Stochastic models — homework problems

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▷ **Exercise 1.** Let $D_n := \mathrm{dist}(X_n, X_0)$ be the distance of SRW from the starting point.

(a) Using the Central Limit Theorem, prove that $\mathbf{E}[\,D_n\,] \asymp \sqrt{n}$ on any $\mathbb{Z}^d$.

(b) Comparing the number of visits to $X_0 = o$ on $\mathbb{T}_k$ and on $\mathbb{Z}$, prove that $\mathbf{E}[\,D_n\,] \sim \frac{k-2}{k} n$, as $n \to \infty$.

▷ **Exercise 2.** Prove that for Green's function of simple random walk on a connected graph, for real $z > 0$,

$$G(x, y|z) < \infty \;\Leftrightarrow\; G(r, w|z) < \infty\,.$$

Therefore, by Pringsheim's theorem, we have that $\mathsf{rad}(x, y)$ is independent of x, y.

▷ **Exercise 3.** Compute $\rho(\mathbb{T}_{k,\ell})$, where $\mathbb{T}_{k,\ell}$ is a tree such that if $v_n \in \mathbb{T}_{k,\ell}$ is a vertex at distance n from the root,

$$\deg v_n = \begin{cases} k & n \text{ even} \\ \ell & n \text{ odd} \end{cases}$$

▷ **Exercise 4** ("Green's function is the inverse of the Laplacian")**.** Let (V, P) be a transient Markov chain with a stationary measure π and associated Laplacian $\Delta = I - P$. Assume that the function $y \mapsto G(x, y)/\pi_y$ is in $L^2(V, \pi)$. Let $f : V \longrightarrow \mathbb{R}$ be an arbitrary function in $L^2(V, \pi)$. Solve the equation $\Delta u = f$.

▷ **Exercise 5.** Give an example of a random sequence $(M_n)_{n=0}^{\infty}$ such that $\mathbf{E}[\,M_{n+1} \mid M_n\,] = M_n$ for all $n \geq 0$, but which is not a martingale w.r.t. the filtration $\mathscr{F}_n = \sigma(M_0, \dots, M_n)$.

▷ **Exercise 6.** Consider asymmetric simple random walk (X_i) on $\mathbb{Z}$, with probability $p > 1/2$ for a right step and $1-p$ for a left step. Find a martingale of the form r^{X_i} for some $r > 0$, and calculate $\mathbf{P}_k[\,\tau_0 > \tau_n\,]$. Then find a martingale of the form $X_i - \mu\, i$ for some $\mu > 0$, and calculate $\mathbf{E}_k[\,\tau_0 \wedge \tau_n\,]$. (Hint: to prove that the second martingale is uniformly integrable, first show that $\tau_0 \wedge \tau_n$ has an exponential tail.)

▷ **Exercise 7.**

(a) For SRW on $\mathbb{Z}^2$, show that the expected number of vertices visited by time n is

$$\mathbf{E}|\{X_0, X_1, \dots, X_n\}| \asymp n/\log n\,.$$

(b) Conclude that on the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{Z}^2$, the distance is $\mathbf{E}\,\mathrm{dist}(Y_0, Y_n) \asymp n/\log n$.

▷ **Exercise 8.**

(a) Prove that, for SRW on any transient transitive graph,

$$\lim_{n\to\infty} \frac{\mathbf{E}|\{X_0, X_1, \dots, X_n\}|}{n} = \mathbf{P}[\,X_k \neq X_0,\ k = 1, 2, \dots\,]\,.$$

(b) Conclude that on the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{Z}^d$, with $d \geq 3$, the expected distance grows linearly.

▷ **Exercise 9.** For SRW on the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{Z}$, show that $p_{2n}(o,o) \geq c_1 \exp(-c_2 n^{1/3})$. (We will later state the theorem that groups of exponential growth have $p_{2n}(o,o) \leq c_2 \exp(-c_3 n^{1/3})$, as well.)

