Generalized Fourier Spectrum and sparse reconstruction in spin systems

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Overview

1. Can transitive and "almost-transitive" functions of iid bits (say, Majority and left-right crossing in percolation) be guessed from some sparse subset of the input bits?

2. Extension to Ising model, say, on tori $\mathbb{Z}_n^d$?

A. Close to optimal result using representation of *subcritical* Ising as a factor of iid process.

B. *Critical* Ising is already very different.

C. Faint hope to get optimal results for *sub- and super-critical* Ising using the FK representation. . .

The clue of small subsets?

Gmail chat from Itai Benjamini: For n iid Bernoulli(1/2) input bits, if we know $o(n)$ of the voters, we still have no clue what the result of majority will be. Is this the same for left-to-right crossing in critical planar percolation? Of course, we ask a predetermined set of voters, in a non-adaptive manner.

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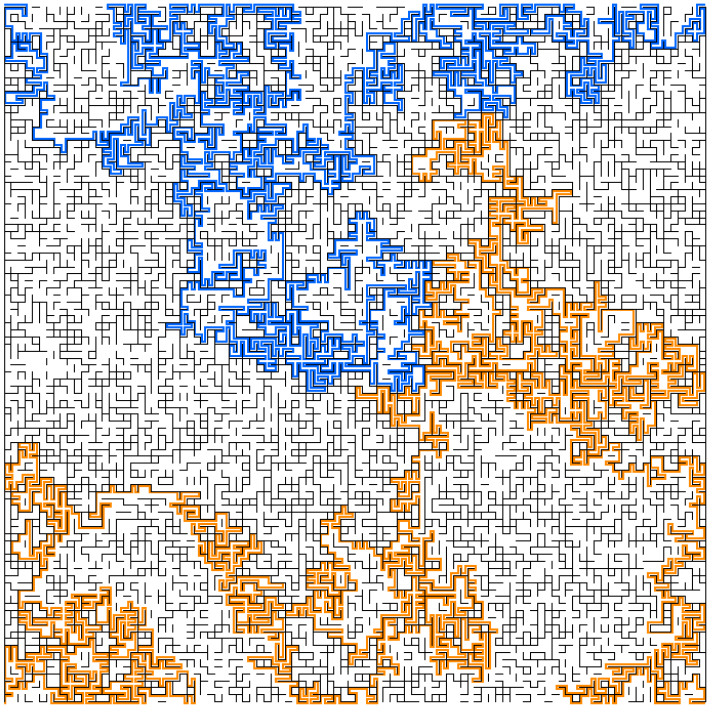

In critical planar percolation, $\mathbf{P}[\,\text{LR crossing in } n * n \text{ box}\,] \sim 1/2$. Can be decided via exploration interface, which has length $n^{2-\delta}$; in fact, $n^{7/4+o(1)}$, proved for site percolation on Δ.

That's why non-adaptive.

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A good definition for clue: how much information we gain (how much the variance decreases on average) if we know the values of the bits of U:

$$\text{clue}_f(U) := \frac{\mathrm{Var}\left[\mathbf{E}[\, f \mid \omega_U \,]\right]}{\mathrm{Var}\, f} = \frac{\mathrm{Var}\, f - \mathbf{E}\left[\, \mathrm{Var}\,[f \mid \omega_U]\,\right]}{\mathrm{Var}\, f}\,.$$

Example. For $\omega \in \{\pm 1\}^n$ and $f_n(\omega) = \sum_{i=1}^n \omega_i$, if $|U| = \epsilon n$, then $\mathbf{E}[\, f_n \mid \omega_U \,] = \sum_{i \in U} \omega_i$, hence $\mathrm{Var}\,\mathbf{E}[\, f_n \mid \omega_U \,] = \epsilon n$, and $\text{clue}_{f_n}(U) = \epsilon$.

Quite similar for majority $\text{Maj}_n(\omega) = \text{sign} f_n$.

What about other transitive functions?

Clue and noise sensitivity for percolation

The answer for percolation LR crossing should be the same, by the noise sensitivity results of Garban, P. & Schramm '10:

A sequence of functions f_n is noise sensitive iff $\forall\, \epsilon > 0$, resampling each bit with probability $\epsilon > 0$ gives Corr$\big[f_n(\omega^\epsilon), f_n(\omega)\big] \to 0$.

