Random walks on percolation clusters, and scale-invariant groups

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Based on: A note on percolation on $\mathbb{Z}^d$: isoperimetric profile via exponential cluster repulsion, *Elect. Comm. Probab.* **13** (2008), [arXiv:math/0702474 math.PR]

and on joint work
with Volodia Nekrashevych (Texas A&M): Scale-invariant groups
Groups, Geometry & Dynamics, to appear, [arXiv:0811.0220 math.GR]

General wisdom:

Algebraic properties of a group
$\longleftrightarrow$ Coarse geometric properties of its Cayley graphs
$\longleftrightarrow$ Behaviour of simple random walk on it

How much of this survives if we pass to an infinite percolation cluster?

This talk:

- On finitely presented groups, anchored isoperimetry survives if $p > 1 - \epsilon$.
- On $\mathbb{Z}^d$, via a new large deviations result (exponential cluster repulsion), everything works nicely for all $p > p_c(\mathbb{Z}^d)$, e.g., $p_n(o, o) \asymp n^{-d/2}$.
- On what groups can one hope to do similar things, especially, percolation renormalization? Scale-invariant groups and tilings.
- A lot of questions.

Isoperimetry, groups, random walks

$\psi(\cdot) \uparrow \infty$. Bounded degree $G(V, E)$ has ψ-isoperimetric inequality $\mathcal{IP}_\psi$ if

$$0 < \iota_\psi(G) := \inf\left\{ \frac{|\partial S|}{\psi(|S|)} : S \subset V(G) \text{ connected finite} \right\}.$$

$\psi(x) = x^{1-1/d}$: d-dimensional isoperimetry $\mathcal{IP}_d$.
$\psi(x) = x$: non-amenability $\mathcal{IP}_\infty$.

A Cayley graph has $|B_n(x)| \leqslant Cn^d$ **iff** $\mathcal{IP}_{d+\epsilon}$ does not hold [Varopoulos 1985, Coulhon-Saloff-Coste 1993]

A Cayley graph has a $d < \infty$ with $|B_n(x)| \leqslant Cn^d$ **iff** the group is almost nilpotent [Gromov 1981].

A Cayley graph has $\mathcal{IP}_d$ **iff** $p_n(x, x) \leqslant cn^{-d/2}$. Varopoulos, Saloff-Coste, Coulhon, Grigoryan, Pittet, Lovász-Kannan, Morris-Peres, etc.
Nash inequalities, Faber-Krahn inequalities, evolving sets, etc.

A group is amenable [von Neumann 1929], i.e., exists invariant mean on all bounded functions, **iff** any Cayley graph of it is amenable [Følner 1955]. Idea 1: Compute averages along almost-invariant sets, take Banach limit. Idea 2: $\mathcal{IP}_\infty$ implies wobbling paradoxical decomposition.

G is non-amenable **iff** spectral radius $\rho = \lim_n p_n(x,y)^{1/n} < 1$ [Kesten 1959, Cheeger 1970]. Almost invariant sets $\longleftrightarrow$ almost invariant functions. Implies that SRW has linear rate of escape.

Conjecture [Benjamini-Schramm 1996]. G is non-amenable iff there is a p with infinitely many infinite clusters.

Amenable examples: Abelian, nilpotent, solvable groups. Any group with subexponential volume growth, e.g., [Grigorchuk 1984]. Basilica group [Bartholdi-Virág 2005].

Non-amenable examples: Anything with an F_2 free subgroup, e.g., $SL_n(\mathbb{Z})$. Gromov hyperbolic groups. Tarski monsters [Olshanskii 1980] and free Burnside groups [Adian 1982].