▷ **Exercise 10.** A simple version of the Tetris game (with no player): on the discrete cycle of length K, unit squares with sticky corners are falling from the sky, at places $[i, i+1]$ chosen uniformly at random ($i = 0, 1, \ldots, K-1$, mod K). Let R_t be the size of the roof after t squares have fallen: those squares of the current configuration that could have been the last to fall. Show that $\lim_{t\to\infty} \mathbf{E} R_t = K/3$.

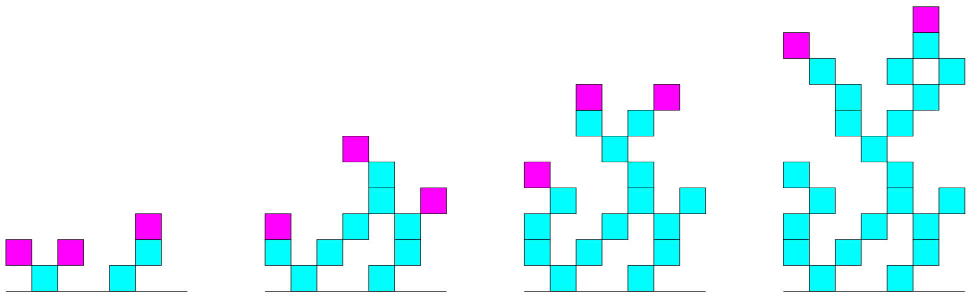

Figure 1: Sorry, this picture is on the segment, not on the cycle.

Remark. If there are two types of squares, particles and antiparticles that annihilate each other when falling on exactly on top of each other, this process is a SRW on a group, and the size of the roof has to do with the speed of the SRW. Here, for $K \geq 4$, the expected limiting size of the roof is already less than $0.32893K$, but this is far from trivial. What's the situation for $K = 3$?

▷ **Exercise 11.*** Show that any harmonic function f on $\mathbb{Z}^d$ with sublinear growth, i.e., one that satisfies $\lim_{\|x\|_2 \to \infty} f(x)/\|x\|_2 = 0$, must be constant.

▷ **Exercise 12.**** Prove via couplings that $\mathbb{Z}^d$ has the strong Liouville property: any positive harmonic function on $\mathbb{Z}^d$ must be constant.

▷ **Exercise 13.** Consider an irreducible Markov chain (V, P).

(a) Assume for the total variation distance that $d_{\mathrm{TV}}\big(p_n(x,\cdot)), p_n(y,\cdot)\big) \to 0$ as $n \to \infty$, for any $x, y \in V$. Show that (V, P) has the Liouville property.

(b) Show that biased nearest-neighbor random walk on $\mathbb{Z}$ has the property of part (a), but nevertheless it does not have the strong Liouville property: it has non-constant positive harmonic functions.

▷ **Exercise 14.*** Show that $\mathbf{E}\big[\, \|\Psi(X_n) - \Psi(X_0)\|^4 \,\big] \leq Cn^2$, using the orthogonality of martingale incremements. Then deduce that $\mathbf{E}\big[\, d(X_0, X_n) \,\big] \geq c\sqrt{n}$. (This improvement over Anna Erschler's argument is due to Bálint Virág. **Hint:** do not be afraid to consider the time-reversal of the random walk when you need to condition on the future.)

▷ **Exercise 15.** Show that the regular trees $\mathbb{T}_k$ and $\mathbb{T}_\ell$ for $k, \ell \geq 3$ are quasi-isometric to each other, by giving explicit quasi-isometries.

▷ **Exercise 16.** Make either definition from class for the **space of ends** of a graph precise as a topological space. Prove that any quasi-isometry of graphs induces naturally a homeomorphism of their spaces of ends.

▷ **Exercise 17** (Hopf 1944)**.**

(a) Show that a group has two ends iff it has $\mathbb{Z}$ as a finite index subgroup.

(b) Show that if a f.g. group has at least 3 ends, then it has continuum many.

▷ **Exercise 18.*** Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.