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1) For $f_n = \mathrm{LR}_n$, $\epsilon_n = n^{-3/4+\epsilon}$ is enough. I.e., if U_n is everything but a sparse random set of density $n^{-3/4+\epsilon}$, then it is asymptotically clueless.

2) If U_n has a Hausdorff-distance scaling limit of Hausdorff-dimension less than 5/4, then it is asymptotically clueless.

3) If U_n is all the vertical bonds, then it is asymptotically clueless.

But, an example with $|U_n| = o(n^2)$ left out: disjoint boxes of radius $n^{3/8+\epsilon}$, distributed randomly, with typical gaps of $n^{3/8+2\epsilon}$.

The proof of 1)-2)-3) uses discrete Fourier analysis, and is quite hard, hence making it more quantitative seemed daunting.

What is the Fourier spectrum and why is it useful?

For $f : \{\pm 1\}^V \longrightarrow \mathbb{R}$, with $\mathbf{P}_{1/2}$ product measure for the input, $(N_\epsilon f)(\omega) := \mathbf{E}[\, f(\omega^\epsilon) \mid \omega\,]$ is the noise operator. Basically the Markov operator for continuous time random walk on the hypercube $\{-1,1\}^V$.

Covariance: $\mathbf{E}[\, f(\omega^\epsilon) f(\omega)\,] - \mathbf{E}[\, f(\omega)\,]\mathbf{E}[\, f(\omega^\epsilon)\,] = \mathbf{E}[\, f(\omega) N_\epsilon f(\omega)\,] - \mathbf{E}[\, f(\omega)\,]^2$. So, we would like to diagonalize the noise operator N_ϵ.

Let χ_i be the function $\chi_i(\omega) = \omega(i)$, $\omega \in \Omega$.

For $S \subset V$, let $\chi_S := \prod_{i \in S} \chi_i$, the parity inside S. Then

$$N_\epsilon \chi_i = (1-\epsilon)\,\chi_i\,; \qquad N_\epsilon \chi_S = (1-\epsilon)^{|S|}\,\chi_S.$$

Moreover, the family $\{\chi_S,\, S \subseteq V\}$ is an orthonormal basis of $L^2(\Omega,\mu)$.

Any function $f \in L^2(\Omega,\mu)$ in this basis (Fourier-Walsh series):

$$\hat{f}(S) := \mathbf{E}[\, f \chi_S\,]\,; \qquad f = \sum_{S \subseteq V} \hat{f}(S)\,\chi_S\,.$$

The covariance:

$$\mathbf{E}[\, f N_\epsilon f \,] - \mathbf{E}[\, f \,]^2 = \sum_S \sum_{S'} \hat{f}(S)\, \hat{f}(S')\, \mathbf{E}\big[\, \chi_S\, N_\epsilon \chi_{S'} \,\big] - \mathbf{E}[\, f \chi_\emptyset \,]^2$$

$$= \sum_{\emptyset \neq S \subseteq V} \hat{f}(S)^2\, (1-\epsilon)^{|S|} = \sum_{k=1}^{|V_n|} (1-\epsilon)^k \sum_{|S|=k} \hat{f}(S)^2.$$

By Parseval, $\sum_S \hat{f}(S)^2 = \mathbf{E}[\, f^2 \,]$. So can define probability measure $\mathbf{P}\big[\, \mathscr{S}_f = S \,\big] := \hat{f}(S)^2 / \mathbf{E}[\, f^2 \,]$, the spectral sample $\mathscr{S}_f \subseteq V$.

Therefore, a sequence f_n of non-degenerate functions, $\liminf_n \operatorname{Var} f_n > 0$, is noise sensitive if $\forall\, k \in \mathbb{Z}^+$, we have

$$\mathbf{P}\big[0 < |\mathscr{S}_n| < k\big] \to 0.$$

Clue and spectral sample

For $U \subseteq V$: $\mathbf{E}\big[\,\chi_S \mid \omega_U\,\big] = \begin{cases} \chi_S & S \subseteq U\,, \\ 0 & \text{otherwise}\,. \end{cases}$

Therefore, $\mathbf{E}\big[\,f \mid \omega_U\,\big] = \sum_{S\subseteq U} \widehat{f}(S)\,\chi_S$, a nice projection.