Anchored isoperimetry

Bounded degree infinite $G(V,E)$, fixed $o \in V(G)$, function $\psi(\cdot) \nearrow \infty$.
$G(V,E)$ satisfies an anchored ψ-isoperimetric inequality $\mathcal{IP}^*_\psi$ if

$$0 < \iota^*_\psi(G) := \lim_{n\to\infty} \inf \left\{ \frac{|\partial S|}{\psi(|S|)} : o \in S \subset V(G),\ S \text{ conn.},\ n \leqslant |S| < \infty \right\}.$$

Does not depend on the anchor o. For G transitive, same as usual $\mathcal{IP}_\psi$.
$\psi(x) = x$: anchored expansion or weak nonamenability, $\mathcal{IP}^*_\infty$.
$\psi(x) = x^{1-1/d}$: d-dimensional anchored isoperimetry $\mathcal{IP}^*_d$.

Definition is by [Thomassen 1992] and [Benjamini-Lyons-Schramm 1999].

Point 1: Unlike $\mathcal{IP}_\psi$, this has a chance to survive percolation.
E.g., supercritical GW trees on non-extinction have $\mathcal{IP}^*_\infty$ [Chen-Peres 2004].

Point 2: Still has many probabilistic implications.

[Thomassen 1992] $\mathcal{IP}^*_{2+\epsilon}$ implies transience (with a "precise ϵ").

Stronger result with very short proof by [Lyons-Morris-Schramm 2006].

[Virág 2000] $\mathcal{IP}^*_\infty$ implies positive liminf speed for SRW, and heat kernel decay $p_n(o,o) \leqslant \exp(-cn^{1/3})$, best possible.

Thomassen and Virág show existence of large subgraph. False for $\mathcal{IP}^*_d$.

Conjecture 1. $\mathcal{IP}^*_d$ implies $p_n(o,o) \leqslant Cn^{-d/2}$. (And there is a general version.)

Conjecture 2. If G is not $\mathcal{IP}^*_\infty$ (so, strongly amenable), then the Green super-level sets $S_t := \{x \in V : \mathcal{G}(o,x) > t\}$ form a Følner sequence.
Open even for groups, conjectured also by C. Pittet.
Maybe, if G is not $\mathcal{IP}^*_\psi$, then the S_t witness this: $|\partial S_t|/\psi(|S_t|) \to 0$?

Conjecture 3. If G satisfies $\mathcal{IP}^*_{\tilde{\psi}}$, with $\tilde{\psi}$ derived from its volume growth, then SRW is not subdiffusive: $\mathbf{E}_o[X_n] \geqslant c\sqrt{n}$. True for groups [Mok-Erschler, Lee-Peres].

Survival of $\mathcal{IP}^*_\psi$

Proposition. If G has $\mathcal{IP}^*_\psi$ with some $\psi \nearrow \infty$, and the exponential decay

$$\mathbf{P}_p\big[|\mathscr{C}_o| < \infty,\ |\partial_E^+ \mathscr{C}_o| = n\big] \leqslant \varrho(p)^n$$

holds, then p-a.s. on the event $|\mathscr{C}_o| = \infty$, also $\mathscr{C}_o$ has $\mathcal{IP}^*_\psi$.

The $\varrho(p)^n$ decay holds for $p > 1 - \epsilon$ if the number of cutsets of size n grows at most exponentially (Peierls argument), e.g., for finitely presented groups.

A bit trickier: $\mathcal{IP}^*_\infty$ for all $p > \frac{1}{\iota^*_\infty(G)+1}$. [my Appendix to Chen-Peres 2004]

However, on $\mathbb{Z}^d$, $d \geqslant 3$, it does not hold for $p \in (p_c, 1-p_c)$.

