Book review for Acta Math. Sci. (Szeged) on

Béla Bollobás: *Random graphs. 2nd edition.* (Cambridge Studies in Advanced Mathematics Vol. 73), xviii+498 pages, Cambridge University Press, Cambridge, 2001.

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This is the second, completely updated, edition of the now classic text from 1985. The book focuses on the **Erdős-Rényi random graph model** $G(n,p)$: n vertices are given, and each of the $\binom{n}{2}$ edges is present with probability p, independently from each other. Now, given a graph property $\mathcal{P}_n$, what is the probability that $G(n,p)$ satisfies this property? A purely combinatorial point of view would be to determine the number of graphs with property $\mathcal{P}_n$, and generating functions would be a usual method for this. Inspite of the obvious connections, the probabilistic theory has questions and answers of a quite different nature. For example, it is clear that most natural graph properties $\mathcal{P}_n$ are monotone, meaning that they are closed upwards (or downwards) with respect to the subgraph relation. Erdős and Rényi found in 1959 that several important examples of monotone properties, such as connectedness, have a **threshold function** $p_c(n)$: if $p \ll p_c(n)$, then $\mathbb{P}_p(G(n,p)$ satisfies $\mathcal{P}_n) \to 0$, while it tends to 1 if $p \gg p_c(n)$, as $n \to \infty$. Consequently, as p increases from 0 to 1, a typical graph of $G(n,p)$ goes through various different stages, with abrupt changes. This process is called the **evolution** of random graphs. By now, the subject has not only become a mature mathematical discipline, but is also an indispensable tool in computer science, electrical engineering and biomathematics, and is deeply connected to the statistical physics of complex systems.

The book starts with a summary of standard tools from probability theory: normal and Poisson approximation, Lindeberg's CLT, sieve formulae, the method of moments to establish convergence in distribution, and the Hoeffding-Azuma martingale inequality. Chapter 2 reviews the basic properties of the models $G(n,p)$, $G(n,m)$ and random regular graphs. Chapter 3 is devoted to the distribution of the degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. As an application, an $O(pn^2)$ time algorithm is given that determines for asymptotically almost every graph of $G(n,p)$ if it is isomorphic to a fixed graph. Chapter 4 determines the threshold function for containing a fixed graph as a subgraph, and it proves that the number of copies is usually well-approximated by a Poisson distribution. Chapter 5 examines the sparse components of $G(n,p)$: trees and unicyclic graphs. If $p \leq \frac{c}{n}$, with a fixed constant $c < 1$, then almost all components are actually small trees. However, as p passes $\frac{1}{n}$, some trees become larger, and merge together to form a unique giant component, which contains a positive proportion of the vertices for $p \geq \frac{c}{n}$, $c > 1$. Almost all the components apart from this giant one are small trees again. Chapter 6 studies these critical and supercritical phases. It also contains the following application: if we assign independent uniform $[0,1]$ weights to each edge, then the minimal spanning tree has total weight converging in probability to $\zeta(3) = \sum_{k=1}^{\infty} k^{-3}$. Note the resemblence to the limit $\zeta(2)$ of the random assignment problem. Chapter 7 explains that the giant component swallows up the whole graph at $p \sim \frac{\log n}{n}$, and the graph becomes connected. Moreover, when the last isolated vertex disappears, the graph immediately admits a perfect matching (missing one vertex if n is odd). Chapter 8 deals with long paths and cycles in $G(n,p)$. In particular, when every vertex has degree at least 2, that is, just a little bit after the graph has become conected, it already contains a Hamiltonian cycle. Moreover, the much sparser random 3-regular graph is also Hamiltonian. Chapter 9 proves the relation $U_{n,m} \sim L_{n,m}/n!$ between the number of unlabelled and labelled graphs with m edges and n vertices, where $\frac{1}{2}n\log n \ll m \ll \binom{n}{2} - \frac{1}{2}n\log n$. This makes it possible to deduce results for random graph models with indistinguishable vertices. The chapter also discusses some further results concerning symmetries of random graphs. Chapter 10 shows that the diameter of $G(n,p)$ is really small: it is at most $\frac{\log n}{\log\log n}$ whenever $G(n,p)$ is connected, and equals 2 when p is bounded away from 0. A new section describes a rigourous version of real life 'small world' models (such as the internet or a network of personal aquaintances); in this model the fraction of vertices with degree d decays like d^{-3}, and the diameter is $\frac{\log n}{\log\log n}$. Chapter

11 studies the interconnected issues of cliques, independent sets, and colourings. Using the martingale method, it is shown that the chromatic number $\chi(G(n,p))$ for $p \sim n^{-\alpha}$, $1/2 < \alpha < 1$, is concentrated on a finite interval, while for $\chi(G(n,\frac{1}{2})) \sim \frac{n}{2\log_2 n}$, this interval has length not much more than $(\log n)n^{1/2}$, which has recently been conjectured to be close to optimal. Chapter 12 discusses Ramsey theorems whose proofs have a random graph theory component. For instance, $R(3,t) \sim \Theta\left(\frac{t^2}{\log t}\right)$. Chapter 13 describes explicit constructions giving random-like graphs, which is really a hard task. The standard example is the Paley graph on the finite field $\mathbb{F}_q$ as the vertex set, where q is a prime power congruent to 1 $(\mod 4)$, and (x,y) is an edge if $x-y$ is a quadratic residue. To prove that this is similar to a random graph one needs A. Weil's famous theorem from the forties: the Riemann hypothesis for algebraic function fields over finite fields. There is also a brief discussion of the Margulis-Gabber-Galil expander construction, and of the equivalence of possible definitions of pseudo-randomness. In Chapter 14 we learn that the threshold function for the connectedness of a random subgraph of the hypercube $\{0,1\}^n$ is $p=1/2$; that the probability that n random vertices of $\{0,1\}^n$ are linearly independent is $O(n^{-1/2})$; how to construct balanced splittings of set systems; and what the structure of a uniform random permutation is. Chapter 15 is about sorting algorithms. The main result of the chapter is the Ajtai-Komlós-Szemerédi sorting algorithm with width $n/2$ and depth $O(\log n)$. Chapter 16 contains tables for properties of small random regular graphs.

The book contains many exercises of varying difficulty after each chapter, and an extensive bibliography. The addition of over 150 references to the previous 750, and the discussion of new results, which increases the text by 50 pages, has brought this account up to date. This is a self-contained, concise and clear presentation of an enormous body of material, by an author who has himself played a prominent role in the development of this field. Clearly, it is impossible to include all the findings of the last fifteen years into a new edition of the same book, so some very important fresh subjects are missing: Aldous's Brownian excursion approach to the emergence of the giant component; the Bourgain-Friedgut-Kalai theory of sharp thresholds; Szemerédi's regularity lemma.

The text is written with great care; the only small annoying problem is the high number of misprints in the newly added parts. The author always states and usually proves the strongest known results on each subject, and includes all details. This precision is truly admirable, but it might have a drawback: I found it difficult to decipher the main points in the theorems and the proofs. Altogether, this is a very good and handy guidebook for researchers, but I would recommend it for self-study only to a very mature and patient student.