$\mathbf{P}\big[\,\mathscr{S}_f \subseteq U\,\big] = \sum_{S\subseteq U} \hat{f}(S)^2 = \mathbf{E}\Big[\Big(\sum_{S\subseteq U} \hat{f}(S)\,\chi_S\Big)^2\Big] = \mathbf{E}\Big[\mathbf{E}\big[\,f \mid \omega_U\,\big]^2\Big].$

Hence

$$\mathrm{clue}_f(U) = \frac{\mathrm{Var}\,\big[\mathbf{E}[\,f \mid \omega_U\,]\big]}{\mathrm{Var}\,f} = \frac{\mathbf{P}[\,\emptyset \neq \mathscr{S}_f \subseteq U\,]}{\mathbf{P}[\,\emptyset \neq \mathscr{S}_f\,]}$$

$$= \mathbf{P}\big[\,\mathscr{S}_f \subseteq U \,\big|\, \mathscr{S}_f \neq \emptyset\,\big]\,.$$

Small subsets are clueless

Proposition. If $f : \{\pm 1\}^V \longrightarrow \mathbb{R}$ is *transitive*, and $U \subset V$, then

$$\mathrm{clue}(U) \cdot \mathrm{Var}\, f = \mathbf{P}[\, \emptyset \neq \mathscr{S} \subseteq U \,] \leqslant \mathbf{P}[\, X \in U \,] = \sum_{u \in U} \mathbf{P}[\, X = u \,] = \frac{|U|}{|V|},$$

where $X \in \mathscr{S}$ is a uniformly chosen random bit from $\mathscr{S} \neq \emptyset$. □

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Consider an $n \times n$ torus, and all n^2 translated ways of cutting it into a square. Let $f_{i,j}$, for $(i,j) \in \{1, \ldots, n\}^2$, be the LR crossing indicators in these squares. Want to join them using a transitive function Φ,

$$F(\omega) = \Phi(f_{1,1}(\omega), \ldots, f_{n,n}(\omega)),$$

and argue that if the $f_{i,j}$'s had small sets with large clue, then F would also have. For this, Φ definitely should not be noise-sensitive.

From LR crossing to a transitive function

$$F(\omega) := \sum_{(i,j)\in\{1,2,\dots,n\}^2} f_{i,j}(\omega)\,, \qquad \mathrm{SD}\, F \asymp n^2\,.$$

$$F_\epsilon(\omega) := \sum_{(i,j)\in\{\epsilon n, 2\epsilon n, \dots\}^2} f_{i,j}(\omega)\,, \qquad \mathrm{SD}\, F_\epsilon \asymp 1/\epsilon^2\,.$$

Claim 1.

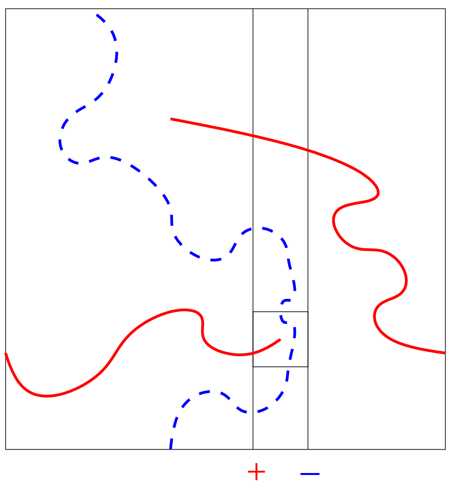
$$\mathrm{Corr}\left[\frac{F_\epsilon}{1/\epsilon^2}, \frac{F}{n^2}\right] \geqslant 1 - O(\epsilon)\,.$$

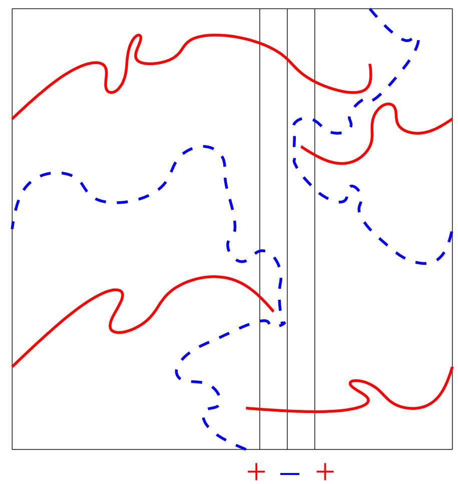
Claim 2. If a small subset $U_{i,j}$ had a positive clue about $f_{i,j}$, then U_ϵ, the union of the $1/\epsilon \times 1/\epsilon$ translates, would have a positive clue about F_ϵ, so also about F, which is impossible, since $|U_\epsilon|/n^2 = \epsilon^{-2}|U_{i,j}|/n^2$ is still small.

