Stochastic models — homework problems

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▷ **Exercise 1.** Prove that for Green's function of simple random walk on a connected graph, for real $z > 0$,

$$G(x, y|z) < \infty \;\Leftrightarrow\; G(r, w|z) < \infty\,.$$

Therefore, by Pringsheim's theorem, we have that $\mathsf{rad}(x, y)$ is independent of x, y.

▷ **Exercise 2.** Compute $\rho(\mathbb{T}_{k,\ell})$, where $\mathbb{T}_{k,\ell}$ is a tree such that if $v_n \in \mathbb{T}_{k,\ell}$ is a vertex at distance n from the root,

$$\deg v_n = \begin{cases} k & n \text{ even} \\ \ell & n \text{ odd} \end{cases}$$

▷ **Exercise 3** ("Green's function is the inverse of the Laplacian"). Let (V, P) be a transient Markov chain with a stationary measure π and associated Laplacian $\Delta = I - P$. Assume that the function $y \mapsto G(x, y)/\pi_y$ is in $L^2(V, \pi)$. Let $f : V \longrightarrow \mathbb{R}$ be an arbitrary function in $L^2(V, \pi)$. Solve the equation $\Delta u = f$.

▷ **Exercise 4.** Give an example of a random sequence $(M_n)_{n=0}^\infty$ such that $\mathbf{E}[\, M_{n+1} \mid M_n\,] = M_n$ for all $n \geq 0$, but which is not a martingale w.r.t. the filtration $\mathscr{F}_n = \sigma(M_0, \dots, M_n)$.

▷ **Exercise 5.** Consider asymmetric simple random walk (X_i) on $\mathbb{Z}$, with probability $p > 1/2$ for a right step and $1 - p$ for a left step. Find a martingale of the form r^{X_i} for some $r > 0$, and calculate $\mathbf{P}_k[\,\tau_0 > \tau_n\,]$. Then find a martingale of the form $X_i - \mu\, i$ for some $\mu > 0$, and calculate $\mathbf{E}_k[\,\tau_0 \wedge \tau_n\,]$. (Hint: to prove that the second martingale is uniformly integrable, first show that $\tau_0 \wedge \tau_n$ has an exponential tail.)

▷ **Exercise 6.** Using the de Moivre-Laplace Central Limit Theorem, show that
(i) for SRW on $\mathbb{Z}$, the expected distance from the starting point after n steps is $\mathbf{E}\,\mathrm{dist}(X_0, X_n) \asymp \sqrt{n}$.
(ii) Same for SRW $Y_0, Y_1, \dots$ on the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{Z}$. For this, first show the following lemma, using the reflection principle: for SRW on $\mathbb{Z}$, let $M_n := \max\{0 = X_0, X_1, \dots, X_n\}$, then

$$\mathbf{P}[\, M_n \geq t\,] \leq 2\mathbf{P}[\, X_n \geq t\,]\,.$$

▷ **Exercise 7.**
(i) For SRW on $\mathbb{Z}^2$, show that the expected number of vertices visited by time n is

$$\mathbf{E}|\{X_0, X_1, \dots, X_n\}| \asymp n/\log n\,.$$

(ii) Show that on the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{Z}^2$, the distance is $\mathbf{E}\,\mathrm{dist}(Y_0, Y_n) \asymp n/\log n$.

▷ **Exercise 8.**

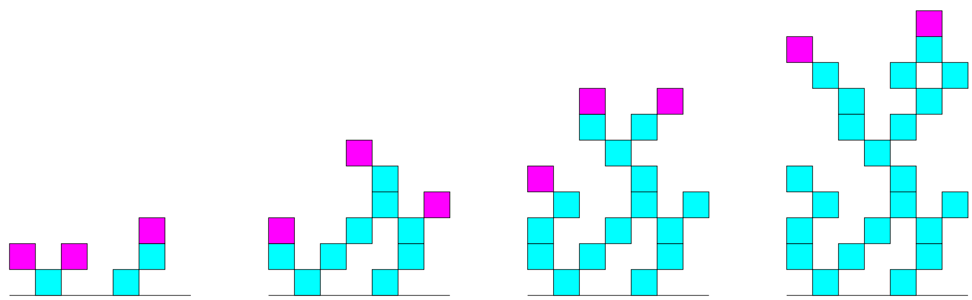
(i) Prove that, for SRW on any transient transitive graph,

$$\lim_{n\to\infty} \frac{\mathbf{E}|\{X_0, X_1, \dots, X_n\}|}{n} = \mathbf{P}[\, X_k \neq X_0,\ k = 1, 2, \dots \,]\,.$$

(ii) Show that on the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{Z}^d$, with $d \geq 3$, the expected distance grows linearly.