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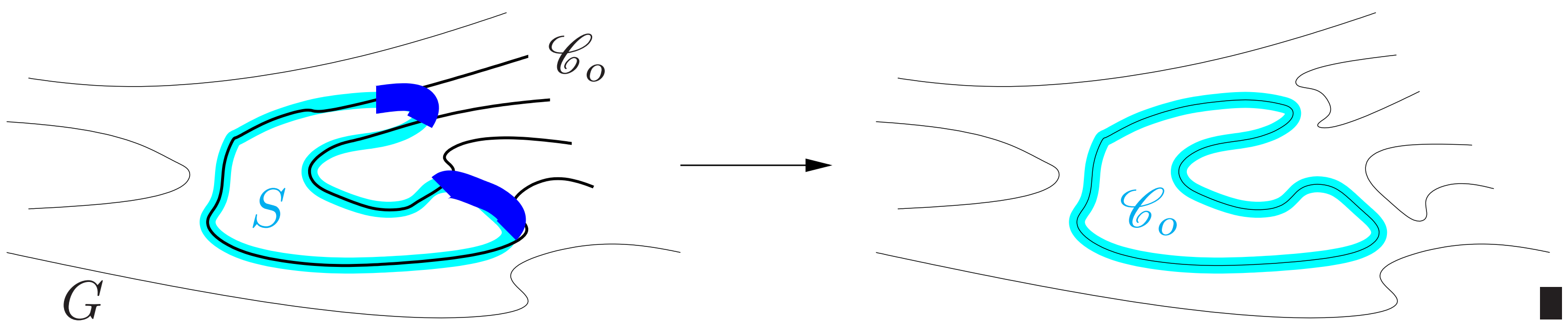
Proof of theorem: If $|\partial_G^+ S| = n$ but $|\partial_{\mathscr{C}_o}^+ S| \leqslant \alpha n$, then can redeclare with a cost $\leqslant (1-p)^{-\alpha n}$, and $\leqslant \binom{n}{\alpha n}$ preimages. Small exponential for α small.



∎

Exponential cluster repulsion on $\mathbb{Z}^d$

Between two clusters, the number of touching edges is $\tau(\mathscr{C}_1, \mathscr{C}_2)$.

Theorem. For $d \geqslant 2$ and any $p > p_c(\mathbb{Z}^d)$, there is a $c_1 = c_1(d,p) > 0$ s.t.

$$\mathbf{P}_p\Big[m \leqslant |\mathscr{C}_o| < \infty \text{ and } \tau(\mathscr{C}_o, \mathscr{C}_\infty) \geqslant t\Big] \leqslant \exp\big(-c_1 \max\{m^{1-1/d}, t\}\big).$$

Setting $t = 0$, the stretched exponential decay we get is a sharp classical result: [Kesten-Zhang 1990] combined with [Grimmett-Marstrand 1990].

Corollaries. For all $p > p_c(\mathbb{Z}^d)$, $\mathscr{C}_\infty$ satisfies $\mathcal{IP}_d^*$ a.s. For giant cluster $\mathscr{C}$ in $[-n,n]^d$, $\exists\ c_2(d,p), \alpha(d,p) > 0$ s.t., a.a.s., for all connected $S \subseteq \mathscr{C}$ with $c_2 (\log n)^{\frac{d}{d-1}} \leqslant |S| \leqslant |\mathscr{C}|/2$, we have $|\partial_{\mathscr{C}} S| \geqslant \alpha |S|^{1-1/d}$. By [Morris-Peres 2005], L^∞-mixing time is $\Theta(n^2)$, while $p_n(o,o) \leqslant C n^{-d/2}$ on $\mathscr{C}_\infty$.

Almost by [Benjamini-Mossel 2003], then actually by [Mathieu-Remy 2004] and [Rau 2006], 40 pages. Gaussian off-diagonal decay by [Barlow 2004].

Exponential cluster repulsion elsewhere?

Conjecture. On any infinite group,

$$\mathbf{P}_p\Big[|\mathscr{C}_o| < \infty \text{ and } \exists \mathscr{C}_\infty \text{ with } \tau(\mathscr{C}_o, \mathscr{C}_\infty) \geqslant t\Big] \leqslant \exp(-ct).$$

The renormalization technique is completely missing.

