Applications of Stochastics — Exercise sheet 3

Cheeger-constant, clustering-coefficient, centrality measures. Size-biasing, bus-paradox, renewal process with rewards.

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November 26, 2021

Bonus exercises are marked with a star. They can be handed in for extra points.

▷ **Exercise 1.** Consider simple symmetric random walk on $\mathbb{Z}$. Starting from 0, calculate the expected time to reach 1. (Hint: write a recursion.)

We defined the Cheeger constant of a finite undirected graph as

$$h(G) := \min \left\{ \frac{|E(S,S^c)|}{\sum_{x \in S} \deg(x)} : S \subset V(G) \text{ with } \sum_{x \in S} \deg(x) \le |E(G)| \right\}.$$

▷ **Exercise 2.**

(a) Let the dumbbell graph K_n—K_n be two complete graphs K_n joined by a single edge (the bridge). Show that $h(K_n\text{—}K_n) \asymp 1/n^2$, and that this is the smallest possible order of magnitude for a simple graph on n vertices.

(b) Consider SRW $(X_t)_{t \ge 0}$ on K_n—K_n, started at some vertex x that is not one of the endpoints of the bridge. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all large enough n, if $t \le \delta n^2$, then the probability that $(X_i)_{i=0}^t$ has ever crossed the bridge is smaller than ϵ. Conclude that, for $t \le \delta n^2$, the L^1-distance of X_t from the stationary distribution $\pi(y) = 1/(2n)$ is

$$\sum_y \Big| \mathbf{P}_x[\, X_t = y \,] - \pi(y) \Big| \ge 1 - 2\epsilon.$$

That is, at δn^2 steps, we are pretty far from stationarity.

▷ **Exercise 3.** Recall that we defined the **clustering coefficient** of an undirected graph as

$$\mathsf{CC} := \frac{\#\text{ paths of length 2 with endpoints connected by an edge}}{\#\text{ paths of length 2}}.$$

With n vertices and $10n$ edges, find a graph with small CC, and another one with large CC.

Linear algebra brush-up:

▷ **Exercise 4.** For $u, v \in \mathbb{C}^n$ column vectors, define the inner product $(u,v) := u^T \overline{v}$, where $\overline{v}$ is coordinate-wise complex conjugation. Let A be a symmetric $n \times n$ real matrix.

(a) Show that $(v,u) = \overline{(u,v)}$, and $(Au, v) = (u, Av)$. Deduce that if $v \in \mathbb{C}^n$ is an eigenvector of A with eigenvalue λ, then $\lambda \in \mathbb{R}$.

(b) From the fundamental theorem of algebra we know that $\det(A - \lambda I)$ has a root $\lambda \in \mathbb{C}$. Recall that this implies that there exists a nonzero $v \in \mathbb{C}^n$ in the kernel of $A - \lambda I$, hence λ is an eigenvalue, with eigenvector v.

(c) Show that $v^\perp := \{u \in \mathbb{C}^n : (u, v) = 0\}$ is a linear subspace, and $Av^\perp \subseteq v^\perp$ (when v is the eigenvector found in the previous part).

(d) Prove by induction that A has an orthonormal basis of eigenvectors $v_1, \dots, v_n \in \mathbb{C}^n$, with all real eigenvalues $\lambda_1, \dots, \lambda_n$.

(e) Show that A also has an orthonormal basis of eigenvectors $u_1, \dots, u_n \in \mathbb{R}^n$, with the same eigenvalues.

If G is an undirected simple graph on the vertex set $\{1, \dots, n\}$, then its adjacency matrix A is a real symmetric $n \times n$ matrix, and $P = AD^{-1}$ is the Markov transition matrix of the associated simple random walk, where D is the diagonal matrix formed by the degrees $\deg(i)$.

▷ **Exercise 5.**

(a) Show that $\|Pf\|_\infty \le \|f\|_\infty$ holds for any $f : V \longrightarrow \mathbb{R}$ considered as a column vector.

(b) A dual statement (this notion of "duality" can be made precise, but you do not have to care right now) is that $\|f^T P\|_1 \le \|f^T\|_1$ holds for any $f : V \longrightarrow \mathbb{R}$. (Here, $\cdot^T$ means transposing, so f^T is a row vector.)

