Applications of Stochastics — Exercise sheet 2: Borel-Cantelli, exponential Markov, recurrence of random walks

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Bonus exercises, marked with $*$, can be handed in for extra points any time before the exam period.

Skorokhod's representation theorem says that if $X_n \xrightarrow{d} X$ for real-valued random variables, then there is a coupling of all of them so that $X_n \xrightarrow{a.s.} X$ holds, as well. To make this more tangible, here is a simple example:

$\triangleright$ **Exercise 1.** Check by definition that $X_n \sim \mathsf{Unif}\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$ converges in distribution to $X \sim \mathsf{Unif}[0,1]$. Construct (by hand, without using the above general theorem) a coupling between $X_1, X_2, \dots, X$ such that $X_n \xrightarrow{a.s.} X$ holds.

The following is called "Monte Carlo integration."

$\triangleright$ **Exercise 2.** Let $f : [0,1] \longrightarrow \mathbb{R}$ be a measurable function with $\int_0^1 |f(x)|^4 dx < \infty$, and let $U_1, U_2, \dots$ be i.i.d. $\mathsf{Unif}[0,1]$ variables. Prove that $(f(U_1) + \cdots + f(U_n))/n$ converges almost surely to $\int_0^1 f(x)dx$.

$\triangleright$ **Exercise 3.** Flip a fair coin 60 times, and let $X \sim \mathsf{Binom}(60, 1/2)$ be the number of heads. Using Markov's inequality for e^{tX} with the best possible t, which can be found by minimizing the convex function $f(t) = \log(1 + e^t) - \frac{5}{6}t$, show that

$$\mathbf{P}\big[\, |X - 30| \geq 20 \,\big] \leq 2 \cdot 3^{60} \cdot 5^{-50} < 10^{-6}.$$

$\triangleright$ **Exercise 4.** Prove that for any $\delta > 0$ there exist $c_\delta > 0$ and $C_\delta < \infty$ such that

$$\mathbf{P}\big[\, |\mathsf{Poisson}(\lambda) - \lambda| > \delta\lambda \,\big] < C_\delta \, e^{-c_\delta \lambda},$$

for any $\lambda > 0$. (Hint: use the moment generating function of $\mathsf{Poisson}(\lambda)$.)

$\triangleright$ **Exercise 5.** Let $\xi_i \sim \mathsf{Expon}(\lambda)$ i.i.d. random variables, and let $S_n := \xi_1 + \cdots + \xi_n$. Prove that for any $\delta > 0$ there exist $c_\delta > 0$ and $C_\delta < \infty$ (also depending on λ, of course) such that

$$\mathbf{P}\big[\, |S_n - \mathbf{E}S_n| > \delta n \,\big] < C_\delta \, e^{-c_\delta n}.$$

Hint: use the moment generating function of Expon or the previous Poisson exercise!

$\triangleright$ **Exercise 6.** Let $p, \alpha \in (0,1)$ be arbitrary.

(a) Using exponential Markov, optimized, show that, if $\alpha \leq p$, then

$$\mathbf{P}\big[\, \mathsf{Binom}(n,p) \leq \alpha n \,\big] \leq \exp\left[\left(\alpha \log \frac{\alpha}{p} + (1-\alpha) \log \frac{1-\alpha}{1-p} \right) (1 + o(1))\, n \right].$$

while if $\alpha \geq p$, then

$$\mathbf{P}\big[\, \mathsf{Binom}(n,p) \geq \alpha n \,\big] \leq \exp\left[\left(\alpha \log \frac{p}{\alpha} + (1-\alpha) \log \frac{1-p}{1-\alpha} \right) (1 + o(1))\, n \right].$$

(b) Now let $\alpha_n \to \alpha$ such that $\alpha_n n \in \mathbb{Z}$ for every n. Using Stirling's formula, show that

$$\lim_{n\to\infty} \frac{-\log \mathbf{P}\big[\,\mathsf{Binom}(n,p) = \alpha_n n\,\big]}{n} = \alpha \log \frac{\alpha}{p} + (1-\alpha)\log\frac{1-\alpha}{1-p}\,.$$

When $\alpha = p$, we are getting that $\mathbf{P}\big[\,\mathsf{Binom}(n,p) = \alpha_n n\,\big]$ is only subexponentially small. In particular, roughly how large is $\mathbf{P}\big[\,\mathsf{Binom}(n,p) = \lfloor pn \rfloor\,\big]$?