Proof of Claim 1

If (i,j) and (k,ℓ) are neighbours in the ϵ-grid, and $f_{i,j} \neq f_{k,\ell}$, then there is a half-plane 3-arm event from distance ϵn to n. Since $\alpha_3^+(\epsilon n, n) \asymp \epsilon^2$, the expected number of such neighbours is $O(1)$. Thus

$$\mathbf{P}\Big[\text{number of neighbours with } f_{i,j} \neq f_{k,\ell} \text{ is } > 1/\epsilon\Big] < O(\epsilon)\,.$$





If there are neighbours (i,j) and (k,ℓ) with $f_{i,j} = f_{k,\ell}$ in the ϵ-grid, but there is an $(u,v) \in \{1,\ldots,n\}^2$ nearby with $f_{u,v}$ different, we have two independent half-plane 3-arm events, from two ϵn-boxes vertically or horizontally aligned. The probability of this happening is $\asymp (\epsilon^2)^2/\epsilon^3 = O(\epsilon)$.

Proof of Claim 1

In summary: with probability $1 - O(\epsilon)$, the ϵ-grid detects all the changes in crossing events, and there are only $1/\epsilon$ changes, hence

$$\left| F_\epsilon \cdot \epsilon^2 n^2 - F \right| < 1/\epsilon \cdot \epsilon^2 n^2 \, .$$

That is,

$$\mathbf{P}\Big[\Big| \frac{F_\epsilon}{1/\epsilon^2} - \frac{F}{n^2} \Big| > \epsilon \Big] < O(\epsilon) \, .$$

Being bounded random variables, this implies that

$$\mathrm{Corr}\Big[\frac{F_\epsilon}{1/\epsilon^2}, \frac{F}{n^2} \Big] > 1 - O(\epsilon) \, .$$

$\square$

Proof of Claim 2 is not hard, either.

Transitive functions of non-iid spins?

Consider a translation invariant Markov random field $\sigma \in \{-1,+1\}^{\mathbb{Z}_n^2}$; e.g., the Ising model at inverse temperature $\beta \in (0,\infty)$:

$$\mu_\beta^{\mathbb{Z}_n^2}(\sigma) := \frac{1}{Z_\beta} \exp\Big(-\beta \sum_{x \sim y} \mathbf{1}_{\sigma(x) \neq \sigma(y)} \Big).$$

If $f : \{-1,+1\}^{\mathbb{Z}_n^2} \longrightarrow \mathbb{R}$ is a transitive function, and $U \subset \mathbb{Z}_n^2$, let

$$\mathrm{clue}_f(U) = \frac{\mathrm{Var}\, \mathbf{E}[\, f \mid \sigma_U \,]}{\mathrm{Var}\, f}.$$

For what measures is it true that $|U_n| = o(n^2)$ implies $\mathrm{clue}(U_n) \to 0$?

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Note: For noise sensitivity with iid spins, there are two key techniques:

(1) Explicit eigenfunctions of the noise operator, indexed by subsets of the spins, giving rise to the Fourier spectral sample.

(2) Hypercontractivity / log-Sobolev inequality for RW on the hypercube, again proved using Fourier, implies things like: a monotone function is noise sensitive iff uncorrelated with majority over any subset.

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Example: Low temperature Ising, $\beta > \beta_c$. Then $\mu_\beta^{\mathbb{Z}_n^2}$ converges weakly to $(\mu_\beta^+ + \mu_\beta^-)/2$: not extremal. And sparse reconstruction is easy: if $|U_n| \to \infty$, then $\mathrm{sign} \sum_{x\in U_n} \sigma(x)$ tells us with large probability if we are in μ_β^+ or μ_β^-, hence has clue close to 1 about $f(\sigma) := \mathrm{sign} \sum_{x\in\mathbb{Z}_n^2} \sigma(x)$.