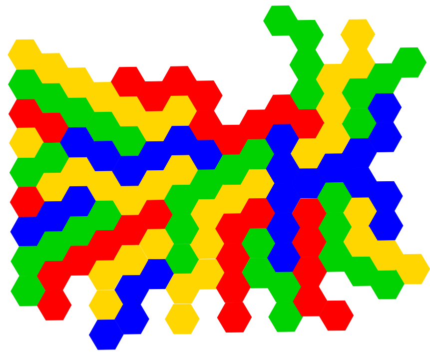

Figure 2: Trying to create at least 7 neighbours for each country.

▷ **Exercise 19.**

(a) Find the edge Cheeger constant $\iota_{\infty,E}$ of the infinite binary tree.

(b) Show that a bounded degree tree without leaves is amenable iff there is no bound on the length of "hanging chains", i.e., chains of vertices with degree 2. (Consequently, for trees, $IP_{1+\epsilon}$ implies IP_∞.)

(c) Give an example of a bounded degree tree of exponential volume growth that satisfies no $IP_{1+\epsilon}$ and is recurrent for the simple random walk on it.

▷ **Exercise 20.*** Show that a bounded degree graph $G(V,E)$ is nonamenable if and only if it has a wobbling paradoxical decomposition: two injective maps $\alpha, \beta : V \longrightarrow V$ such that $\alpha(V) \sqcup \beta(V) = V$ is a disjoint union, and both maps are at a bounded distance from the identity, or wobbling: $\sup_{x \in V} d(x, \alpha(x)) < \infty$. (Hint: State and use the locally finite infinite bipartite graph version of the Hall marriage theorem, called the Hall-Rado theorem.)

▷ **Exercise 21.** Show by examples that, in directed weighted graphs, the measure $(C_x)_{x \in V}$ might be non-stationary, and might be stationary but non-reversible. Can the walk associated to a finite directed weighted graph (with at least one non-symmetric weight) have a reversible measure?

▷ **Exercise 22.** Show that effective resistances (as defined in class, (6.3) of PGG) add up when combining networks in series, while effective conductances add up when combining networks in parallel.

▷ **Exercise 23.** Let $G(V,E,c)$ be a transitive network (i.e., the group of graph automorphisms preserving the edge weights have a single orbit on V). Show that, for any $u, v \in V$,

$$\mathbf{P}_u[\,\tau_v < \infty\,] = \mathbf{P}_v[\,\tau_u < \infty\,]\,.$$

▷ **Exercise 24.**** `http://xkcd.com/356/`

▷ **Exercise 25.** Prove the following claims made and vaguely explained in class about total variation distance:

(a) $d_{\mathrm{TV}}(\mu,\nu) = \min\big\{\mathbf{P}[\,X \neq Y\,] : (X,Y) \text{ is a coupling of } \mu \text{ and } \nu\big\}$.

(b) $d(t) \le \bar{d}(t) \le 2d(t)$.

(c) Using part (a), show that $\bar{d}(t+s) \le \bar{d}(t)\,\bar{d}(s)$.

▷ **Exercise 26.** Let (V, P) be a reversible, finite Markov chain, with stationary distribution $\pi(x)$. Recall that P is self-adjoint with respect to $(f, g) = \sum_{x \in V} f(x)g(x)\pi(x)$. Show:

(a) If $f : V \longrightarrow \mathbb{R}$ is a right eigenfunction of P, then $x \mapsto g(x) = f(x)\pi(x)$ is a left eigenfunction, with the same eigenvalue.

(b) All eigenvalues λ_i satisfy $-1 \le \lambda_i \le 1$.

(c) If we write $-1 \le \lambda_n \le \cdots \le \lambda_1 = 1$, then $\lambda_2 < 1$ if and only if (V, P) is connected (the chain is irreducible).

(d) $\lambda_n > -1$ if and only if (V, P) is not bipartite. (Recall here the easy lemma that a graph is bipartite if and only if all cycles are even.)