On non-amenable groups, $\tau(\mathscr{C}^i_\infty, \mathscr{C}^j_\infty) < \infty$ a.s. for all i, j. [Timár 2006]

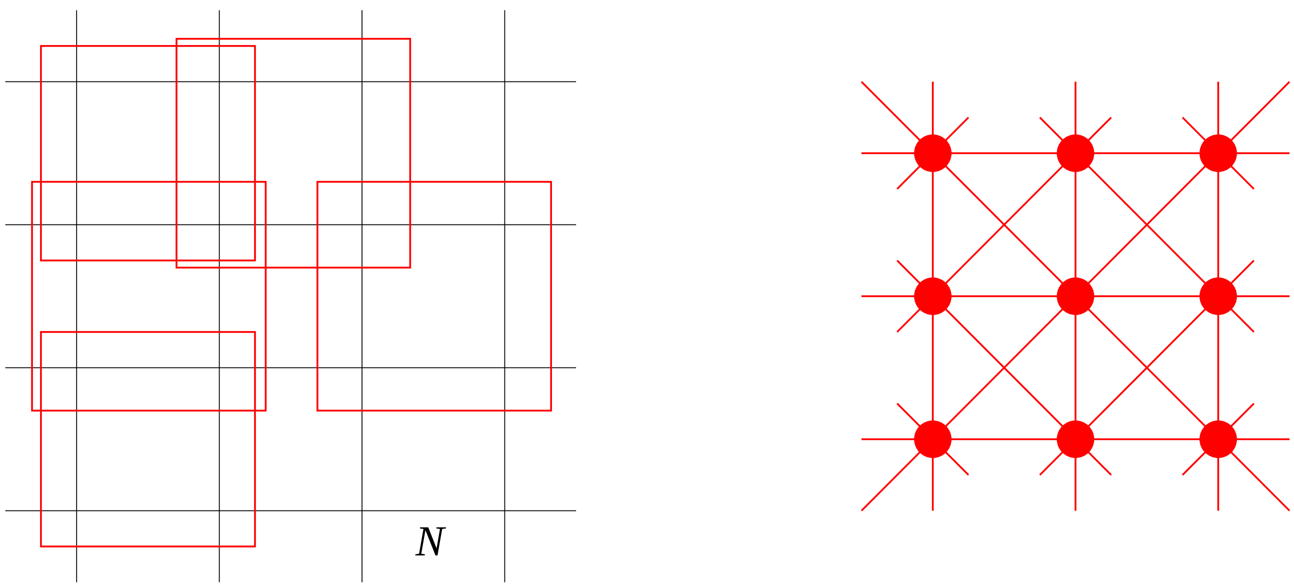
Conjecture. Bond percolation $p > \frac{1+\epsilon}{d}$ on the hypercube $\{0,1\}^d$. Then SRW on the giant cluster has mixing time $d^{O_\epsilon(1)}$. Even $O_\epsilon(d \log d)$?

Would follow from $\mathbf{P}_p\Big[\exists \mathscr{C} \text{ with } \tau(\mathscr{C}_o, \mathscr{C}) \geqslant t\Big] \leqslant \exp(-ct/d^{O_\epsilon(1)})$.

Note that if $|\mathscr{C}_o| > d^{O_\epsilon(1)}$, then it should already be the giant cluster. But none of [AjKoSz82], [BoKoŁu92], [BoChvdHSlSp05] helps.

Proof idea of exponential cluster repulsion

Blocks $B_x = \{y \in \mathbb{Z}^d : \|y - Nx\|_\infty \leqslant 3N/4\}$. B is good if it has a cluster connecting its $(d-1)$-dim faces, while its other clusters have diam $< N/5$.
Renormalization: $\forall p > p_c$, $\lim_N \mathbf{P}_p(B \text{ is good}) = 1$. [Antal-Pisztora 1996]



