullet\bullet$ Compute the spectral radius $\rho(\mathbb{T}_{k,\ell})$, where $\mathbb{T}_{k,\ell}$ is a tree such that if $v_n \in \mathbb{T}_{k,\ell}$ is a vertex at distance n from the root,

$$\deg v_n = \begin{cases} k & n \text{ even} \\ \ell & n \text{ odd}\,. \end{cases}$$

▷ **Exercise 5.** Two basic exercises about martingales:

(a) •• Show that if $\{M_i\}_{i=0}^{\infty}$ is a martingale, then the differences $X_i = M_i - M_{i-1}$ satisfy the uncorrelatedness condition $\mathbf{E}\big[\, X_{i_1} \cdots X_{i_k} \,\big] = 0$, for any $k \in \mathbb{Z}_+$ and $i_1 < i_2 < \cdots < i_k$.

(b) •••• Give an example of a random sequence $(M_n)_{n=0}^{\infty}$ such that $\mathbf{E}[\, M_{n+1} \mid M_n \,] = M_n$ for all $n \geq 0$, but which is not a martingale w.r.t. the filtration $\mathscr{F}_n = \sigma(M_0, \ldots, M_n)$.

▷ **Exercise 6.** •••••• Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.

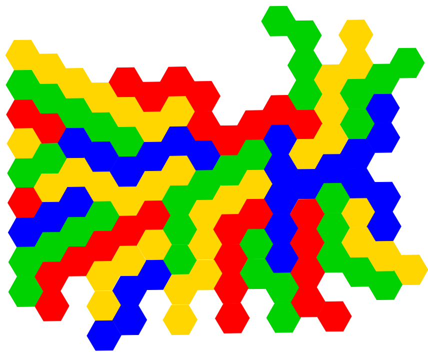

Figure 1: Trying to create at least 7 neighbours for each country.

▷ **Exercise 7.** Recall that being non-amenable means satisfying the strong isoperimetric inequality IP_∞.

(a) •• Show that a bounded degree tree without leaves is amenable iff there is no bound on the length of "hanging chains", i.e., chains of vertices with degree 2. (Consequently, for trees, $IP_{1+\epsilon}$ implies IP_∞.)

(b) •••• Give an example of a bounded degree tree of exponential volume growth that satisfies no $IP_{1+\epsilon}$ and is recurrent for the simple random walk on it.

▷ **Exercise 8.** •••••• Show that a bounded degree graph $G(V, E)$ is nonamenable if and only if it has a wobbling paradoxical decomposition: two injective maps $\alpha, \beta : V \longrightarrow V$ such that $\alpha(V) \sqcup \beta(V) = V$ is a disjoint union, and both maps are at a bounded distance from the identity, or wobbling: $\sup_{x \in V} d(x, \alpha(x)) < \infty$. (Hint: State and use the locally finite infinite bipartite graph version of the Hall marriage theorem, called the Hall-Rado theorem.)

▷ **Exercise 9.** Recall that the universal covering tree T of a graph G is the unique tree for which there exists a surjective graph-homomorphism $\pi : T \longrightarrow G$ that locally, restricted to the radius 1 neighborhood of any vertex of G, is an isomorphism.

(a) •• Show that the universal covering tree of any finite graph is quasi-transitive (that is, its automorphism group has finitely many orbits).

(b) ••• Give an example of a quasi-transitive infinite tree that is not the universal covering tree of any finite graph.

▷ **Exercise 10.** ••• Consider the graph G with 6 vertices and 7 edges that looks like a figure 8 on a digital display. Consider the uniform measure on the 15 spanning trees of G, denoted by UST, and the uniform measure on the 7 connected subgraphs with 6 edges (one more than a spanning tree), denoted by $\mathsf{UST}+1$. Find an explicit monotone coupling between the two measures (i.e., with $\mathsf{UST} \subset \mathsf{UST}+1$).

Remark. I do not know if such a monotone coupling exists for any finite graph. A proof or a counterexample would earn you at least 15 points and would be the basis of a great MSc thesis.

▷ **Exercise 11.** ••• Find the order of magnitude of the threshold function $p_{1/2}(n)$ for the random graph $G(n,p)$ containing a copy of the cycle C_4.

