Book review for Acta Sci. Math. (Szeged) on

H. Bass – A. Lubotzky: *Tree Lattices.* With appendices by Bass, L. Carbone, Lubotzky, G. Rosenberg and J. Tits. (Progress in Mathematics Vol. 176), xii+233 pages, Birkhäuser Boston, 2001.

Gábor Pete, 15. November 2001.

Let X be a locally finite tree; then the goup of its graph-automorphisms, $G = \operatorname{Aut} X$ is a locally compact group. A subgroup $\Gamma \subseteq G$ is discrete iff the vertex stabilizers Γ_x are finite. Such a Γ is called a **tree lattice**, or X-lattice, if $\operatorname{Vol}(\Gamma\backslash\backslash X) := \sum_{x\in X} 1/|\Gamma_x|$ is finite. A tree lattice is said to be **uniform** if $\Gamma\backslash X$ is a finite graph. More generally, if $H \leq G$ is a closed subgroup, acting on X without inverting the two vertices of any edge, then we can form the factor graph $A = H\backslash X$, equipped with edge-weights in the following way. For any edge e, we can consider both its orientations, and if it is directed from x to y, say, then let $i(e) := [H_x : H_e]$. Now this pair (A, i) is called an **edge-indexed graph**, and $(A, \{H_x\}_x, \{H_e\}_e)$ is a **graph of groups**. A basic result is that for any collection of groups satisfying $H_{(xy)} \leq H_x$ for all $x, y \in V(A)$, there is a tree X and group H acting on X with the given stabilizers, and factor A; this X in fact depends only on the edge-indices $\{i(e)\}_e$. When the additional structure is emphasized, we write $H\backslash\backslash X$ or (A, i). Sometimes we are interested in the group G_H of deck transformations, which are the automorphisms of X leaving A invariant. Also note that to define $\operatorname{Vol}(H\backslash\backslash X)$ in the general case, we need to use a Haar-measure on H, and H has to be unimodular.

This research monograph is a fairly systematic study of group theoretical properties of lattices and geometric properties of their actions and factor graphs. Group actions on trees furnish a unified geometric way of recasting the chapter of combinatorial group theory dealing with free groups, amalgams, and HNN extensions, and recently we can encounter them everywhere, from computer science through probability theory to geometry. (A famous example: the extraordinary group of Grigorchuk is a group of automorphisms of the binary tree — but this is not a lattice at all!) To quote the oldest theorem of this kind: *A group is free iff it admits a free action on a tree. Note that this tree must be infinite: any automorphism of a finite tree has a fixed point or an inverted edge.* Considering graphs of groups as above is a natural step forward from this theorem.

The principal examples for the theory are the rank 1 simple Lie groups over a non-archimedean local field acting on their Bruhat-Tits buildings, which are bi-regular trees in this case. So the authors try to keep their theory parallel to that of lattices in semi-simple Lie groups, but this serves only as motivation for some of the questions: the presentation is essentially elementary and self-contained.

Let us now see some of the main results discussed in the book.

There are results on the algebraic/topological properties of the full automorphism group G, e.g. on the simplicity of large subgroups of G — vaguely speaking, G is not very far from being simple. One of the appendices, by Bass and Tits, gives simple combinatorial criteria for the discreteness of G, and more generally, of G_H. In this case X or A is called **rigid**.

The very basic question about the existence of lattices has been settled only in this book. The **Uniform Existence Theorem** was known before: there exists a uniform X-lattice iff X is the universal cover of a finite connected graph iff G is unimodular and $G\backslash X$ is finite iff there is a uniform X-lattice Γ with $\Gamma\backslash X = G\backslash X$. Moreover, for any closed subgroup H, any two uniform X-lattices in G_H are commensurable. This implies **Leighton's finite common covering theorem**: if A_1 and A_2 are connected finite graphs with a common covering, then they have a common finite covering, as well. However, the case of non-uniform lattices is much more difficult, and in every respect, much more complex and varied phenomena occur. Thus the book focuses on this case; in particular, most of the book can be viewed as a preparation for the general **Lattice Existence Theorem**, which is finally proved in the appendix written with Bass, Carbonne and Rosenberg: There exists an X-lattice $\Gamma \leq G_H$ iff H is unimodular and $\operatorname{Vol}(H\backslash\backslash X) < \infty$. Note that the unimodularity of H is equivalent to a simple condition about its factor graph (A, i): for any closed path in A, the product of edge-indices along the path equals to the product along the path directed backwards.

The basic results about the structure of lattices are: a group is isomorphic to a uniform tree lattice iff it is finitely generated and virtually free. A non-uniform lattice Γ is never finitely generated, and it contains arbitrary large finite subgroups, and can be e.g. simple. By investigating the possible actions of Γ on X (hyperbolicity, ellipticity, parabolicity, action on ends, cusps, etc.), detailed descriptions can be given on the structure of centralizers, normalizers and commensurators of tree lattices. For example, the following analogue of the **Borel density theorem** is proved: if the action of Γ on X is minimal (i.e. it has no nontrivial invariant subtree), or equivalently, the action of G is minimal, then Γ has a trivial centralizer in G, except in 2 obvious cases. For uniform lattices the commensurator of Γ is dense in G. This, together with the uniform commensurability theorem we mentioned in the previous paragraph, led Lubotzky to ask a version of the **Congruence Subgroup Theorem**: if two uniform X-lattices have the same commensurators, does it follow that they are strictly commensurable? This is known to be true for regular trees. For non-uniform lattices the commensurator can be dense or small, e.g. can be equal to Γ; examples are coming from **Nagao lattices**.

Finally, what are the geometric properties of the graph $\Gamma\backslash X$, when Γ is a non-uniform tree lattice? In general, everything can happen: any connected locally finite graph is of the form $\Gamma\backslash X$. Even for a d-regular tree T_d with $d \geq 3$, there are lots of possibilities: $\Gamma\backslash T_d$ can have any concievable number of cusps, and $\mathrm{Vol}\,(\Gamma\backslash\backslash T_d)$ can take any positive real value with $\Gamma\backslash T_d$ being a half-line.

To read the book the only prerequisites needed are some usual notions of group theory, like “unimodular”, “inverse limit”, “residually finite”. However, this work is in many ways a sequel to **H. Bass – R. Kulkarni:** Uniform tree lattices *J. Amer. Math. Soc.* **3** (1990), 843-902, whose results are regularly quoted, so some acquaintance with that paper is quite useful. Also, familiarity with the standard questions and answers of the theory of semi-simple Lie group lattices helps a lot in following the material, but again — this is not necessary at all. A good deal of attention is given to the construction and study of diverse examples. The style is clear and nice, but the authors constantly cite results from later chapters, which makes it difficult to follow the development of the theory. Altogether, this book is an excellent resource to researchers in the field, and may be helpful and exciting for graduate students interested in geometric methods in group theory.