Disease Process and Bootstrap Percolation

Gábor Pete

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Gábor Pete
gpete@math.u-szeged.hu

Bolyai Institute
Aradi vértanúk tere 1.
Szeged 6720 Hungary

0. Introduction

In this paper we consider some modifications and generalizations of the following nice combinatorial exercise.

Each square of an 8×8 chessboard can have one of two states: "clean" or "weedy". We have some originally weedy squares, and each square of the board can change its state in time according to the following expansion rule: a weedy square remains weedy forever, and if on the previous day a square was clean, then it stays clean if and only if at most one of its neighbours was weedy. (Adjacency among the squares is defined by having a common edge.) What is the minimum number of weedy squares one needs to make the whole chessboard weedy? Unfortunately, we do not know the origin of this elementary problem.

We call an $n \times n$ chessboard with each of its squares of a specific state a *configuration*. In the later parts of the paper we will refer to weedy squares as "black squares" and to clean squares as "white squares"; so a configuration is just a coloured board. We start with an *initial configuration* and a *painting rule* that tells us which white squares become black the next day. The expansion rule in the original problem was the 2-*neighbour rule*: a white square becomes black iff it has at least two black neighbours.

The initial configuration and the rule define a finite or infinite sequence of configurations. We call this sequence a *disease process*. The initial configuration is called *contagious* or *successful* if the corresponding sequence has the all-weedy board as a member.

Now let $G(n)$ be the minimal number of weedy (or black) squares in a contagious configuration. The solution of the initial exercise is $G(n) = n$, and if we consider a torus board of n^2 squares, we need $n - 1$; see *Fact 1.1* and *Fact 1.2* in *Chapter 1*.

One can easily generalize the problem to the k-dimensional $n \times \ldots \times n$ chessboard with an l-neighbour painting rule, where $1 \leq l \leq 2k$, the corresponding function is denoted by $G_{k,l}(n)$. In [P], published in Hungarian, we determined the exact order of magnitude of $G_{k,l}(n)$ for all k, l fixed, and gave precise asymptotics for the lower dimensional cases. (See also the *Appendix* to [BP].) These results and their proofs, together with some open problems, are described in *Chapter 1*.

A possible way to modify the original problem would be to determine the minimal number of initial black squares needed, if we want to paint the whole board with arbitrary arrangement of these squares, and not only in a special clever way. But it would not be too interesting: one can immediately see that we would need almost all the squares to be black at the beginning, as very ugly initial configurations exist. Nevertheless, it may happen that these bad accidents are rare, and it is worth examining what the probability of the complete painting is, if we put in our initial black squares randomly. Or, what is almost equivalent: for a starting configuration we colour black each square of the board independently with a probability $p = p(n)$. We say that the configuration we obtain in this way is *p-random*. Then we have our deterministic painting or expansion rule, and the question is the behaviour of the disease process determined by this rule of spreading. (For a comparison of the possible different models for such random structures see Chapter II in [Bo]; another standard reference for probabilistic combinatorics is [AS]. For a review on percolation models see [G].)

In particular, how large must $p(n)$ be to paint the whole chessboard black? The main topic of *Chapter 2* is the determination of the threshold function in the fundamental case of this *random disease problem*. This randomized model for the infinite square grid was first introduced in statistical physics as *bootstrap percolation* in 1979 by [CLR], and with mathematically rigorous results in 1987 by [E]. The finite version was thoroughly discussed by *M. Aizenman* and *J. L. Lebowitz* in [AiL]. Then the problem was rediscovered in [BP] from the combinatorial respect of [P]. *Chapter 2* contains a brief survey of the most important results in this problem and almost the whole of the paper [BP]. Further problems for future research are also posed.

Of course, the painting rule can itself be randomized: a white square will be black with a probability proportional to the number of its already black neighbours. This process may remind us of spreading of opinions in communities, or the mechanism of democratic societies, and similar physical phenomena occur, as well. For instance, the so-called *voter model* is a well-known problem (see e.g. [CG]): on the k-dimensional infinite grid each square changes its colour to its opposite with a probability of the ratio of the opposite colour among the $2k$ neighbours of the square. (Thus the black part can also decrease in this model, which is very reasonable: each opinion has an opposite to spread.) Or, reaching the idea of regressing the population of one colour, we refer to J. H. Conway's *Game of life*, where the main question is how an initial configuration changes in time (see [BCG] or [T]). Besides, the time taken until reaching the final configuration in our original process could be also of interest, perhaps it would bring new light to the problem. Finally, we can examine our process on other underlying graphs instead of the square grid, or on random graphs; some connections with usual graph parameters (connectivity, expander property, etc.) may occur.

In this paper we are using the following notations:

If for some functions $f(n)$ and $g(n)$ there is a constant $c > 0$ such that $f(n) \leq cg(n)$ for all sufficiently large n, then we write $f(n) = O(g(n))$, if $\exists\, d > 0$ such that $f(n) \geq dg(n)$ for all sufficiently large n, then $f(n) = \Omega(g(n))$, if both conditions hold, then $f(n) = \Theta(g(n))$, and if the constants c and d can be arbitrarily close to each other, i.e. $lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, then we can speak about asymptotical equality, $f(n) \sim cg(n)$.

The k-dimensional cube has one k-dimensional face (the cube itself), $2k$ pieces of the $(k-1)$-dimensional faces, and so on, 2^k pieces of the 0-dimensional ones (the vertices). The number of the i-dimensional faces is denoted by $f_k(i)$.

The abbreviation 'w.h.p.' stands for 'with high probability' and means 'with a probability tending to 1'. We also remark that in *Chapter 2* we may and will assume, whenever needed, that n is sufficiently large.

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