▷ **Exercise 27.**

(a) For $f : V \longrightarrow \mathbb{R}$, let $\mathrm{Var}_\pi[f] := \mathbf{E}_\pi[f^2] - (\mathbf{E}_\pi f)^2 = \sum_x f(x)^2\pi(x) - \big(\sum_x f(x)\pi(x)\big)^2$. Show that $g_{\mathrm{abs}} > 0$ implies that $\lim_{t\to\infty} P^t f(x) = \mathbf{E}_\pi f$ for all $x \in V$. Moreover,

$$\mathrm{Var}_\pi[P^t f] \le (1 - g_{\mathrm{abs}})^{2t}\,\mathrm{Var}_\pi[f]\,,$$

with equality at the eigenfunction corresponding to the λ_i giving $g_{\mathrm{abs}} = 1 - |\lambda_i|$. Hence t_{relax} is the time needed to reduce the standard deviation of any function to $1/e$ of its original standard deviation.

(b) Show that if the chain (V, P) is transitive, then

$$4\,d_{\mathrm{TV}}\big(p_t(x,\cdot), \pi(\cdot)\big)^2 \le \left\| \frac{p_t(x,\cdot)}{\pi(\cdot)} - \mathbf{1}(\cdot) \right\|_2^2 = \sum_{i=2}^n \lambda_i^{2t}\,.$$

For instance, recall the spectrum of the lazy walk on the hypercube $\{0,1\}^k$, and prove the bound $d\big(1/2\,k\ln k + c\,k\big) \le e^{-2c}/2$ for $c > 1$ on the TV distance. (This is sharp even regarding the constant $1/2$ in front of $k \ln k$.) Also, recall the spectrum of the cycle C_n, and show that $t_{\mathrm{mix}}^{\mathrm{TV}}(C_n) = O(n^2)$.

▷ **Exercise 28.** You may accept here that transitive expanders exist.

(a) Give a sequence of d-regular transitive graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$ that mix rapidly, $t_{\mathrm{mix}}^{\mathrm{TV}}(1/4) = O(\log |V_n|)$, but do not form an expander sequence.

(b) In a similar manner, give a sequence $G_n = (V_n, E_n)$ satisfying $t_{\mathrm{relax}} \asymp t_{\mathrm{mix}}^{\mathrm{TV}}(1/4)^\alpha \asymp \log^\alpha |V_n|$, with some $0 < \alpha < 1$.

The next few exercises have no probability content, only geometric group theory.

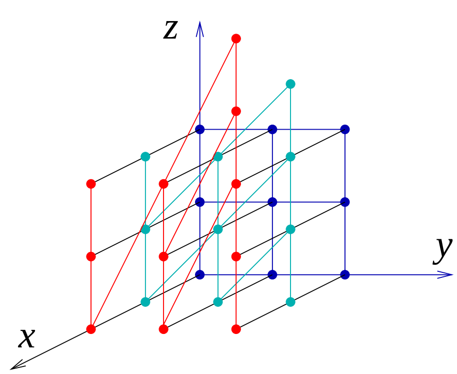

Figure 3: The Cayley graph of the Heisenberg group with generators X, Y, Z.

The **3-dimensional discrete Heisenberg group** is the matrix group

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,:\, x,y,z \in \mathbb{Z} \right\}.$$

If we denote by X, Y, Z the matrices given by the three permutations of the entries $1, 0, 0$ for x, y, z, then $H_3(\mathbb{Z})$ is given by the presentation

$$\big\langle X, Y, Z \,\big|\, [X,Z] = 1, [Y,Z] = 1, [X,Y] = Z \big\rangle.$$

▷ **Exercise 29.** Show that the discrete Heisenberg group has 4-dimensional volume growth.

A group homomorphism $\varphi : \Gamma_1 \longrightarrow \Gamma_2$ is an **expanding virtual isomorphism** if it is expanding (hence injective), and $[\Gamma_2 : \varphi(\Gamma_1)] < \infty$. For instance, for the Heisenberg group $H_3(\mathbb{Z})$, the map

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \overset{\varphi_{m,n}}{\longmapsto} \begin{pmatrix} 1 & mx & mnz \\ 0 & 1 & ny \\ 0 & 0 & 1 \end{pmatrix}$$

is an expanding virtual automorphism, with index $[H_3(\mathbb{Z}) : \varphi_{m,n}(H_3(\mathbb{Z}))] = m^2 n^2$.