(c) Deduce from part (a) or (b) that all the eigenvalues of P have absolute value at most 1.

(d) Note that it is clear what $D^{-1/2}$ means. Show that $B = D^{-1/2} A D^{-1/2}$ is a symmetric matrix that is conjugate to P, hence has the same eigenvalues. Deduce that all the eigenvalues of B (and P) are real, are between 1 and -1, and that the vector $D^{1/2}\mathbf{1} = \left(\sqrt{\deg(i)}\right)_{1 \le i \le n}$ is an eigenvector (both left and right, apart from transposing) for the eigenvalue 1.

(e) Find a left eigenvector and a right eigenvector for P with eigenvalue 1.

(f) Generalizing (e), show that if B has a basis of real eigenvectors $\{\varphi_i\}_{i=1}^n$, orthonormal w.r.t. the inner product $\langle \varphi, \tilde{\varphi} \rangle := \sum_j \varphi(j)\tilde{\varphi}(j)$, as in Exercise 4, then P has a basis of right real eigenvectors $\psi_i := D^{1/2}\varphi_i$, orthonormal w.r.t. the inner product

$$\left\langle \psi, \tilde{\psi} \right\rangle_\pi := \sum_j \psi(j)\tilde{\psi}(j)\pi(j),$$

and left eigenvectors $\tilde{\psi}_i^T := \varphi_i^T D^{-1/2}$, orthonormal w.r.t. the same inner product.
(Note how natural this inner product $\langle \cdot, \cdot \rangle_\pi$ is: vertices are weighted according to how much time the stationary chain spends there. Also, if you know what the self-adjointness of an operator means: P is self-adjoint w.r.t. this inner product. And one can prove, similarly to (a) and (b), that P is a contraction in the corresponding L^2-norm.)

Remark. Graph theorists prefer B to P because it is symmetric, and sometimes to A because it is normalized to have spectrum between 1 and -1.

▷ **Exercise 6.** Let P be the Markov transition matrix for the simple random walk on a finite undirected simple graph G. Write $-1 \le \lambda_n \le \cdots \le \lambda_1 = 1$ for its eigenvalues (see the previous exercise).

(a) Show that $\lambda_2 < 1$ if and only if G is connected (the chain is irreducible), and this is precisely when P has a unique stationary distribution.

(b) Show that $\lambda_n > -1$ if and only if G is not bipartite. (Recall here the easy lemma that a graph is bipartite if and only if all cycles are even.)

(c) Let $\pi_t := \pi_0 P^t$ be the distribution of the random walker after t steps. Show that π_t converges coordinate-wise to the unique stationary distribution precisely when $\lambda_2 < 1$ and $\lambda_n > -1$.

Now back to general directed graphs and their associated Markov transition matrix P.

▷ **Exercise 7.** In Google's **PageRank** for a graph on n vertices, the iteration $\overline{x}_{t+1} := \alpha \overline{x}_t P + (1-\alpha)\mathbf{1}/n$ is used, with some $\alpha \in (0,1)$. Show that, for any starting vector $\overline{x}_0$, the sequence $\overline{x}_t$ converges to $\frac{1-\alpha}{n}\mathbf{1}\,(I - \alpha P)^{-1}$.

(Hint: use the Banach fixed point theorem, with an appropriate notion of distance; see part (b) of Exercise 5. In order to have a strict contraction, don't forget to use that $\alpha < 1$. Also, note that part (c) implies that $I - \alpha P$ is invertible for any $\alpha \in (0,1)$.)

▷ **Exercise 8.** Consider the undirected graph on the vertex set $\{1,2,3,4\}$, where $1,2,3$ form a triangle, and 1 and 4 are also connected by an edge.

(a) Calculate the Eigenvector centrality of the four vertices.

(b) Calculate the PageRank scores, for several values of α.

You are welcome to use Mathematica or other software.

▷ **Exercise 9.** Let G be a directed graph on 3 vertices, where there is an undirected path through vertices $1,2,3$, plus a directed edge from 1 to 3. Let A be its adjacency matrix.