Lemma (Lanford & Ruelle '69). For Markov fields, non-extremal $\Leftrightarrow$ not tail-trivial $\Leftrightarrow$ spin reconstruction from a large distance.

Spectral sample for non-iid spins?

Can we define a random set $\mathscr{S} = \mathscr{S}_f$, based on clue:

$$\mathbf{P}[\,\mathscr{S} \subseteq U\,] = \left\|\mathbf{E}[\,f \mid \sigma_U\,]\right\|^2, \qquad \frac{\mathbf{P}[\,\emptyset \neq \mathscr{S} \subseteq U\,]}{\mathbf{P}[\,\emptyset \neq \mathscr{S}\,]} = \mathrm{clue}_f(U)\,?$$

Eigenfunctions of the Glauber dynamics noise operator are typically not indexed by subsets of bits, hence this would be a different generalization of Fourier transform to non-iid measures.

Can try inclusion-exclusion formula:

$$\mathbf{P}[\,\mathscr{S} = S\,] := \sum_{T \subseteq S} (-1)^{|S|-|T|}\,\mathbf{P}[\,\mathscr{S} \subseteq T\,]\,.$$

Issue: why would this be non-negative for all S?

Product measures: Efron-Stein decomposition

Theorem (Efron & Stein '81). For $f \in L^2(\Omega^n, \pi^{\otimes n})$, there is a unique decomposition

$$f = \sum_{S \subseteq [n]} f^{=S},$$

where $f^{=S}$ depends only on the bits in S, and $(f^{=S}, f^{=T}) = 0$ for $S \neq T$.

Namely, letting $f^{\subseteq S} := \mathbf{E}[\, f \mid \mathscr{F}_S \,]$, the inclusion-exclusion definition works:

$$f^{=S} := \sum_{T \subseteq S} (-1)^{|S|-|T|} f^{\subseteq T}.$$

Remark. For $\Omega = \{\pm 1\}$, just $f^{=S} = (f, \chi_S)$.

Consequently, $\mathbf{P}[\, \mathscr{S} = S \,] := \| f^{=S} \|^2 / \| f \|^2$ is a good spectral sample, and the one-line Small Clue Theorem works.

Why are we happy about this?

Subcritical Ising as a factor of iid

A measure μ on $\{-1,+1\}^{\mathbb{Z}^d}$ is a factor of iid if there is a measurable map $\psi : [0,1]^{\mathbb{Z}^d} \longrightarrow \{-1,+1\}$ such that if $\omega \sim \mathrm{Unif}[0,1]^{\mathbb{Z}^d}$, then

$$\sigma(x) := \psi\big(\omega(x+\cdot)\big)\,, \quad x \in \mathbb{Z}^d\,,$$

is distributed w.r.t. μ. This factor map is finitary if there is a random coding radius $R(\omega) < \infty$ such that $\psi(\omega)$ and $R(\omega)$ are determined by $\big\{\omega(x) : x \in [-R,R]^d\big\}$.

Using exponential convergence of the Ising Glauber dynamics for $\beta < \beta_c$ (Martinelli & Olivieri '94), and the Coupling From The Past perfect sampling algorithm (Propp & Wilson '96):

Theorem (van den Berg & Steif '99). For $\beta < \beta_c$, the unique Ising measure μ on $\mathbb{Z}^d$ is a finitary factor of $\mathrm{Unif}[0,1]^{\mathbb{Z}^d}$, with coding radius $\mathbf{P}[\,R > t\,] < \exp(-ct)$.

Small Clue Theorem for subcritical Ising

Theorem. For any transitive function f of the Ising spins $\{\sigma(x) : x \in \mathbb{Z}_n^d\}$, and any subset $|U_n| = o(n^d / \log^d n)$, we have $\mathrm{clue}_f(U_n) \to 0$.

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Proof. We can get σ as a finitary factor ψ of iid, with coding radii

$$\mathbf{P}\big[\, R_u < r_n \text{ for all } u \in U_n \,\big] > 1 - |U_n| \exp(-c r_n)\,,$$

which is $1 - o(1)$ if $r_n = C \log n$ with large enough C, while $|U_n|\, r_n^d = o(n^d)$ still holds. Thus, taking $V_n = \bigcup_{u \in U_n} B_{C \log n}(u)$, we have

- ω_{V_n} determines σ_{U_n} with probability $1 - o(1)$;
- $|V_n| = o(n^2)$, hence for $g = f \circ \psi$, we have $\mathrm{clue}_g(V_n) = o(1)$.