▷ **Exercise 30.** Prove that if a finitely generated group has an expanding virtual automorphism, then it has polynomial growth.

▷ **Exercise 31.** *** Assume Γ is a finitely generated group and has a virtual isomorphism φ such that

$$\bigcap_{n \geq 1} \varphi^n(\Gamma) = \{1\}.$$

(This is the case, e.g., when φ is expanding.) Does this imply that Γ has polynomial growth?

A condition weaker than in the last exercise is the following: a group Γ is called **scale-invariant** if it has a chain of subgroups $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ such that $[\Gamma : \Gamma_n] < \infty$ and $\bigcap \Gamma_n = \{1\}$. This notion was introduced by Itai Benjamini, and he had conjectured that it implies polynomial growth of Γ. However, this was disproved by Nekrashevych and myself: the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is a counterexample.

The following exercise is why we know that Grigorchuk's group has intermediate volume growth.

▷ **Exercise 32.** If Γ is a group with growth function $v_\Gamma(n)$ and there exists an expanding virtual isomorphism

$$\underbrace{\Gamma \times \Gamma \times \cdots \times \Gamma}_{m \geq 2} \longrightarrow \Gamma,$$

then $\exp(n^{\alpha_1}) \preceq v_\Gamma(n) \preceq \exp(n^{\alpha_2})$ for some $0 < \alpha_1 \leq \alpha_2 < 1$. (Hint: $\Gamma^m \hookrightarrow \Gamma$ implies the existence of α_1, since $v(n)^m \leq C\, v(kn)$ for all n implies that $v(n)$ has superpolynomial growth. The expanding virtual isomorphism gives the existence of α_2.)

Now back to probability.

▷ **Exercise 33.** Show that the uniform random d-regular bipartite multigraph on $2n$ vertices with $d \geq 3$ has 4-cycles with a positive probability, and no 4-cycles with a positive probability, uniformly in n.

▷ **Exercise 34.** Let $G(V,E)$ be any bounded degree infinite graph, and $S_n \nearrow V$ an exhaustion by finite connected subsets. Is it true that, for $p > p_c(G)$, we have

$$\lim_{n\to\infty} \mathbf{P}_p[\,\text{largest cluster for percolation inside } S_n \text{ is the subset of an infinite cluster}\,] = 1\,?$$

▷ **Exercise 35.**

(a) Show that for percolation on any infinite graph, the event {there are exactly three infinite clusters} is Borel measurable.

(b) Give an $\mathrm{Aut}(\mathbb{Z}^2)$-invariant and $\mathbb{Z}^2$-ergodic percolation on $\mathbb{Z}^2$ with infinitely many ∞ clusters.

▷ **Exercise 36.** Give an $\mathrm{Aut}(\mathbb{Z}^2)$-invariant and $\mathbb{Z}^2$-ergodic percolation on $\mathbb{Z}^2$ with exactly two ∞ clusters.

▷ **Exercise 37.** ***

(a) Is there an ergodic deletion-tolerant $\mathbb{Z}^2$-invariant percolation on $\mathbb{Z}^2$ with exactly two infinite clusters?

(b) And what about infinitely many infinite clusters?

▷ **Exercise 38.** Assume that $\pi : G' \twoheadrightarrow G$ is a topological covering between infinite graphs, or in other words, G is a factor graph of G'. Show that $p_c(G') \le p_c(G)$.

▷ **Exercise 39.**

(a) Show that if in a graph G the number of minimal edge-cutsets (a subset of edges whose removal disconnects a given vertex from infinity, minimal w.r.t. containment) of size n is at most $\exp(Cn)$ for some $C < \infty$, then $p_c(G) \le 1 - \epsilon(C) < 1$.