(a) Find the eigenvalues of A and an orthonormal basis of eigenvectors.

(b) Consider the iteration $\overline{x}_{t+1} := \overline{x}_t A$, with $\overline{x}_0 = \mathbf{1}$. Find a sequence of scalars c_t such that $c_t \overline{x}_t$ converges to a nonzero vector.

▷ **Exercise 10.** If X is a non-negative random variable with finite expectation, then its **size-biased version** $\widehat{X}$ is defined by

$$\mathbf{P}[\widehat{X} \in A] = \frac{\mathbf{E}[\, X\, \mathbf{1}_{\{X \in A\}}\,]}{\mathbf{E}[\, X\,]}, \qquad \text{for all measurable } A \subset [0,\infty).$$

If this looks incomprehensible to you, think of just two special cases: when X is discrete, with possible values $\{x_k\}_{k\geq 1}$, then $\mathbf{P}[\widehat{X} = x_k] = x_k \mathbf{P}[X = x_k]/\mathbf{E}X$; when X has a density function $f_X(x)$, then $\widehat{X}$ has density $f_{\widehat{X}}(x) = x f(x)/\mathbf{E}X$.

(a) Show that the size-biased version of $\mathsf{Poi}(\lambda)$ is just $\mathsf{Poi}(\lambda) + 1$.

(b) Show that the size-biased version of $\mathsf{Expon}(\lambda)$ is the sum of two independent $\mathsf{Expon}(\lambda)$'s.

(c) * Take a Poisson point process of intensity λ on $\mathbb{R}$. Condition on the interval $(-\epsilon, \epsilon)$ to contain at least one arrival. As $\epsilon \to 0$, what is the point process we obtain in the limit? What does this have to do with parts (a) and (b)?

The next exercise proves the renewal paradox for the renewal process $T_n := \xi_1 + \cdots + \xi_n$ in the case when the interevent time distribution ξ is arithmetic.

▷ **Exercise 11.** Let $\mathbf{P}[\,\xi = k\,] = p_k$, for $k = 1, 2\ldots$ and $\sum_{k\geq 1} p_k = 1$. Let $N_t := \min\{n \geq 0 : T_n > t\}$, and let $\delta_t := t - T_{N_t - 1} \geq 0$ be the current lifetime. Note that $\delta_0 = 0$.

(a) Show that $(\delta_t)_{t=0}^{\infty}$ is an irreducible aperiodic Markov chain, and find its transition probabilities.

(b) Show that δ_t converges in distribution to $\mathsf{Unif}\{0, 1, \ldots, \hat{\xi} - 1\}$, where $\hat{\xi}$ is the size biased version of ξ.

▷ **Exercise 12.** Mr Smith likes the brand UniCar. These cars break down after a uniform $\mathsf{Uni}[0,2]$ years of use, independently of everything. Mr Smith wants to replace each of his old cars after a fixed T years of use, or the time of breakdown, whichever happens earlier. When a car breaks down, there is a cost of USD 1000 for towing it from the road and getting rid of it, and a new car costs USD 12000. If he replaces a car when it still works, he gets a discount at the store for the old car, so the new car costs only USD 10000 (and there is no extra cost of getting rid of the old car). How should Mr Smith choose T to optimize his spendings on the long run?

▷ **Exercise 13.** Let $\xi_1, \xi_2, \ldots$ be the i.i.d. lifetimes of the light bulbs, with $\mathbf{E}\xi_i = \mu \in (0, \infty]$, and we have a janitor who visits the corridor at times given by a Poisson process with intensity λ, and if he sees that the bulb is dead, he replaces it by a new one. Thus the times $\tau_1, \tau_2, \ldots$ passing between the death of a light bulb and the next visit of the janitor are thus i.i.d. $\mathsf{Expon}(\lambda)$ variables.

This is an alternating renewal process, and after a little thought, one can think of it as a renewal process with rewards.

(a) At what rate are bulbs replaced? (I.e., what is the number of replacements per unit time, during a large time t?)

(b) What is the almost sure limiting fraction of visits by the janitor on which the bulb is working?

(c) What is the limiting fraction of time that the light works?