Let $\mathscr{G}$ be the sigma-algebra generated by $\Big\{\omega_{B_{R_u}(u)},\ u \in U_n\Big\}$ and ω_{V_n}. Then $\mathrm{Var}\,\mathbf{E}[\, g \mid \mathscr{G}\,] \geqslant \mathrm{Var}\,\mathbf{E}[\, f \mid \sigma_{U_n}\,]$. On the other hand,

$$\big\|\mathbf{E}[\, g \mid \mathscr{G}\,]\big\|^2 = \big\|\mathbf{E}[\, g \mid \omega_{V_n}\,]\big\|^2 + \big\|\mathbf{E}[\, g \mid \mathscr{G}\,] - \mathbf{E}[\, g \mid \omega_{V_n}\,]\big\|^2 .$$

Both terms on the right are $o(\mathrm{Var}\, g)$, hence $\mathrm{Var}\,\mathbf{E}[\, f \mid \sigma_{U_n}\,] = o(\mathrm{Var}\, f)$. □

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What about critical and supercritical Ising?

Theorem (van den Berg & Steif '99).

- At $\beta = \beta_c$, using $\sum_{x \in \mathbb{Z}^d} \mathbf{E}[\,\sigma(0)\sigma(x)\,] = \infty$, the unique Ising measure μ cannot be a finitary factor with $\mathbf{E}[\,R^d\,] < \infty$.
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Intuition 1: The infinite susceptibility $\sum_{x\in\mathbb{Z}^d} \mathbf{E}[\,\sigma(0)\sigma(x)\,] = \infty$ translates to $\mathrm{Var}\,[\mathrm{Maj}_n] \gg n^d$. Then, majority over a random not too sparse sample, which forgets the geometry, will have a high correlation with the full majority. But this is only non-constructive sparse reconstruction.

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Intuition 2: the discrete magnetization field has a scaling limit that is measurable w.r.t. macroscopic cluster structure of the FK random cluster representation underlying the Ising model (Camia, Garban, Newman '13). That is, from macrosopic info only, can guess microscopic magnetization. Can similarly guess the sparse magnetization, and if ϵ is small enough for M_n^ϵ to be supported everywhere, then it will have the same scaling limit, and must be close to the full magnetization.

What about critical and supercritical Ising?

Theorem (van den Berg & Steif '99).

- At $\beta = \beta_c$, using $\sum_{x \in \mathbb{Z}^d} \mathbf{E}[\, \sigma(0)\sigma(x) \,] = \infty$, the unique Ising measure μ cannot be a finitary factor with $\mathbf{E}[\, R^d \,] < \infty$.
- At $\beta > \beta_c$, the $+$ measure μ^+ is not a finitary factor of iid.

Theorem. At $\beta = \beta_c$ on $\mathbb{Z}_n^2$, the total magnetization $M_n(\sigma) := \sum_x \sigma(x)$ can be guessed with high precision from the sparse magnetization $M_n^\epsilon(\sigma) := \sum_{n^\epsilon | x} \sigma(x)$, as long as $\epsilon < 7/8$. This implies $\mathrm{clue}_{M_n}(n^\epsilon\text{-grid}) = 1 - o(1)$.

Proof (with help from Christophe Garban): From Wu '66 and Chelkak, Hongler, Izyurov '12, we know $\mathbf{E}[\, \sigma(x)\sigma(y) \,] \sim c \, \|x - y\|^{-1/4}$.
Thus $\mathrm{Var}\,[M_n] = \sum_{x,y} \mathbf{E}[\, \sigma(x)\sigma(y) \,] \asymp n^{4-1/4}$.
Also, $\mathrm{Var}\,[M_n^\epsilon] \asymp n^{2-2\epsilon} + n^{4-4\epsilon-1/4}$.
On the other hand, $\mathrm{Cov}\big[\, M_n, M_n^\epsilon \,\big] = \sum_{x, n^\epsilon | y} \mathbf{E}[\, \sigma(x)\sigma(y) \,] = n^{4-2\epsilon-1/4}$.

