MAXIMAL TORI OF COMPACT LIE GROUPS

GÁBOR PETE 7. DEC. 1998.

In this short note we sketch a generalization of the fact that any unitary matrix $A \in U(n)$ is diagonalizable. The group $U(n)$ will be replaced by an arbitrary compact connected Lie group, and the subgroup of diagonal matrices by maximal tori.

1. The exponential map of (Abelian) Lie Groups

A *Lie group* G is a real or complex differentiable manifold together with a group structure, where the group operations (multiplication and inverse) are differentiable. E.g., $U(n) = \{A \in GL(n, \mathbb{C}) \mid A^{-1} = A^*\}$ is a real (and not complex!) Lie group.

The tangent space T_eG at the unit element e of G reflects the structure of the Lie group through the derivative of the so-called *adjoint representation* (not defined here), giving rise to the *Lie bracket* $[\,\cdot\,,\,\cdot\,]$, a bilinear anti-commutative form on T_eG, satisfying the Jacobi identity. A vector space T_eG together with this bracket is the *Lie algebra* $\mathfrak{g}$ of G. If $X \in T_eG$ then it can be extended to a vector field $X(g) := (\mathrm{d}m_g)_e X$ through the left multiplication $m_g : G \longrightarrow G$. With this extension we can describe the Lie bracket as the Poisson bracket $[X, Y]f = X(Y(f)) - Y(X(f))$, where f is the germ of a function on G.

If $X \in T_eG$ and $X(g)$ is its extended vector field, then we have a one-parameter subgroup $\varphi_X : \mathbb{R} \longrightarrow G$ satisfying $\varphi'_X(t) = X(\varphi_X(t))$, $\varphi_X(0) = e$. So $\varphi'_X(0) = X$. Now we define the *exponential map* $\exp : \mathfrak{g} \longrightarrow G$ by $\exp(X) := \varphi_X(1)$. The main point of the whole construction is the following:

Proposition 1.1. *If $f : G \longrightarrow H$ is a Lie group homomorphism, then its derivative $(df)_e : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism, and $\exp((df)_e(X)) = f(\exp(X))$. Moreover, a linear map $\mathfrak{g} \longrightarrow \mathfrak{h}$ is the derivative of a Lie group homomorphism iff it is a Lie algebra homomorphism, and a group homomorphism is uniquely determined by its derivative.*

The derivative of the exponential map at the origin is $\mathrm{Id}_{\mathfrak{g}}$, so it is a local bijection between some neigbourhoods of $0 \in \mathfrak{g}$ and $e \in G$. If G is Abelian then the multiplication $\mu : G \times G \longrightarrow G$ is a homomorphism, and considering its derivative $\mathfrak{g} \oplus \mathfrak{g} \longrightarrow \mathfrak{g}$: $(\mathrm{d}\mu)_e(X, Y) = X + Y$, we get by Proposition 1.1 that the exponential map is a homomorphism $\exp(X + Y) = \exp(X)\exp(Y)$, so we get not only a nbhd of $e \in G$ but the whole generatum of it, that is the connected component of $e \in G$. Thus exp is *surjective* in the case of a connected Abelian group. Its kernel is a discrete subgroup in $\mathfrak{g}$ because it is locally bijective, so it is a lattice isomorphic to $\mathbb{Z}^k$ for some k. Thus, writing $\mathfrak{g} \simeq \mathbb{R}^n$, we have proved:

Theorem 1. *Any connected Abelian Lie group is the product of a torus $\mathbb{T}^k \simeq \mathbb{R}^k/\mathbb{Z}^k$ and a vectorspace $\mathbb{R}^{n-k}$.* $\square$

If G is a subgroup of $GL(V)$, then $\mathfrak{g}$ is a subalgebra of $gl(V)$, and the Lie bracket is simply the commutator of matrices, and $\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$.

For a closed (and so compact) subgroup G of $U(n)$ we can define a real inner product on $\mathfrak{g}$ as $(X, Y) := \Re\mathrm{Tr}(XY^*)$, which is invariant under conjugation by the elements of G, and so invariant under the Lie bracket of $\mathfrak{g}$ in the sense that $([X, Y], Z) + (Y, [X, Z]) = 0$. It is easy to see that the orthogonal complement of an ideal $\mathfrak{b}$ of $\mathfrak{g}$ w.r.t. this inner product is also an ideal.

2. The Kronecker-Weyl equidistribution theorem

In this section we'd like to generalize and strengthen the well-known statement that for any irrational real number α the set $\{n\alpha - \lfloor n\alpha \rfloor \mid n \in \mathbb{Z}\}$ is dense in the unit interval.

It is easy to see that every continuous homomorphism $\mathbb{T}^1 \longrightarrow S^1$ is of the form $x \mapsto \exp(2\pi i x k)$, $k \in \mathbb{Z}$, just look at its closed kernel. For $\mathbb{T}^1$ we use the additive, for S^1 the multiplicative notation. Then it is not difficult to prove:

Proposition 2.1. *The character group of $\mathbb{T}^n$ consisting of all continuous homomorphisms $\mathbb{T}^n \longrightarrow S^1$ is $\mathbb{Z}^n$.*

Proposition 2.2. *All the continuous automorphisms of the torus $\mathbb{T}^n$ are of the form $x \mapsto Ax$, where $A \in \pm SL(n, \mathbb{R})$.*

We have the normalized Lebesgue measure $\mu(\mathbb{T}^n) = 1$. Then for any $\gamma \in \mathbb{T}^n$ the translation $T_\gamma(x) := x + \gamma$ is measure preserving.

