Book review for Acta Math. Sci. (Szeged) on

G. Davidoff – P. Sarnak – A. Valette: *Elementary number theory, group theory, and Ramanujan graphs.* London Mathematical Society Student Texts Vol. 55, x+144 pages, Cambridge University Press, Cambridge, 2003.

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The existence of expanders, which are sparse graphs with good connectivity properties, follows easily from the probabilistic method of combinatorics. However, constructing them explicitly, especially Ramanujan graphs, which are the optimal ones from some point of view, is an extremely difficult and interesting problem that was solved by Lubotzky-Phillips-Sarnak and Margulis in 1988. The problem comes from combinatorics and information theory, but it has turned out to be a meeting point of Lie-groups, representation theory, geometry and number theory. Moreover, there are real-life applications to network-designs, complexity theory, derandomization, coding theory and cryptography. The present wonderful book gives an *elementary and self-contained* treatment of constructing almost-optimal examples, and thus is intended for a general mathematical audience. In particular, for adventurous undergraduates this book can be an excellent introduction to major modern subjects, by showing how such deep theories are synthesized to solve an easily understandable problem.

For a fixed integer $k \geq 3$ and $c > 0$ small, a k-regular graph $X(V,E)$ on n vertices is called a (n,k,c)**-expander**, if for any subset of vertices $U \subset V$, there are many edges connecting U with its complement: $\frac{E(U,V\setminus U)}{|U||V\setminus U|} > c$. The minimum of these "boundary to volume" ratios is called the **isoperimetric** or **Cheeger constant** $h(X)$ of the graph. Most of mathematics comes into play after noting that the largeness of $h(X)$ is closely related to the largeness of the so-called **spectral gap** of X. The adjacency matrix of X is symmetric, with n real eigenvalues $k = \mu_0 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq -k$. If the graph is connected, then $k > \mu_1$, and $\mu_{n-1} = -k$ iff X is bipartite. Inspired by differential geometry, Alon-Milman and Dodziuk showed that $(k-\mu_1)/2 \leq h(X) \leq \sqrt{2k(k-\mu_1)}$. Thus, finding good expanders is almost the same as maximizing the spectral gap. Alon-Boppana and Serre showed that for fixed k, any sequence X_n of k-regular connected graphs on n vertices has $\liminf_n \mu_1(X_n) \geq 2\sqrt{k-1}$, and if the girth sequence (the length of the shortest cycle) satisfies $g(X_n) \to \infty$, then also $\limsup_n \mu_{n-1}(X) \leq -2\sqrt{k-1}$. This motivates the following definition: a finite connected k-regular graph is called a **Ramanujan graph** if all eigenvalues with $|\mu| < k$ satisfy $|\mu| \leq 2\sqrt{k-1}$.

The probabilistic method shows that for any $k \geq 3$ and $c > 0$ small enough, when n is large, most k-regular graphs on n vertices are (n,k,c)-expanders. But the first explicit construction of an infinite family of (k,c)-expanders by Margulis in 1973 used Kazhdan's property (T), and the original proof in 1988 that some well-chosen Cayley graphs of the groups $\mathrm{PSL}_2(q)$ are Ramanujan, make free use of the theory of algebraic groups, modular forms, theta correspondences, and the Riemann hypothesis for curves over finite fields. In particular, the main ingredient was Eichler's 1954 proof of the Ramanujan conjecture about the order of magnitude of coefficients of modular cusp forms in weight 2 — hence the name of these graphs. So it must be clear how valuable an elementary approach is — even if the authors prove only that the constructed Cayley graphs are expanders with a good c, and not that they are actually Ramanujan.

Chapter 1 of the book contains the graph theoretical background. It proves the basic facts about the spectrum of the adjacency matrix; the Alon-Milman and Dodziuk inequalities; the asymptotic optimality of $2\sqrt{k-1}$ via Chebyshev polynomials; the fact that a k-regular Ramanujan graph has chromatic number at least $\frac{k}{2\sqrt{k-1}}$. It also gives Erdős' probabilistic proof that there exist graphs with both arbitrary large girth and chromatic number.

Chapter 2 is devoted to elementary number theory. Using the ring $\mathbb{Z}[i]$ of Gaussian integers, it gives Legendre's formula on the number of ways an integer can be written as a sum of two squares. Then it proves the law of quadratic reciprocity, and Jacobi's formula on the number of ways an integer can be written as a sum of four squares (which is at least one for any positive integer). Finally, we learn the algebraic structure underlying the representation as a sum of four squares: the arithmetic of the quaternion ring $\mathbb{H}(\mathbb{Z})$.

Chapter 3 bears the title $\mathrm{PSL}_2(q)$. Firstly, it shows that $\mathrm{PSL}_2(q)$ is a simple group if $q > 3$, then it proves that any proper subgroup with more than 60 elements is metabelian. We are introduced to the linear representation theory of finite groups in 17 pages, followed by Frobenius' result that, for $q \geq 5$, the degree of any non-trivial representation of $\mathrm{PSL}_2(q)$ is at least $\frac{q-1}{2}$.

Chapter 4 ties everything together. For p, q distinct odd primes, q large enough, $X^{p,q}$ is a $(p+1)$-regular Cayley graph of $\mathrm{PSL}_2(q)$, constructed via an isomorphism between $\mathbb{H}(\mathbb{F}_q)$ and the algebra of 2×2 matrices over $\mathbb{F}_q$. This is the point where quaternionic number theory and group theory merge. However, it is very hard to show that the $(p+1)$-element set defining the Cayley graph actually generates the whole group, and thus that $X^{p,q}$ is connected. Instead of this, certain Cayley graphs $Y^{p,q}$ are constructed as quotients of the $(p+1)$-regular tree. These are obviously connected, but it is not clear if the vertex set is the whole of $\mathrm{PSL}_2(q)$ or only a subgroup. Here comes the result of Chapter 3: it is easy to show that $Y^{p,q}$ does not have small girth, while any Cayley graph of a metabelian group does. Therefore, we must have $X^{p,q} = Y^{p,q}$. Then the earlier representation theory implies that the multiplicity of any nontrivial eigenvalue of $X^{p,q}$ is at least $\frac{q-1}{2}$. Using that this value is fairly large compared to q^3, the approximate number of vertices, the authors deduce that there must be a spectral gap.

The book ends by indicating how Eichler's theorem implies that the $X^{p,q}$ are Ramanujan graphs, and with an Appendix containing Margulis' beautiful elementary construction of a family of 4-regular graphs with almost optimally large girth. These are Cayley graphs of $\mathrm{SL}_2(q)$.

To read the book, only basic knowledge of algebra and number theory is required as a background, such as the mere definition of the matrix group $\mathrm{GL}_2(q)$ — everything else is developed quickly but clearly. It is very nice to see that this quickness never makes the proofs rushed. Each section ends with several nice exercises, over 70 in total, which both deepen and widen the reader's understanding of the material. The notes on further results are helpful for researchers. An undergraduate reader certainly has to be quite enthusiastic and committed to master all of the book's material; on the other hand, I can hardly imagine a better source to help enthusiasm and maturity grow. This book will be an exciting reading and a pleasure for anyone.