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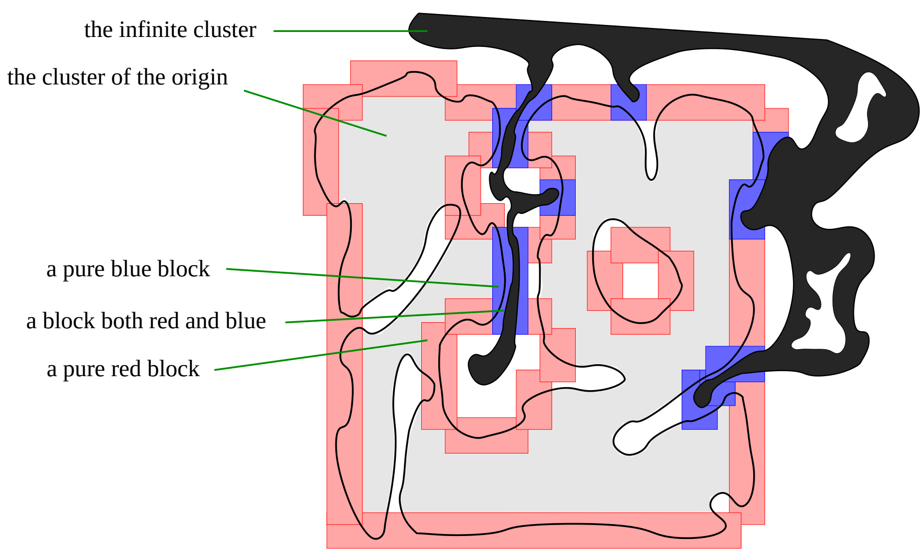
B is $\mathscr{C}$-substantial if $\mathscr{C} \cap B$ has a component of diam $\geqslant N/5$. Assume $o \notin \mathscr{C}_\infty$. A block is RED: $\mathscr{C}_o$-substantial but has a non-$\mathscr{C}_o$-substantial neighbor. BLUE: both $\mathscr{C}_o$- and $\mathscr{C}_\infty$-substantial. Each touching edge is in at least one blue block, in at most 2^d. Observe: a colored block is never good.



Main Lemma. On the event $\{|\mathscr{C}_o| = m \text{ and } \tau(\mathscr{C}_o, \mathscr{C}_\infty) \geqslant t\}$, the set of blocks RED $\cup$ BLUE has a $*$-connected subset of size

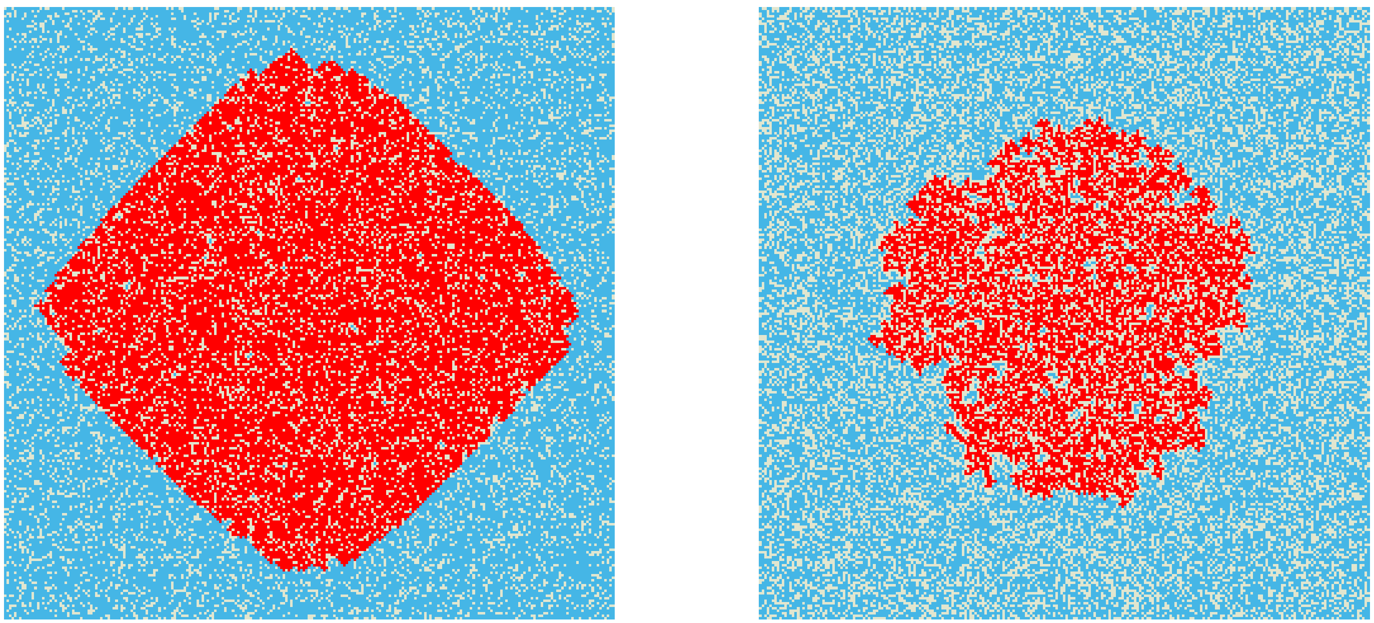
$$\geqslant c(N,d) \max\{m^{1-1/d}, t\},$$

