Applications of Stochastics — Exercise sheet 1: Erdős-Rényi, 1st a 2nd Moment Method, modes of convergence

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Notation. The probability measure for the Erdős-Rényi random graph $G(n,p)$ is denoted by $\mathbf{P}_{n,p}$ or $\mathbf{P}_p$.

Subsets of a base set S are sometimes denoted by $\omega \in \{0,1\}^S$, thinking that $\omega(s) = \mathbf{1}_{s\in\omega}$.

The comparisons $\sim$, $\asymp$, $\ll$, $\gg$ are used as agreed in class.

Bonus exercises are marked with $*$, and can be handed in for extra points any time before the exam period.

▷ **Exercise 1.** Let $S_n := X_1 + \cdots + X_n$ be simple random walk on $\mathbb{Z}$; i.e., $X_i = \pm 1$ with probability 1/2 each, iid. Show that $\mathbf{E}|S_n|/\sqrt{n} \to \sqrt{2/\pi}$. (Caveat: convergence in distribution does not automatically imply the convergence of expectations! You have to use something like the Dominated Convergence Theorem.)

The next exercise gives an example where the Weak Law of Large Numbers fails for a symmetric random variable with non-existing expectation.

▷ **Exercise 2.** Recall the standard Cauchy density: $\frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$. Accepting the fact that if $X_1,\dots,X_n$ are i.i.d. Cauchy variables, then the sum $S_n = X_1+\cdots+X_n$ has the distribution of nX_1, show the following:

(a) $S_n/n \xrightarrow{\mathbf{P}} 0$ does not hold.

(b) For any $\epsilon > 0$, the expected number of returns to the interval $(-\epsilon,\epsilon)$ by the Cauchy walk S_n is infinite.

A small calculus lemma, immensely useful in probability:

▷ **Exercise 3** (Partly repeating from class)**.** Prove that $1 - x \le e^{-x}$ for all $x \in \mathbb{R}$, and $e^{-x} \le 1 - x/2$ for all $0 < x < \epsilon$, if $\epsilon > 0$ is small enough. Conclude that, for any sequence $\epsilon_n \in (0,1)$, we have: $\sum_n \epsilon_n = \infty \iff \prod_n (1-\epsilon_n) = 0$.

▷ **Exercise 4.*** Let $X_1, X_2, \dots$ be iid variables with $\mathbf{E}X_i = \mu$ and $\mathrm{Var}X_i = \sigma^2 < \infty$, and let $Z_n := \frac{X_1+\cdots+X_n-n\mu}{\sqrt{n}\sigma}$. Show that $\liminf_n Z_n = -\infty$ and $\limsup_n Z_n = \infty$ almost surely. That is, we do not have almost sure convergence in the Central Limit Theorem.

Just to get used to the notion of couplings:

▷ **Exercise 5.** Let $S_n, n \ge 0$ be a SRW on $\mathbb{Z}$ started at $S_0 = 0$, while $\tilde{S}_n, n \ge 0$ be a SRW on $\mathbb{Z}$ started at $\tilde{S}_0 = 2$. Give a coupling of the two processes such that $S_n = \tilde{S}_n$ never happens. (We will see soon that the independent coupling does not work for this!)

The next three exercises define explicitly what monotone couplings and stochastic domination are, which implicitly appeared in class.

▷ **Exercise 6** (Repeating from class)**.** An event for the Erdős-Rényi random graph, $A \subset \{0,1\}^{\binom{n}{2}}$, is called *upward closed* or *increasing* if, whenever $\omega \in A$ and $\omega' \supseteq \omega$, then also $\omega' \in A$. Show that, for any such event A, other than the empty or the complete set, the function $p \mapsto \mathbf{P}_p[\,A\,]$ is a strictly increasing polynomial of degree at most $\binom{n}{2}$, with $\mathbf{P}_p[\,A\,] = p$ for $p \in \{0,1\}$. In particular, there exists a unique p

such that $\mathbf{P}_p[A] = 1/2$; this value is usually called the *critical* (or *threshold*) *density*, and will be denoted by $p_c(n) = p_c^A(n)$.

▷ **Exercise 7** (Basically repeating from class)**.** Let $(X, \le)$ be a partially ordered set; think of $(\mathbb{R}, \le)$ or $(\{0,1\}^S, \subseteq)$. Let $\mathcal{B}$ be a sigma-algebra on X, and π a probability measure on $X \times X$ with the product sigma-algebra, with the property that $\pi\big(\{(x,y) \in X \times X : x \le y\}\big) = 1$. Let the first marginal of π be $\mu(A) := \pi(A \times X)$ and the second marginal be $\nu(A) := \pi(X \times A)$ for any $A \in \mathcal{B}$. So, π is a **monotone coupling** of μ and ν. Then, ν **stochastically dominates** μ: for any increasing (upward closed) set $A \in \mathcal{B}$, we have $\mu(A) \le \nu(A)$.