▷ **Exercise 9.** For the return probabilities on the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{Z}$, show that $p_{2n}(o,o) \geq c_1 \exp(-c_2 n^{1/3})$.

▷ **Exercise 10.** A simple version of the Tetris game (with no player): on the discrete cycle of length K, unit squares with sticky corners are falling from the sky, at places $[i, i+1]$ chosen uniformly at random ($i = 0, 1, \dots, K-1$, mod K). Let R_t be the size of the roof after t squares have fallen: those squares of the current configuration that could have been the last to fall. Show that $\lim_{t\to\infty} \mathbf{E} R_t = K/3$.



Remark. If there are two types of squares, particles and antiparticles that annihilate each other when falling on exactly on top of each other, this process is a SRW on a group, and the size of the roof has to do with the speed of the SRW. Here, for $K \geq 4$, the expected limiting size of the roof is already less than $0.32893K$, but this is far from trivial. What's the situation for $K = 3$?

▷ **Exercise 11.** * Show that any harmonic function f on $\mathbb{Z}^d$ with sublinear growth, i.e., satisfying $\lim_{\|x\|_2\to\infty} f(x)/\|x\|_2 = 0$, must be constant.

▷ **Exercise 12.** ** Prove that any positive harmonic function f on $\mathbb{Z}^d$ must be constant.

▷ **Exercise 13.** Show that a Markov chain (V, P) has a reversible measure if and only if for all oriented cycles $x_0, x_1, \dots, x_n = x_0$, we have $\prod_{i=0}^{n-1} p(x_i, x_{i+1}) = \prod_{i=0}^{n-1} p(x_{i+1}, x_i)$.

▷ **Exercise 14.** Show by examples that, in directed weighted graphs, the measure $(C_x)_{x\in V}$ might be non-stationary, and might be stationary but non-reversible. Can the walk associated to a finite directed weighted graph have a reversible measure?

▷ **Exercise 15.** Show that effective resistances (as defined in class, (6.3) of PGG) add up when combining networks in series, while effective conductances add up when combining networks in parallel.

▷ **Exercise 16.**

(a) Show that for the voltage function $f(x) = G^Z(o,x)/C_x$ considered in class, the associated current flow has unit strength, hence $\mathcal{R}(o \leftrightarrow Z) = G^Z(o,o)/C_o$.

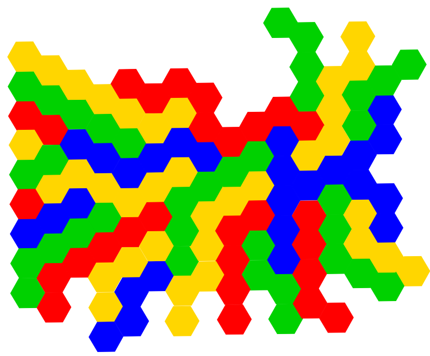
(b) Using part (a), show that $\mathcal{C}(a \leftrightarrow Z) = C_a\, \mathbf{P}_a[\, \tau_Z < \tau_a^+ \,]$, where τ_a^+ is the first positive hitting time on a.

▷ **Exercise 17.** Let $G(V, E, c)$ be a transitive network (i.e., the group of graph automorphisms preserving the edge weights have a single orbit on V). Show that, for any $u, v \in V$,

$$\mathbf{P}_u[\, \tau_v < \infty \,] = \mathbf{P}_v[\, \tau_u < \infty \,]\,.$$

▷ **Exercise 18.** Show that the regular trees $\mathbb{T}_k$ and $\mathbb{T}_\ell$ for $k, \ell \geq 3$ are quasi-isometric to each other, by giving explicit quasi-isometries.

▷ **Exercise 19.*** Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.



Trying to create at least 7 neighbours for each country.

▷ **Exercise 20.**

(a) Find the edge Cheeger constant $\iota_{\infty,E}$ of the infinite binary tree.

(b) Show that a bounded degree tree is amenable iff there is no bound on the length of "hanging chains", i.e., chains of vertices with degree 2. (Consequently, for trees, $IP_{1+\epsilon}$ implies IP_∞.)