▷ **Exercise 12.**

(a) •• Using the 2nd Moment Method, show that for $p = \frac{\lambda \ln n}{n}$, with $\lambda < 1$ fixed, there exist isolated vertices in $G(n,p)$ with probability tending to 1.

(b) •• Let $I_\lambda(n)$ be the expected number of isolated vertices in the previous part. Show that if $0 < \lambda' < \lambda < 1$, and $k(n) = I_{\lambda'}(n) \gg I_\lambda(n)$, then the probability that there exists a component (or a union of components) of size $k(n)$ in $G(n, \frac{\lambda \ln n}{n})$ is going to 0. This is an indication that isolated vertices are indeed the main obstacles to connectivity.

▷ **Exercise 13.** Consider a GW process with offspring distribution ξ, $\mathbf{E}\xi = \mu$. Let Z_n be the size of the nth level, with $Z_0 = 1$, the root. Recall that Z_n/μ^n is a martingale.

(a) ••• Assuming that $\mu > 1$ and $\mathbf{E}[\,\xi^2\,] < \infty$, first show that $\mathbf{E}\big[\,Z_n^2\,\big] \le C(\mathbf{E}Z_n)^2$. (Hint: use the conditional variance formula $\mathbf{D}^2[Z_n] = \mathbf{E}\big[\,\mathbf{D}^2[Z_n \mid Z_{n-1}]\,\big] + \mathbf{D}^2\big[\mathbf{E}[\,Z_n \mid Z_{n-1}\,]\big]$.) Then, using the Second Moment Method, deduce that the GW process survives with positive probability.

(b) •• Extend the above to the case $\mathbf{E}\xi = \infty$ or $\mathbf{D}\xi = \infty$ by a truncation $\xi \mathbf{1}_{\xi<K}$ for K large enough.

▷ **Exercise 14.** If X is a non-negative random variable with finite expectation, then its size-biased version $\widehat{X}$ is defined by $\mathbf{P}[\widehat{X} \in A] = \mathbf{E}[\,X\,\mathbf{1}_{\{X \in A\}}\,]/\mathbf{E}X$.

(a) • Show that the size-biased version of $\mathsf{Poi}(\lambda)$ is just $\mathsf{Poi}(\lambda) + 1$.

(b) • Show that the size-biased version of $\mathsf{Expon}(\lambda)$ is the sum of two independent $\mathsf{Expon}(\lambda)$'s.

(c) ••• Take Poisson point process of intensity λ on $\mathbb{R}$. Condition on the interval $(-\epsilon, \epsilon)$ to contain at least one arrival. As $\epsilon \to 0$, what is the point process we obtain in the limit? What does this have to do with parts (a) and (b)?

▷ **Exercise 15.** ••••• Show that $\mathsf{Binom}(n-1, \lambda/n)$ is stochastically dominated by $\mathsf{Poi}(\lambda)$.

▷ **Exercise 16.**

(a) •• Take a Möbius map from the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{H}$ that takes the center $0 \in \mathbb{D}$ to $i \in \mathbb{H}$. Compute the pushforward of the uniform measure on $\partial\mathbb{D}$ to $\mathbb{R} = \partial\mathbb{H}$, and get the Cauchy distribution.

(b) •• Now understand what the conformal map $z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$ does to the unit disk $\mathbb{D}$ and its complement $\mathbb{C} \setminus \mathbb{D}$. Compute again the pushforward of the uniform measure on $\partial\mathbb{D}$. Interpret the result as the hitting distribution of a free electron performing 2-dimensional Brownian motion, coming from infinitely far, and observe that this distribution is a key to how lightning rods work.

▷ **Exercise 17.** ••• Consider some random walk on $\mathbb{R}$, denoted by $S_n = X_1 + \cdots + X_n$, for $n = 0, 1, \ldots$. Show that if $\mathbf{P}[\,S_n \in (-2\epsilon, 2\epsilon) \text{ infinitely often}\,] < 1$ for some $\epsilon > 0$, then the expected number of returns of S_n to $(-\epsilon, \epsilon)$ is finite. Therefore, our computation in class that the latter expectation for Cauchy jumps is infinite for any $\epsilon > 0$ shows that this walk is recurrent.