(b) Fix $o \in V(G)$ in a graph with maximal degree Δ. Prove that the number of connected sets $o \in S \subset V(G)$ of size n is at most $\Delta(\Delta-1)^{2n-3}$. (Hint: any S has a spanning tree, and one can go around a tree visiting each edge twice.) Conclude that $\mathbb{Z}^d$, $d \ge 2$, has an exponential bound on the number of minimal cutsets. In particular, $p_c(\mathbb{Z}^d) < 1$, although we already knew that from $\mathbb{Z}^2 \subseteq \mathbb{Z}^d$.

▷ **Exercise 40.** ** Let $\lambda(G) := \limsup_{n\to\infty} |\{S \subset V(G) : o \in S \text{ connected}, \ |S| = n\}|^{1/n}$ denote the exponential growth rate of the number of "lattice animals". We saw in part (b) of the previous exercise that $\lambda(G) \le (\Delta-1)^2$ for any graph of maximal degree Δ. What is the smallest possible upper bound here? Kesten's book has a beautiful argument proving $\lambda(G) \le (\Delta-1)e$: for site percolation at $p = 1/(\Delta-1)$, write the probability that the cluster of o is finite using lattice animals and their outer vertex boundaries.

▷ **Exercise 41** (Galton-Watson duality)**.** * Either by computing generating functions directly, or by using a Doob transform argument, show the following duality of super- and sub-critical GW trees. Consider a supercritical GW_ξ tree, with generating function $f(z) = \mathbf{E}[\, z^\xi \,]$ and extinction probability $q = f(q)$.

(a) Condition GW_ξ on non-extinction, and take the subtree of those vertices that have an infinite line of descent. Show that this is a GW tree with offspring distribution ξ^*, where

$$\mathbf{P}[\, \xi^* = k \,] = \sum_{j=k}^{\infty} \binom{j}{k} (1-q)^{k-1} q^{j-k} \mathbf{P}[\, \xi = j \,] .$$

Deduce that the generating function $f^*(z) = \mathbf{E}[\, z^{\xi^*} \,]$ is obtained by taking the part of $f(z)$ in the $[q,1]^2$ square and rescaling it to the square $[0,1]^2$. Note that $\mathbf{P}[\, \xi^* = 0 \,] = 0$ and $\mathbf{E}\xi^* = \mathbf{E}\xi$.

(b) Condition GW_ξ on extinction. Show that we get a subcritical GW tree, with offspring distribution $\tilde{\xi}$, whose generating function $\tilde{f}(z)$ is obtained by taking the part of $f(z)$ in the $[0,q]^2$ square and rescaling it to the square $[0,1]^2$. Note that $\mathbf{E}\tilde{\xi} = f'(q) < 1$.

▷ **Exercise 42.** Consider a spherically symmetric tree T where each vertex on the n^{th} level T_n has $d_n \in \{k, k+1\}$ children, such that $\lim_{n\to\infty} |T_n|^{1/n} = k$, but $\sum_{n=0}^{\infty} k^n/|T_n| < \infty$. Using the second moment method, show that $p_c = 1/k$ and $\theta(p_c) > 0$.

To study percolation on general locally finite rooted trees T, Russ Lyons (1990) defined an "average **branching number**"

$$\mathsf{br}(T) := \sup \left\{ \lambda \geq 1 : \inf_{\Pi} \sum_{e \in \Pi} \lambda^{-|e|} > 0 \right\}, \tag{0.1}$$

where the infimum is taken over all cutsets $\Pi \subset E(T)$ separating the root $o \in V(T)$ from infinity, and $|e|$ denotes the distance of the edge e from o. The following exercises help digest what this notion measures:

▷ **Exercise 43.** Let T be a locally finite infinite tree with root o.

(a) Show that $\mathsf{br}(T)$ does not depend on the choice of the root o.

(b) Show that the $d+1$-regular tree has $\mathsf{br}(\mathbb{T}_{d+1}) = d$.

(c) Define the lower growth rate of T by $\underline{\mathsf{gr}}(T) := \liminf_n |T_n|^{1/n}$, where T_n is the set of vertices at distance exactly n from o. Show that $\mathsf{br}(T) \leq \underline{\mathsf{gr}}(T)$.