Theorem 2. *If $\gamma_1, \ldots, \gamma_n, 1$ are rationally independent (so for almost all γ), then the orbit of T_γ is dense in $\mathbb{T}^n$, moreover, T_γ is ergodic, i.e. every invariant subset of $\mathbb{T}^n$ has measure 0 or 1.*

Proof. Ergodicity is clearly more than the denseness of the orbit, because every open subset has positive measure. Now let χ be the characteristic function of an invariant set. By the Stone-Weierstrass thm for the character group of $\mathbb{T}^n$ we get the following Fourier expansion:

$$\chi(x_1, \ldots, x_n) = \sum_{(k_1,\ldots,k_n)\in\mathbb{Z}^n} \chi_{k_1,\ldots,k_n} \exp\left(2\pi i \sum_{j=1}^{n} k_j x_j\right).$$

Now $\chi(T_\gamma x) = \chi(x)$ by the invariance of χ, and comparing the Fourier coefficients, which are unique, we get

$$\chi_{k_1,\ldots,k_n}\left(1 - \exp\left(2\pi i \sum_{j=1}^{n} k_j \gamma_j\right)\right) = 0$$

for all $k_1, \ldots, k_n$. So the condition of rationally independence implies $\chi_{k_1,\ldots,k_n} = 0$ except for possibly $k_1 = \cdots = k_n = 0$. Thus $\chi = 0$ or $\chi = 1$ almost everywhere, and we are done. $\square$

Thus we have a lot of $x \in \mathbb{T}^n$ such that the orbit $\{k \cdot x \mid k \in \mathbb{Z}\}$ is dense in $\mathbb{T}^n$. We call such an element a *topological generator* of the torus.

The name "Equidistribution Theorem" is a consequence of the **Birkhoff Ergodic Theorem**: If $f : X \longrightarrow X$ is an egodic μ-preserving transformation with $\mu(X) = 1$, and $A \subseteq X$ is a measurable set, then for almost every point $x \in X$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{I}[f^k(x) \in A] = \mu(A).$$

3. Maximal tori of compact Lie groups

Let G be a closed connected subgroup of $U(n)$ (but could be any compact connected Lie group), with the invariant inner product on $\mathfrak{g}$ described in Section 1. A maximal torus T of it is a connected closed Abelian subgroup, not contained properly in any other such subgroup. It is easy to see that the subgroup T is a maximal torus in G iff its Lie algebra $\mathfrak{h}$ is a maximal Abelian subalgebra of $\mathfrak{g}$. From Theorem 1 we know that $T \simeq \mathbb{T}^k$ for some k.

Proposition 3.1. *If T is a maximal torus with Lie algebra $\mathfrak{h}$, then $\mathfrak{g} = \cup_{g\in G}\, g\mathfrak{h}g^{-1}$.*

Proof. Take $X \in \mathfrak{g}$, and choose $Y \in \mathfrak{h}$ such that $\exp(Y)$ is a topological generator for T, known to exist by Theorem 2. Thus by the maximality of $\mathfrak{h}$ the centralizer of Y in $\mathfrak{g}$ is $\mathfrak{h}$. Now consider the function $f(g) = \|gXg^{-1} - Y\|^2$ on G. By the compactness of G it has a minimum at some g_0, and replacing X by $g_0Xg_0^{-1}$ we may assume that $g_0 = e$. Fix any $A \in \mathfrak{g}$, and take the derivative of the function $F(t) = f(\exp(At))$ for small $t \in \mathbb{R}$ at $t = 0$. We have $F'(0) = 2([A,X], X - Y) = 0$. By $[X,X] = 0$ and the invariance of the inner product we get $(A, [X,Y]) = 0$ for all A, so $[X,Y] = 0$, and $X \in \mathfrak{h}$. $\square$

By similar but longer tricks one can get from the previous result the following

Theorem 3. *For a maximal torus T we have $G = \cup_{g\in G}\ gTg^{-1}$.*

So any two maximal tori are conjugate, and every element of G is contained in a maximal torus. Thus the exponential map $\mathfrak{g} \longrightarrow G$ is surjective for compact connected Lie groups. The centralizer of T in G is T itself, and its Lie-algebra $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

If $N(T)$ is the normalizer of T in G then the *Weyl group* $W := N(T)/T$ can be naturally embedded into the discrete group $\mathrm{Aut}(T) = \pm SL(k,\mathbb{Z})$, so it is finite, and two elements of T are conjugate in G iff they are conjugate under the action of W.

For $U(n)$, the standard maximal torus is the subgroup of diagonal matrices, isomorphic to $\mathbb{T}^n$. So Theorem 3 is the generalization of the fact that every matrix in $U(n)$ can be diagonalized. The Weyl group of $U(n)$ and $SU(n)$ is S_n.

The Weyl group plays a central role in representation theory.

References

I can recommend two good books, T. Bröcker — T. tom Dieck: *Representations of compact Lie groups*, Springer, 1985, and P. Walters: *An introduction to ergodic theory*, Springer, 1982.