(c) Conclude from the above that, for any $0 < p < \alpha < \beta \le 1$, if $X_n \sim \mathsf{Binom}(n,p)$, then

$$\mathbf{P}\big[\,X_n/n < \beta \;\big|\; X_n \ge \alpha n\,\big] \to 1.$$

This is a usual large deviations behaviour: the variables fulfill the extreme conditioning by doing not more than what was asked from them.

The next bonus exercise contains some analytic details regarding the moment generating function. The main tool will be the *Dominated Convergence Theorem (DCT)*: if $\{X_n\}_{n\ge1}$ and X and Y are random variables on the same probability space, with the almost sure pointwise convergence $\mathbf{P}[\,X_n \to X\,] = 1$, plus $|X_n| \le Y$ holds almost surely for all n, where $\mathbf{E}Y < \infty$, then $\mathbf{E}|X_n - X| \to 0$, and thus $\mathbf{E}X_n \to \mathbf{E}X < \infty$.

▷ **Exercise 7.*** Assume that $m_X(t) := \mathbf{E}[\,e^{tX}\,] < \infty$ for some $t = t_0 > 0$, and let $\kappa_X(t) := \log m_X(t)$.

(a) Show that $e^{tx} < 1 + e^{t_0 x}$ for all $0 \le t \le t_0$ and $x \in \mathbb{R}$. Deduce that $m_X(t) < \infty$ for all $0 \le t \le t_0$.

(b) Using part (a) and the DCT, show that if $t_n \to t$, all of them in $[0, t_0]$, then $m_X(t_n) \to m_X(t)$. Thus $m_X(t)$ and $\kappa_X(t)$ are continuous functions of $t \in [0, t_0]$.

(c) Show that $x < e^{tx}/t$ for any $t > 0$ and $x \in \mathbb{R}$. Deduce that $\mathbf{E}\big[\,Xe^{tX}\,\big] < \infty$ if $0 < t \le t_0/2$.

(d) Using that $e^b - e^a = \int_a^b e^y\,dy$, show that $(e^{tx} - 1)/t \le xe^{tx}$ for any $t > 0$ and $x \in \mathbb{R}$. Using part (c) and the DCT, show that $m'_X(0) = \mathbf{E}X < \infty$.

(e) Deduce that $\kappa'_X(0) = \mathbf{E}X$. Deduce that if $\alpha > \mathbf{E}X$, then $\kappa_X(t) - \alpha t < 0$ for some $t \in (0, t_0)$.

▷ **Exercise 8.** Consider the following comb graph, a subgraph of $\mathbb{Z}^d$. Let e_i, for $i = 1, 2, \dots, d$, be the ith coordinate axis, each is a copy of the graph $\mathbb{Z}$, and let $\mathbf{S}$ be their union, an infinite star graph with $2d$ rays. Now, for each vertex $\mathbf{x} = (x, 0, \dots, 0) \in e_1$, take the translate $\mathbf{x} + \mathbf{S}$, and let G be the union of all these star graphs. Show that simple random walk on this G is recurrent, for any $d \ge 1$.

▷ **Exercise 9.** Consider SRW on $\mathbb{Z}$, started at $S_0 = 1$. The goal of this exercise is to write down in detail what we have briefly discussed in class:

$$\mathbf{P}_1[\,\tau_0 > n\,] \asymp \mathbf{P}_1[\,\tau_{\sqrt{n}} < \tau_0\,] = 1/\sqrt{n},$$

where the subscript for $\mathbf{P}$ shows where the walk is started.

(a) For any $0 < \ell < m$, recall or prove that $\mathbf{P}_\ell[\,\tau_m < \tau_0\,] = \ell/m$. This proves the second equality above.

(b) Using the strong Markov property, show that $\mathbf{P}_1[\,\tau_0 > n \mid \tau_{\sqrt{n}} < \tau_0\,] \ge \mathbf{P}_0[\,M_n < \sqrt{n}\,] > c_1$, for an absolute constant $c_1 > 0$, where $M_n := \max\{S_i : 0 \le i \le n\}$.

(c)* Show that the conditioned process $(S_i)_{i\ge0} \,|\, \{\tau_0 > n\}$ stochastically dominates the unconditioned process $(S_i)_{i\ge0}$, hence $\mathbf{P}_1[\,\tau_{\sqrt{n}} < \tau_0 \mid \tau_0 > n\,] > \mathbf{P}_1[\,M_n > \sqrt{n}\,] > c_2 > 0$.

From (b) and (c), deduce the first $\asymp$ in the display above.