For $\epsilon < 7/8$, get $\mathrm{Corr}\big[\, M_n, M_n^\epsilon \,\big] > c > 0$. With a more careful argument, using the scaling limit, can get $1 - o(1)$. □

Generalized Divide and Colour Models

Consider a partition of $[n]$. More precisely, $\pi : [n] \longrightarrow [n]$, giving the partition by the inverse images. Then flip an independent fair coin for each part, or equivalently, let $\sigma(i) := \omega(\pi(i))$, where ω is an iid fair sequence.

If f is a function of the spins σ, define $f_\pi(\omega) := f(\omega \circ \pi)$. It turns out that

$$\widehat{f_\pi}(T) = \sum_{S \,:\, \pi_\oplus(S) = T} \widehat{f}(S),$$

where $\pi_\oplus(S) = \oplus_{j \in S}\, \pi(j)$, understood as mod 2 addition in $\{0,1\}^{[n]}$. Clearly,

$$\mathbf{E}[\, f(\sigma)^2 \,] = \mathbf{E}[\, f_\pi(\omega)^2 \,] = \sum_{T \subset [n]} \widehat{f_\pi}^2(T)\,.$$

We can now define $\mathscr{S}_f^\pi$ via $\widehat{f_\pi}(T)^2$, and then, for any subset U of the spins,

$$\frac{\mathrm{Var}\, \mathbf{E}[\, f \mid \sigma(U), \pi \,]}{\mathrm{Var}\, [f \mid \pi]} = \mathbf{P}\big[\, \mathscr{S}_f^\pi \subset U \;\big|\; \mathscr{S}_f^\pi \neq \emptyset \,\big]. \tag{1}$$

This is for any specific π. If π is an invariant random Π, as in the FK random cluster model, then $\sigma(U)$ may contain information about Π, so the LHS of (1) has a non-trivial term in the conditional variance formula, and we cannot just average over Π on the RHS. For $\mathbf{E}f = 0$,

$$\mathbf{P}[\,\mathscr{S}_f^\Pi \subset U \mid \mathscr{S}_f^\Pi \neq \emptyset\,] \leqslant \frac{\operatorname{Var}\mathbf{E}[\,f \mid \sigma(U)\,]}{\operatorname{Var} f} \leqslant \mathbf{P}[\,\mathscr{S}_f^\Pi \subset U\,].$$

Of course, $\mathbf{P}[\,\mathscr{S}_f^\Pi = \emptyset \mid \Pi\,] = \mathbf{E}[\,f \mid \Pi\,]$. If this is close to $\mathbf{E}f = 0$ for most Π, then the lower and upper bounds are close.

In general:

$$\operatorname{clue}_f(\sigma(U)) \leqslant \mathbf{E}\max_i |\Pi^{-1}(i)|\frac{|U|}{n} + \mathbf{P}[\,\mathscr{S}_f^\Pi = \emptyset\,].$$

In nice cases, such as FK, *should be*

$$\operatorname{clue}_f(\sigma(U)) \leqslant C\,\mathbf{E}|\Pi^{-1}(1)|\frac{|U|}{n}.$$

Note that $\mathbf{E}|\Pi^{-1}(1)| = \sum_i \mathbf{E}[\,\sigma(1)\sigma(i)\,]$, susceptibility again.

Some questions

1. If a Markov random field on $\mathbb{Z}^d$ is a finitary factor of iid with finite expected coding volume, $\mathbf{E}[\,R^d\,] < \infty$, and thus finite susceptibility:

   a. No sparse reconstruction for Majority?

   b. No sparse reconstruction for any transitive function?

   In particular, does it hold for subcritical Ising in its sharpest form?

   For generalized DaC models, $\mathbf{E}|\Pi^{-1}(1)| \ll \infty$ implies no sparse reconstruction for Majority, but other functions can sometimes be reconstructed!

2. Supercritical Ising μ^+ is not a finitary factor of iid. Is there a transitive function with sparse reconstruction? This time, magnetization probably does not work.

3. For 2-dimensional Ising at β_c, prove that LR-crossing of spins with $+ - +-$ boundary condition has sparse reconstruction. (Reason: no pivotals, positive correlation with magnetization field.)