▷ **Exercise 18.** Consider asymmetric simple random walk (X_i) on $\mathbb{Z}$, with probability $p > 1/2$ for a right step and $1-p$ for a left step.

(a) •• Find a martingale of the form r^{X_i} for some $r > 0$, and calculate $\mathbf{P}_k[\,\tau_0 > \tau_n\,]$.

(b) ••• Find a martingale of the form $X_i - \mu\, i$ for some $\mu > 0$, and calculate $\mathbf{E}_k[\,\tau_0 \wedge \tau_n\,]$. (Hint: to prove that this second martingale is uniformly integrable, first show that $\tau_0 \wedge \tau_n$ has an exponential tail.)

▷ **Exercise 19.** ••••• Using the exploration Markov chain for GW trees and a Doob transform, show that if we condition the GW tree with offspring distribution Poisson(λ) on extinction, where $\lambda > 1$, then we get a GW tree with offspring distribution Poisson(μ) with $\mu < 1$, where $\lambda e^{-\lambda} = \mu e^{-\mu}$.

▷ **Exercise 20.** ••••• For the GW tree with offspring distribution Poisson$(1+\epsilon)$, show that the survival probability is asymptotically 2ϵ, as $\epsilon \to 0$.

▷ **Exercise 21.** •••• Prove the Bollobás-Thomason threshold theorem: for any sequence monotone events $\mathcal{A} = \mathcal{A}_n$ and any ϵ there is $C_\epsilon < \infty$ such that $\left|p^{\mathcal{A}}_{1-\epsilon}(n) - p^{\mathcal{A}}_{\epsilon}(n)\right| < C_\epsilon \left(p^{\mathcal{A}}_{\epsilon}(n) \wedge (1 - p^{\mathcal{A}}_{1-\epsilon}(n))\right)$. (Hint: take many independent copies of low density to get success with good probability at a larger density.)

▷ **Exercise 22.** •••• In the random graph $G(n,p)$ with $p = \lambda/n$, for $\mathcal{A}_n = \{\text{containing a triangle}\}$, show directly that the expected number of pivotal edges is $\asymp n$ (with factors depending on λ), and hence, by Russo's formula, the threshold window is of size $p^{\mathcal{A}}_{1-\epsilon}(n) - p^{\mathcal{A}}_{\epsilon}(n) \asymp 1/n$, as we also saw earlier.

▷ **Exercise 23.** For functions $f : \{-1,1\}^n \longrightarrow \mathbb{R}$ of n bits, consider the inner product $(f,g) := \mathbf{E}_p[\, fg\,]$, where each bit is 1 with probability p and -1 with probability $1-p$, independently.

(a) •• For $p = 1/2$ show that $\left\{\chi_S(\omega) := \prod_{i\in S} \omega(i) : S \subseteq [n]\right\}$ is an orthonormal basis for this inner product space.

(b) •• Find a similar orthonormal basis for general p.

▷ **Exercise 24.** With the notation of the previous exercise, define the **Fourier-Walsh coefficients** $\widehat{f}(S) := \mathbf{E}_{1/2}\left[\, f(\omega)\, \chi_S(\omega)\,\right]$. We will consider monotone Boolean functions with values in $\{-1,1\}$ (instead of the usual $\{0,1\}$, because our formulas will be simpler this way).

(a) •• Show that the probability that the kth bit is pivotal for f is exactly $\widehat{f}(\{k\})$.

(b) ••• Using Cauchy-Schwarz and Parseval, deduce that the expected number of pivotals at $p = 1/2$ is at most $\sqrt{n}$.

(c) •• Show by the example of majority, $\mathsf{Maj}(x_1,\dots,x_{2k+1}) = \operatorname{sign}(x_1 + \cdots + x_{2k+1})$, that this is sharp.