contained in the box $B_m(o)$. ∎

Wulff shape inside the cluster?

Balls inside $\mathscr{C}_\infty$ $\overset{?}{\longleftrightarrow}$ Isoperimetrically optimal sets inside $\mathscr{C}_\infty$ $\overset{?}{\longleftrightarrow}$ Classical Wulff shape (large finite clusters)

Conjecture. Limit shapes exist. (Are they the same?) In the plane, these converge to a Euclidean ball, as $p \downarrow p_c$. (For balls, asked by Itai Benjamini.)



Site percolation on $\mathbb{Z}^2$, at densities 0.8 and 0.65, the ball of radius 120.

Scale-invariant groups

A group G is scale-invariant if there is a subgroup chain $G = G_0 \supset G_1 \supset G_2 \supset \dots$ with $G_n \simeq G$ and $\bigcap_{n \geqslant 0} G_n$ finite. Benjamini **conjectured** that such a G has polynomial growth, hence is almost nilpotent [Gromov 1981].

Reason for definition: renormalization. For conjecture: If $G_n = \varphi^{\circ n}(G)$, then φ "looks like" expanding: $d(\varphi(x), \varphi(y)) > (1+\epsilon) d(x,y)$. But then $|B_{(1+\epsilon)r}| \lesssim [G : \varphi(G)]\, |B_r|$. [Franks 1970, Farkas 1981, Gelbrich 1985]

Not scale-inv: free group F_r has $s - 1 = [F_r : F_s](r-1)$. More generally, if there is a non-zero Euler characteristic $\chi(G)$, e.g., if Betti$_1^{(2)} > 0$ (the G-dimension of harmonic Dirichlet functions).
Using this and a theorem of Zlil Sela: torsion-free hyperbolic groups. What about relatively hyperbolic groups?

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Not scale-inv: $\text{Betti}_1^{(2)} > 0$. Torsion-free (relatively?) hyperbolic groups.

Theorem. Let H be scale-invariant, $\bigcap_{n \geqslant 0} H_n = \{1\}$, and A an automorphism group of H leaving all H_n invariant. Assume A is faithful on each H_n, and $A \ltimes H_n \simeq A \ltimes H$. Then $G := A \ltimes H$ is scale-invariant.

Corollary. The following groups are scale-invariant:
The lamplighter groups $\mathbf{F} \wr \mathbb{Z}$, where $\mathbf{F}$ is any finite Abelian group.
The solvable Baumslag-Solitar groups $BS(1,m) = \langle a, b \mid bab^{-1} = a^m \rangle$.
The affine groups $A \ltimes \mathbb{Z}^d$ with $A \leqslant GL(\mathbb{Z}, d)$.