(Strassen's theorem says that the converse also holds: if ν stochastically dominates μ, then a monotone coupling does exist.)

▷ **Exercise 8** (Partly repeating from class)**.** By constructing monotone couplings, show the following:

(a) If $0 \le p \le q \le 1$, then $\mathsf{Binom}(n,q)$ stochastically dominates $\mathsf{Binom}(n,p)$.

(b) If $n \le m$, then $\mathsf{Binom}(m,p)$ stochastically dominates $\mathsf{Binom}(n,p)$.

▷ **Exercise 9.** Prove carefully that choosing M edges one-by-one between n vertices, always uniformly at random, independently of previous choices, but resampling the edge if a multiple edge was created, we get the model $G(n,M)$.

▷ **Exercise 10.** Find the order of magnitude of the critical density $p_c(n)$ for the random graph $G(n,p)$ containing a copy of the cycle C_4. Same with K_4. (Hint: as in class, use the 1st and 2nd Moment Methods.)

▷ **Exercise 11.** Let H be the following graph with 5 vertices and 7 edges: a complete graph K_4 with an extra edge from one of the four vertices to a fifth vertex. Show that if $5/7 > \alpha > 4/6$, and $p = n^{-\alpha}$, then the expected number of copies of H in $G(n,p)$ goes to infinity, but nevertheless the probability that there is at least one copy goes to 0. What goes wrong with the 2nd Moment Method?

▷ **Exercise 12.** Let $X_k(n)$ be the number of degree k vertices in the Erdős-Rényi random graph $G(n,\lambda/n)$, with any $\lambda \in \mathbb{R}_+$ fixed. Show that $X_k(n)/n$ converges in probability, as $n \to \infty$, to $\mathbf{P}[\,\mathsf{Poisson}(\lambda) = k\,]$. (Hint: the 1st moment of $X_k(n)$ should be clear; then use the 2nd moment method.)

▷ **Exercise 13.** Assume that Y_n, $n = 1,2,\dots$, are non-negative integer valued random variables, with $Y_n/n \xrightarrow{\mathbf{P}} 0$. Show that $\mathsf{Binom}\left(n - Y_n, \frac{\lambda}{n}\right) \xrightarrow{d} \mathsf{Poi}(\lambda)$, as $n \to \infty$.

(Hint: there are two sources of randomness: Y_n and Binom. If Y_n is not too big, the usual argument should work. And it is unlikely that Y_n is too big.)

The goal of the last bonus exercise is to present one way to pass from $G(n,p)$ to the $G(n,M)$ model.

▷ **Exercise 14.*** Fix $\delta > 0$ arbitrary, and let $p_n \in (0,1)$ and $M_n \in \left\{0,1,\dots,\binom{n}{2}\right\}$ be two sequences satisfying $\binom{n}{2}p_n \to \infty$ and $(1+\delta)\binom{n}{2}p_n < M_n$ for all n. Let $A_n \subset \{0,1\}^{\binom{n}{2}}$ be a sequence of upward closed events such that $\mathbf{P}_{p_n}[\,A_n\,] \to 1$. Prove that $\mathbf{P}\big[\,G(n,M_n) \text{ satisfies } A_n\,\big] \to 1$, as $n \to \infty$.

In more detail:

(a) Show that $\mathbf{P}\big[\,\mathsf{Binom}\big(\binom{n}{2}, p_n\big) < M_n\,\big] \to 1$.

(b) Let $\mathcal{E}_n$ denote the number of edges in $G(n,p)$. Deduce from part (a) that $\mathbf{P}_{p_n}[\,A_n \mid \mathcal{E}_n < M_n\,] \to 1$.

(c) Show that, for any $M \in \left\{0,1,\dots,\binom{n}{2}\right\}$, we have $\mathbf{P}_{p_n}\big[\,A_n \,\big|\, \mathcal{E}_n = M\,\big] = \mathbf{P}\big[\,G(n,M) \text{ satisfies } A_n\,\big]$.

(d) Deduce from part (c) that $\mathbf{P}_{p_n}\big[\,A_n \,\big|\, \mathcal{E}_n < M_n\,\big] \le \mathbf{P}\big[\,G(n,M_n) \text{ satisfies } A_n\,\big]$.

Combining parts (b) and (d) concludes the exercise.