Applications of Stochastics — Exercise sheet 2

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▷ **Exercise 1.** As in the class discussing the Barabási-Albert preferential attachment graphs, let $\nu_1, \nu_2, \dots$ be positive reals satisfying the recursion

$$\nu_{t+1} = \left(1 - \frac{\alpha_t}{t}\right) \nu_t + \frac{\beta_t}{t},$$

where α_t, β_t are positive reals converging to some positive α and β, respectively. Is this information enough to find $\lim_{t\to\infty} \nu_t$?

▷ **Exercise 2.** Recall that we defined the **clustering coefficient** of an undirected graph as

$$\mathsf{CC} := \frac{\#\text{ paths of length 2 with endpoints connected by an edge}}{\#\text{ paths of length 2}}.$$

With n vertices and $10n$ edges, find a graph with small CC, and another one with large CC.

For a (possibly directed) graph, the adjacency matrix is $A_{u,v} = \mathbf{1}_{u\to v}$. The probability transition matrix for the corresponding Markov chain is $P_{u,v} = A_{u,v}/\sum_w A_{u,w}$. For an undirected graph on the vertex set $\{1,\dots,n\}$, we know from the Stochastic Processes course (and it is straightforward to verify) that P has a left eigenvector $\pi(i) = \deg(i)$, $1 \le i \le n$, with eigenvalue 1; i.e., it is a stationary measure. The leading eigenvector of A is a bit more mysterious, but clearly relevant for measuring how central which vertex is — it is sometimes called the Eigenvector Centrality.

▷ **Exercise 3.**

(a) When P is the Markov transition matrix for any finite directed graph $G(V,E)$, show that $\|Pf\|_\infty \le \|f\|_\infty$ holds for any $f : V \longrightarrow \mathbb{R}$.

(b) Deduce that $|\lambda| \le 1$ for any eigenvalue $\lambda \in \mathbb{C}$ of P.

▷ **Exercise 4.** Let A be the symmetric $n \times n$ adjacency matrix of an undirected finite graph on the vertex set $\{1,\dots,n\}$. Let D be the diagonal matrix formed by the degrees $\deg(i)$, and note that it is clear what $D^{-1/2}$ means. Let $B = D^{-1/2} A D^{-1/2}$. Observe that B and the Markov transition matrix P are conjugate matrices, hence they have the same eigenvalues. Show that they are all real, between 1 and -1, and that the vector $\left(\sqrt{\deg(i)}\right)_{1\le i\le n}$ is an eigenvector for the eigenvalue 1.

Remark: Graph theorists prefer B to P because it is symmetric, and to A because it is normalized to have spectrum between 1 and -1.

▷ **Exercise 5.** In Google's **PageRank**, we are considering the iteration $\overline{x}_{t+1} := \alpha \overline{x}_t P + (1-\alpha)\mathbf{1}$, with some $\alpha \in (0,1)$. Show that, for any starting vector $\overline{x}_0$, the sequence $\overline{x}_t$ converges to $(1-\alpha)\mathbf{1}\,(I - \alpha P)^{-1}$. (Hint: use the Banach fixed point theorem, with an appropriate notion of distance. See part (a) of Exercise 3.)

▷ **Exercise 6.** Consider the undirected graph on the vertex set $\{1,2,3,4\}$, where $1,2,3$ form a triangle, and 1 and 4 are also connected by an edge.

(a) Calculate the Eigenvector Centrality scores.

(b) Calculate the PageRank scores, for several values of α.

You are welcome to use Mathematica or other software.

▷ **Exercise 7.** If X is a non-negative random variable with finite expectation, then its **size-biased version** $\widehat{X}$ is defined by $\mathbf{P}[\widehat{X} \in A] = \mathbf{E}[\, X\, \mathbf{1}_{\{X \in A\}}\,]/\mathbf{E}X$.

(a) Show that $\mathbf{E}[\,\widehat{X}\,] \geq \mathbf{E}[\,X\,]$.

(b) * Show that $\widehat{X}$ *stochastically dominates* X in the sense that $\mathbf{P}[\,\widehat{X} > t\,] \geq \mathbf{P}[\,X > t\,]$ for any $t \geq 0$.

(c) Show that the size-biased version of $\mathsf{Poi}(\lambda)$ is just $\mathsf{Poi}(\lambda) + 1$.

(d) Show that the size-biased version of $\mathsf{Expon}(\lambda)$ is the sum of two independent $\mathsf{Expon}(\lambda)$'s.

(e) * Take Poisson point process of intensity λ on $\mathbb{R}$. Condition on the interval $(-\epsilon, \epsilon)$ to contain at least one arrival. As $\epsilon \to 0$, what is the point process we obtain in the limit? What does this have to do with parts (c) and (d)?

▷ **Exercise 8.** Let $X_0, X_1, X_2, \ldots$ be simple random walk on the infinite d-regular tree $\mathbb{T}_d$, and let $D_n := \mathrm{dist}(X_n, X_0)$ be the graph distance from the starting point $X_0 = o$.

(a) Give a coupling of the process D_n with the biased random walk Y_n on $\mathbb{Z}$ that goes $+1$ with probability $\frac{d-1}{d}$, and -1 with probability $\frac{1}{d}$, in such a way that $Y_n \leq D_n$ for all n, almost surely.

(b) State a large deviations bound for the biased random walk $\{Y_n\}$ above, and deduce that the return probability $p_n(o,o) := \mathbf{P}[\,X_n = o \mid X_0 = o\,]$ is exponentially small in n. Deduce that the walk on $\mathbb{T}_d$ is transient.

(c) Again by stating a large deviations bound for the biased random walk and Borel-Cantelli, deduce that $\lim_{n\to\infty} \frac{D_n}{n} = \frac{d-2}{d}$ almost surely (the speed of escape of the random walk $\{X_n\}$).

▷ **Exercise 9.** As in class, let $\xi_1, \xi_2, \ldots$ be the i.i.d. lifetimes of the light bulbs, with $\mathbf{E}\xi_i = \mu \in (0,\infty]$, and we have a janitor who visits the corridor at times given by a Poisson process with intensity λ, and if he sees that the bulb is dead, he replaces it by a new one. Thus the times $\tau_1, \tau_2, \ldots$ passing between the death of a light bulb and the next visit of the janitor are i.i.d. $\mathsf{Expon}(\lambda)$ variables.

(a) At what rate are bulbs replaced?

(b) What is the almost sure limiting fraction of visits by the janitor on which the bulb is working?

(c) Now assume that $\xi_i \sim \mathsf{Expon}(1/\mu)$. What is the limiting fraction of time that the light works? (This part might require a bit of thinking and an application of Borel-Cantelli.))

▷ **Exercise 10.** Mr Smith likes the brand UniCar. These cars break down after a uniform $\mathsf{Uni}[0,2]$ years of use, independently of everything. Mr Smith wants to replace each of his old cars after a fixed T years of use, or the time of breakdown, whichever happens earlier. When a car breaks down, there is a cost of USD 1000 for towing it from the road and getting rid of it, and a new car costs USD 12000. If he replaces a car when it still works, he gets a discount at the store for the old car, so the new car costs only USD 10000 (and there is no extra cost of getting rid of the old car). How should Mr Smith choose T to optimize his spendings on the long run?

Finally, a technical lemma used in the proof of the Elementary Renewal Theorem:

▷ **Exercise 11.** If $a_K(t) \geq 0$, monotone decreasing in K for any fixed t, then

$$\limsup_{K\to\infty} \limsup_{t\to\infty} a_K(t) \geq \limsup_{t\to\infty} \limsup_{K\to\infty} a_K(t)\,.$$