Sketch of proof

Coset tree $\mathcal{T} = \{(x_0 x_1 x_2 \dots) \colon H_{n+1} x_{n+1} \subset H_n x_n\}$. Since H_n is A-invariant, $G := A \ltimes H$ acts on $\mathcal{T}$ by the affine transformations $(H_n x_n)^{(\alpha,h)} = H_n\,\alpha(x_n) h$. Extends to a continuous action on $\partial\mathcal{T}$.

$$\mathrm{St}_G(H_n x_n) = \Big\{\big(\alpha,\, \alpha(x_n)^{-1} h_n x_n\big) \colon \alpha \in A,\ h_n \in H_n\Big\} \simeq A \ltimes H_n \simeq A \ltimes H\,.$$

For $x = (x_0, x_1, x_2, \dots) \in \partial\mathcal{T}$, we have $\mathrm{St}_G(x) = \bigcap_{n \geqslant 0} \mathrm{St}_G(H_n x_n)$. For $(\alpha, h) \neq (1,1)$, $\big\{x \in \partial\mathcal{T}$ not stabilized by $(\alpha, h)\big\}$ is open and dense, so, by Baire's category theorem, exists ray with trivial stabilizer. ∎

The isomorphisms $\varphi_{x_n} : G \longrightarrow \mathrm{St}_G(H_n x_n)$ satisfy $\varphi_{x_n} = \varphi_{x_{n-1}} \circ \varphi_{x_{n-1},x_n}$, where x_n is parent of x_{n-1}, and $\varphi_{x_{n-1},x_n} \in \{\varphi_1, \dots, \varphi_t\}$.

In our examples, the ray with trivial stabilizer has to be an "irrational ray", never periodic! For quite different reasons in the three examples. . .

Question. If $G_n = \varphi^{\circ n}(G)$ scale-inv, is then G of polynomial growth?

How is Lamplighter Group an example?

$H < \mathbb{Z}_2[[t]]$ additive group of finite Laurent polynomials of (1+t), $\psi(F(t)) = tF(t)$, $[H : \psi(H)] = 2$.
Coset tree is same as natural representation of $\mathbb{Z}_2[[t]]$ as binary tree.

$A = \mathbb{Z}$ acts on H, multiplication by $1 + t$. $G = A \ltimes H$.

$$(m, f) : \quad F(t) \mapsto (1 + t)^m F(t) + \sum_{k \in \mathbb{Z}} f(k)(1 + t)^k$$

Wreath generators: $s : F(t) \mapsto F(t) + 1$ and $R : F(t) \mapsto (1 + t)F(t)$.

With $a = Rs$, $b = R$, self-similar action: $a = (b, a)\epsilon$, $b = (b, a)$.

Diestel-Leader graph is Cayley graph w.r.t. both $\langle Rs, R\rangle$ and $\langle sR, Rs\rangle$.

Good tilings and tiles?

Scale-invariant tiling $\{T_i : i \in I\}$ of transitive Γ: finite connected $T_i \simeq T$, tiling graph $(i \sim j : \exists x \in T_i, y \in T_j, x \sim y) \simeq \Gamma$, and can iterate $T^{(n)} \nearrow \Gamma$.

SI of non-Abelian G **does not** give SI tiling of Γ! But expanding homomorphism of Heisenberg Lie group induces nice contracting self-similar action on coset tree, producing a SI tiling. Still, the following remains:
Question. Does existence of SI tiling imply polynomial growth?

In our amenable examples, exists tiling sequence $\{\gamma T^{(n)} : \gamma \in G_n\}$ such that $T^{(n)}$ is connected Følner, so converges locally to Γ. [G. Elek]

Question. Γ amenable transitive graph, $p > p_c(\Gamma)$. Does there exist a connected Følner sequence $F_n \nearrow \Gamma$ s.t.

$$\lim_{n\to\infty} \frac{\text{largest component}(F_n \cap \mathscr{C}_\infty)}{|F_n|} = \lim_{n\to\infty} \frac{|F_n \cap \mathscr{C}_\infty|}{|F_n|} = \theta(p)\ a.s.?$$

This would be the main percolation lemma for renormalization.