Corner, trixor, odd-trixor, quaxor: Linear entropy planar percolation models without and with (conjectured) conformal invariance

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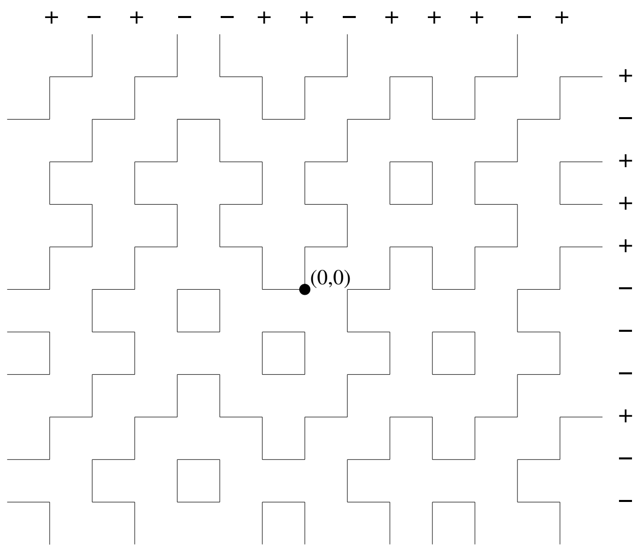
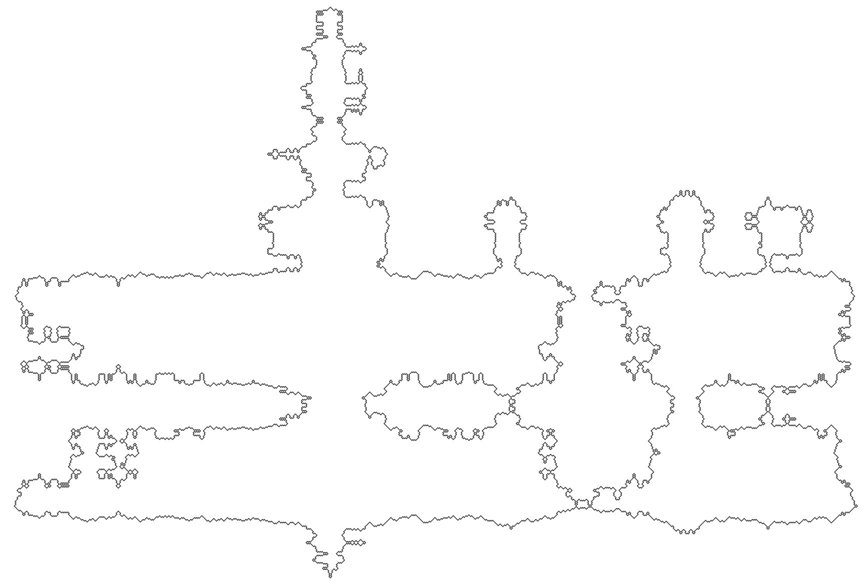
Corner percolation is a strongly dependent 4-vertex model due to Bálint Tóth.





Theorem (P., 2005). Almost surely, all components are finite, and each vertex is surrounded by infinitely many cycles. The exponents

$$\mathbb{P}(\text{the diameter of the cycle of the origin } > n) \approx n^{-\gamma},$$

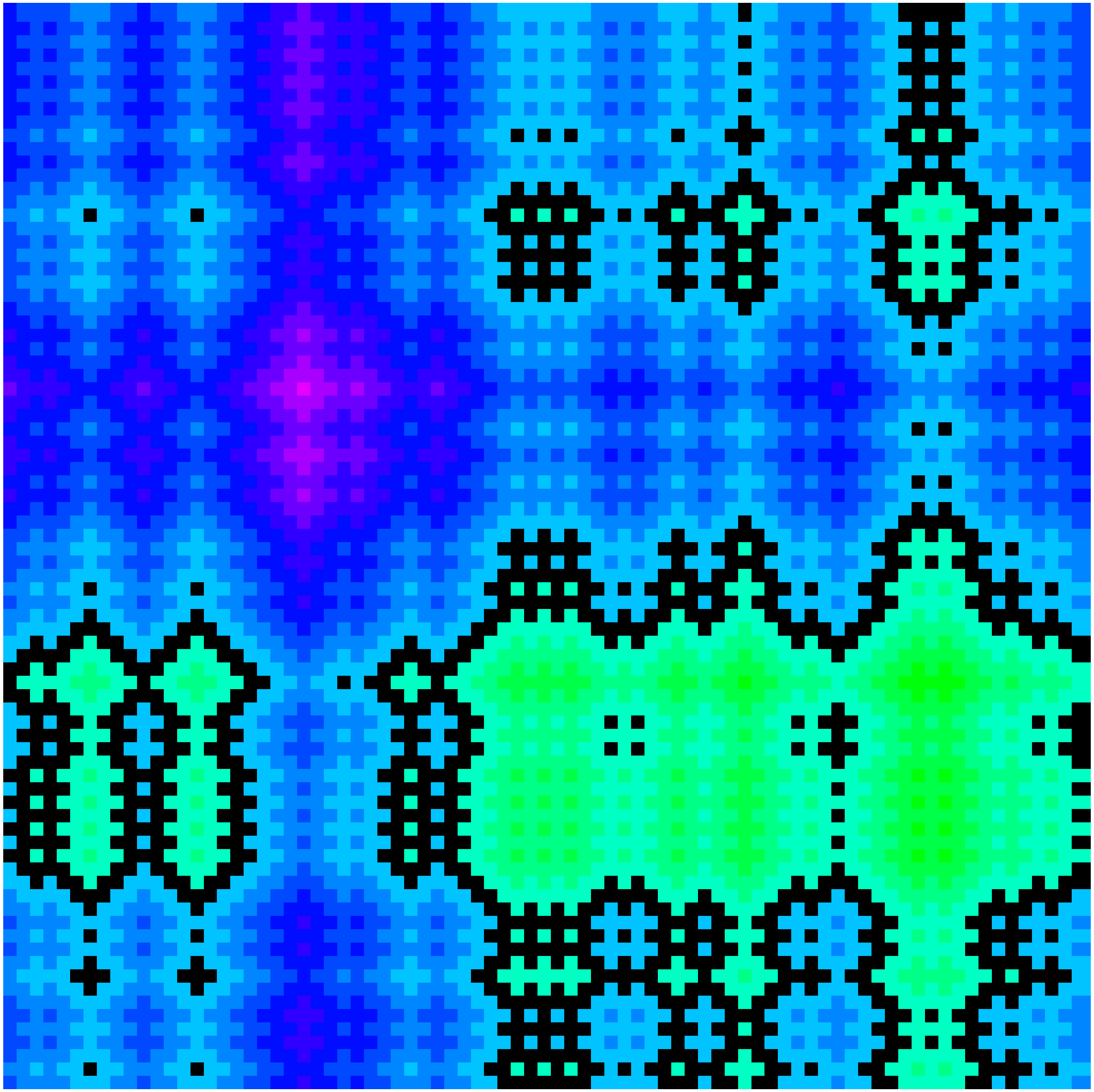
$$\mathbb{E}(\text{length of a typical cycle with diameter } n) \approx n^{\delta}$$