(c) Give an example of a bounded degree tree of exponential volume growth that satisfies no $IP_{1+\epsilon}$ and is recurrent for the simple random walk on it.

▷ **Exercise 21.*** Show that a bounded degree graph $G(V, E)$ is nonamenable if and only if it has a wobbling paradoxical decomposition: two injective maps $\alpha, \beta : V \longrightarrow V$ such that $\alpha(V) \sqcup \beta(V) = V$ is a disjoint union, and both maps are at a bounded distance from the identity, or wobbling: $\sup_{x \in V} d(x, \alpha(x)) < \infty$. (Hint: State and use the locally finite infinite bipartite graph version of the Hall marriage theorem, called the Hall-Rado theorem.)

▷ **Exercise 22.** Let (V, P) be a reversible, finite Markov chain, with the stationary distribution $\pi(x)$. Note that P is self-adjoint with respect to $(f, g) = \sum_{x \in V} f(x)g(x)\pi(x)$. Show:

(a) All eigenvalues λ_i satisfy $-1 \leq \lambda_i \leq 1$;

(b) If we write $-1 \leq \lambda_n \leq \cdots \leq \lambda_1 = 1$, then $\lambda_2 < 1$ if and only if (V, P) is connected (the chain is irreducible);

(c) $\lambda_n > -1$ if and only if (V, P) is not bipartite. (Recall here the easy lemma that a graph is bipartite if and only if all cycles are even.)

▷ **Exercise 23.**

(a) For $f : V \longrightarrow \mathbb{R}$, let $\mathrm{Var}_\pi[f] := \mathbf{E}_\pi[f^2] - (\mathbf{E}_\pi f)^2 = \sum_x f(x)^2 \pi(x) - \left(\sum_x f(x)\pi(x)\right)^2$. Show that $g_{\mathrm{abs}} > 0$ implies that $\lim_{t \to \infty} P^t f(x) = \mathbf{E}_\pi f$ for all $x \in V$. Moreover,

$$\mathrm{Var}_\pi[P^t f] \leq (1 - g_{\mathrm{abs}})^{2t} \, \mathrm{Var}_\pi[f] \,,$$

with equality at the eigenfunction corresponding to the λ_i giving $g_{\mathrm{abs}} = 1 - |\lambda_i|$. Hence t_{relax} is the time needed to reduce the standard deviation of any function to $1/e$ of its original standard deviation.

(b) Show that if the chain (V,P) is transitive, then

$$4\, d_{\mathrm{TV}}\big(p_t(x,\cdot),\pi(\cdot)\big)^2 \le \left\| \frac{p_t(x,\cdot)}{\pi(\cdot)} - \mathbf{1}(\cdot) \right\|_2^2 = \sum_{i=2}^{n} \lambda_i^{2t}\,.$$

For instance, recall the spectrum of the lazy walk on the hypercube $\{0,1\}^k$, and prove the bound $d\big(1/2\,k\ln k + c\,k\big) \le e^{-2c}/2$ for $c > 1$ on the TV distance. (This is sharp even regarding the constant $1/2$ in front of $k \ln k$.) Also, recall the spectrum of the cycle C_n, and show that $t_{\mathrm{mix}}^{\mathrm{TV}}(C_n) = O(n^2)$.

▷ **Exercise 24.** Why it is hard to construct large expanders:

(a) If $G' \longrightarrow G$ is a covering map of infinite graphs, then the spectral radii satisfy $\rho(G') \le \rho(G)$, i.e., the larger graph is more non-amenable. In particular, if G is an infinite k-regular graph, then $\rho(G) \ge \rho(\mathbb{T}_k) = \frac{2\sqrt{k-1}}{k}$.

(b) If $G' \longrightarrow G$ is a covering map of finite graphs, then $\lambda_2(G') \ge \lambda_2(G)$, i.e., the larger graph is a worse expander.

▷ **Exercise 25.** Show that a uniform random d-regular bipartite graph on $2n$ vertices with $d \ge 3$ has 4-cycles with a positive probability that is independent of n.

▷ **Exercise 26.** In the random graph $G(n,p)$ with $p = \lambda/n$, for $\mathcal{A}_n = \{\text{containing a triangle}\}$, show directly that the expected number of pivotal edges is $\asymp n$ (with factors depending on λ). (Hence, by Russo's formula, the threshold window is of size $p_{\mathcal{A}}^{1-\epsilon}(n) - p_{\mathcal{A}}^{\epsilon}(n) \asymp 1/n$, as we already saw on class.)