(d) Let us denote the set of non-backtracking infinite rays starting from o by ∂T, the boundary of the tree, equipped with the metric $d(\xi, \eta) := e^{-|\xi \wedge \eta|}$, where $\xi \wedge \eta$ is the last common vertex of the two rays, and $|\xi \wedge \eta|$ is its distance from o. Show that

$$e^{\dim_H(\partial T, d)} = \mathsf{br}(T) \qquad \text{and} \qquad e^{\underline{\dim}_M(\partial T, d)} = \underline{\mathsf{gr}}(T),$$

where $\dim_H$ is Hausdorff dimension and $\underline{\dim}_M$ is lower Minkowski dimension.

▷ **Exercise 44.** Find the branching number of the following two trees (see Figure 4):

(a) The quasi-transitive tree with degree 3 and degree 2 vertices alternating.

(b) The so-called 3-1-tree, which has 2^n vertices on each level n, with the left 2^{n-1} vertices each having one child, the right 2^{n-1} vertices each having three children; the root has two children.

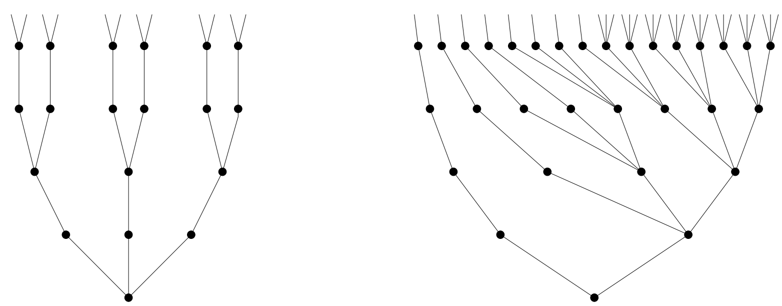

Figure 4: A quasi-transitive tree and the 3-1 tree.

The key theorem Lyons proved (using a version of the 2nd Moment Method) is that $p_c(T) = 1/\mathsf{br}(T)$. This easily implies that $\mathsf{br}(\mathsf{GW}_\xi) = \mathbf{E}\xi$ a.s. on nonextinction, which is a nice "proof" that this is a good definition of average branching. Moreover, the branching number turns out to govern the behavior of most stochastic processes on trees. For instance, if we take λ-biased **homesick random walk**, where the edge going towards the starting point o has weight λ compared to the outgoing edges that have weight 1, the walk is recurrent for $\lambda > \mathsf{br}(T)$ and transient for $\lambda < \mathsf{br}(T)$.

▷ **Exercise 45.** Prove the last statement on transience and recurrence using flows and cutsets in electric networks.

▷ **Exercise 46.** Prove that for any sequence monotone events $\mathcal{A} = \mathcal{A}_n$ and any ϵ there is $C_\epsilon < \infty$ such that $\left|p^{\mathcal{A}}_{1-\epsilon}(n) - p^{\mathcal{A}}_{\epsilon}(n)\right| < C_\epsilon\, p^{\mathcal{A}}_{\epsilon}(n) \wedge (1 - p^{\mathcal{A}}_{1-\epsilon}(n))$. (Hint: take many independent copies of a low density percolation to get success with good probability at a larger density.)

▷ **Exercise 47.** In the random graph $G(n,p)$ with $p = \lambda/n$, for $\mathcal{A}_n = \{\text{containing a triangle}\}$, show directly that the expected number of pivotal edges is $\asymp n$ (with factors depending on λ). (Hence, by Russo's formula, the threshold window is of size $p_{\mathcal{A}}^{1-\epsilon}(n) - p_{\mathcal{A}}^{\epsilon}(n) \asymp 1/n$, as we already saw on class.)

▷ **Exercise 48.** Find the order of magnitude of the threshold function $p_c(n)$ for the random graph $G(n,p)$ containing a copy of **(a)** the complete graph K_4, and **(b)** the cycle C_4.