▷ **Exercise 25.** ••• For a subset A of the hypercube $\{0,1\}^n$, let $B(A,t) := \left\{x \in \{0,1\}^n : \operatorname{dist}(x,A) \le t\right\}$. Let $\epsilon, \lambda > 0$ be constants satisfying $\exp(-\lambda^2/2) = \epsilon$. Prove using Azuma-Hoeffding that

$$|A| \ge \epsilon\, 2^n \implies \left|B(A, 2\lambda\sqrt{n})\right| \ge (1-\epsilon)\, 2^n \,.$$

That is, even small sets become huge if we enlarge them a little.

▷ **Exercise 26.** ••• Is there a graph property (a subset of $\{0,1\}^{\binom{n}{2}}$ that is closed under graph isomorphisms) for which the edge exposure martingale is a random walk on $\mathbb{R}$, or even SRW on $\mathbb{Z}$, started somewhere?

▷ **Exercise 27.** •• Show that for percolation on any infinite graph, the event {there are exactly three infinite clusters} is Borel measurable.

▷ **Exercise 28.** •••• Let $G(V,E)$ be any bounded degree infinite graph, and $S_n \nearrow V$ an exhaustion by finite connected subsets. Is it true that, for $p > p_c(G)$, we have

$$\lim_{n\to\infty} \mathbf{P}_p[\,\text{largest cluster for percolation inside } S_n \text{ is the subset of an infinite cluster}\,] = 1\,?$$

▷ **Exercise 29.** •• As in class, a trifurcation point of an infinite cluster is a vertex whose removal breaks the cluster into at least 3 infinite components. Show carefully the claim we used in the Burton-Keane theorem: if $\mathscr{C}_\infty$ denotes the union of all the infinite clusters in some percolation on G, and $U \subset V(G)$ is finite, then the size of $\mathscr{C}_\infty \cap \partial_V^{\mathrm{out}} U$ is at least the number of trifurcation points of $\mathscr{C}_\infty$ in U, plus 2.

▷ **Exercise 30.**

(a) •••• Give an $\mathrm{Aut}(\mathbb{Z}^2)$-invariant and $\mathbb{Z}^2$-ergodic percolation on $\mathbb{Z}^2$ with infinitely many ∞ clusters.

(b) •••••• Give an $\mathrm{Aut}(\mathbb{Z}^2)$-invariant and $\mathbb{Z}^2$-ergodic percolation on $\mathbb{Z}^2$ with exactly two ∞ clusters.

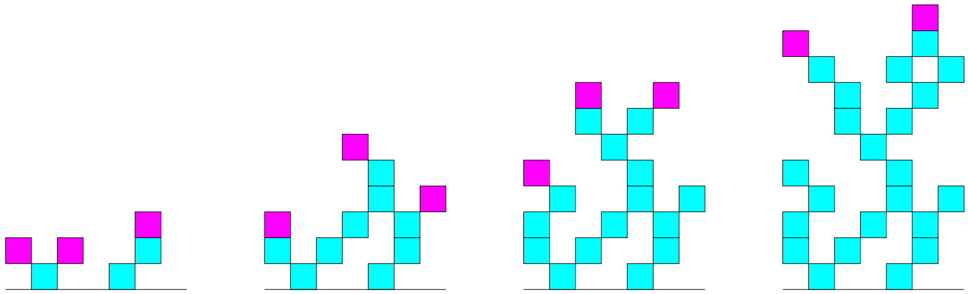

Figure 2: Sorry, this picture is on the segment, not on the cycle.

▷ **Exercise 31.** •••• A simple version of the Tetris game (with no player): on the discrete cycle of length K, unit squares with sticky corners are falling from the sky, at places $[i, i+1]$ chosen uniformly at random ($i = 0, 1, \ldots, K-1$, mod K). Let R_t be the size of the roof after t squares have fallen: those squares of the current configuration that could have been the last to fall. Show that $\lim_{t\to\infty} \mathbf{E} R_t = K/3$.

Remark. If there are two types of squares, particles and antiparticles that annihilate each other when falling on exactly on top of each other, this process is a SRW on a group, and the size of the roof has to do with the speed of the SRW. Here, for $K \geq 4$, the expected limiting size of the roof is already less than $0.32893K$, but this is far from trivial. What's the situation for $K = 3$?