▷ **Exercise 49.** Consider the d-ary canopy tree of Figure 5: infinitely many leaves on level 0, grouped into d-tuples, each tuple having a parent on level -1, which are grouped again in d-tuples, and so on, along infinitely many levels. Let $p_T := \inf\{p : \mathbf{E}_p|\mathscr{C}(\varrho)| = \infty\}$, where the expectation is both over the random root ϱ and p-percolation. It is clear that $p_T \le p_c$, and there is a theorem that, for transitive graphs, there is equality. However, show that here $p_T = 1/\sqrt{d}$ while $p_c = 1$.

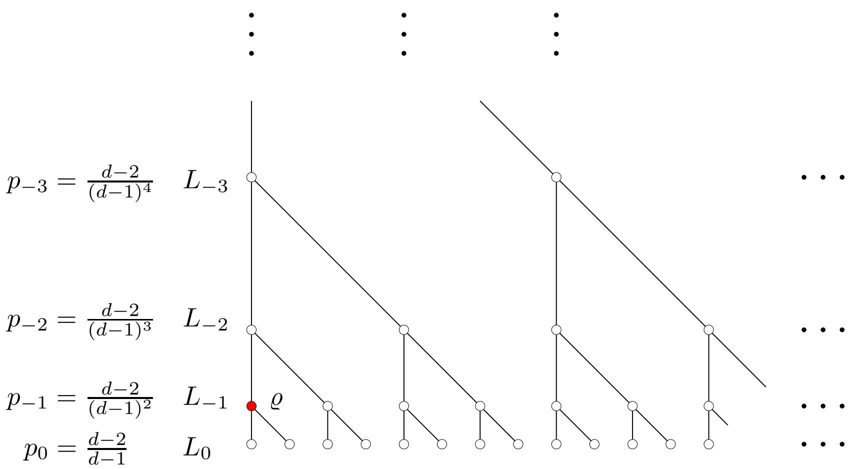

Figure 5: The "canopy tree" T_d^* with a random root ϱ (now on level L_{-1}), which is the local weak limit (as defined below) of the balls in the d-regular tree $\mathbb{T}_d$, for $d = 3$.

▷ **Exercise 50.** Prove using subadditivity that $\sigma(p) := \lim_{n\to\infty} \frac{-1}{n} \log \mathbf{P}_p[\, o \longleftrightarrow \partial B_n(o)\,]$ exists in any transitive graph.

▷ **Exercise 51.**

(a) Show that the "conditional FKG-inequality" does not hold: find three increasing events A, B, C in some $\mathsf{Ber}(p)$ product measure space such that $\mathbf{P}_p[\, AB \mid C\,] < \mathbf{P}_p[\, A \mid C\,]\,\mathbf{P}_p[\, B \mid C\,]$.

(b) Show that the conditional FKG-inequality would imply that $\mathbf{P}_p[\,\cdot \mid 0 \longleftrightarrow \partial B_{n+1}(o)\,]$ stochastically dominates $\mathbf{P}_p[\,\cdot \mid 0 \longleftrightarrow \partial B_n(o)\,]$ restricted to any box $B_m(0)$ with $m < n$. (However, this monotonicity is not known and might be false, and hence it was proved without relying on it that, for $p = p_c(\mathbb{Z}^2)$, these measures have a weak limit as $n \to \infty$, the IIC.)

As we defined on one of the classes, a sequence of finite graphs G_n is said to converge to a random rooted graph (G, ϱ) in the **Benjamini-Schramm sense** (also called **local weak convergence** if for every $r \in \mathbb{N}_+$ the distribution of the r-neighbourhood around a uniformly chosen random root ϱ_n of G_n converges weakly to the distribution of the r-ball around ϱ in G.

▷ **Exercise 52.** Show that a transitive graph G has a sequence G_n of subgraphs converging to it in the local weak sense iff it is amenable.

▷ **Exercise 53.** Show that for all $\lambda \in \mathbb{R}_+$, the local weak limit of the Erdős-Rényi random graphs $G(n, \lambda/n)$ is the Galton-Watson tree with offspring distribution Poisson(λ), usually denoted by $\mathsf{PGW}(\lambda)$.