exist, with values $\gamma = (5-\sqrt{17})/4 = 0.219...$ and $\delta = (\sqrt{17}+1)/4 = 1.28....$

$\gamma + \delta = 3/2$ corresponds to having a **height function** in the model, with scaling limit $\mathcal{H}(t,s) = W_t + W'_s$, the Additive Brownian Motion, whose **level sets** have dim$=3/2$.

Colour-coded height function for corner percolation.

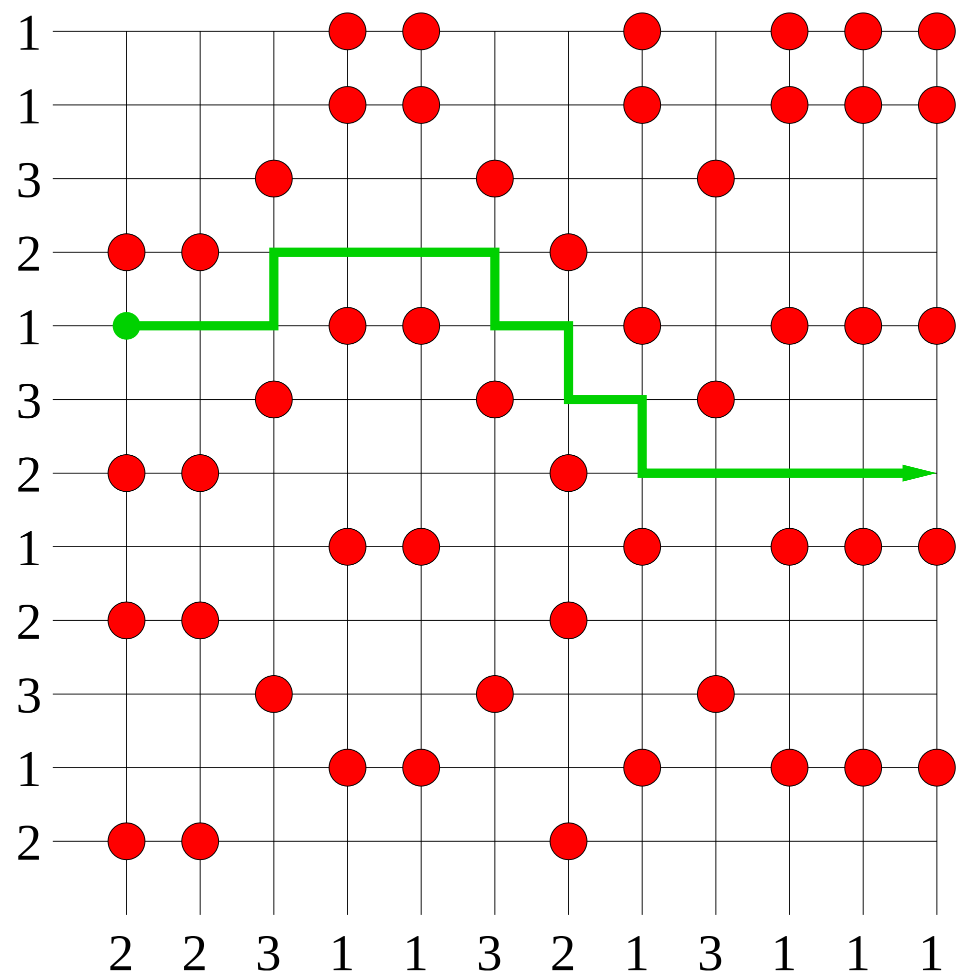
$H(n,m) = \lceil \frac{X_n + Y_m}{2} \rceil$, where $\{X_n\}_{-\infty}^{+\infty}$ and $\{Y_m\}_{-\infty}^{+\infty}$ are two independent SRWs on $\mathbb{Z}$.



A universality class for linear entropy percolation?

Winkler's percolation:
k letters $\{1, 2, \ldots, k\}$ uniformly i.i.d.

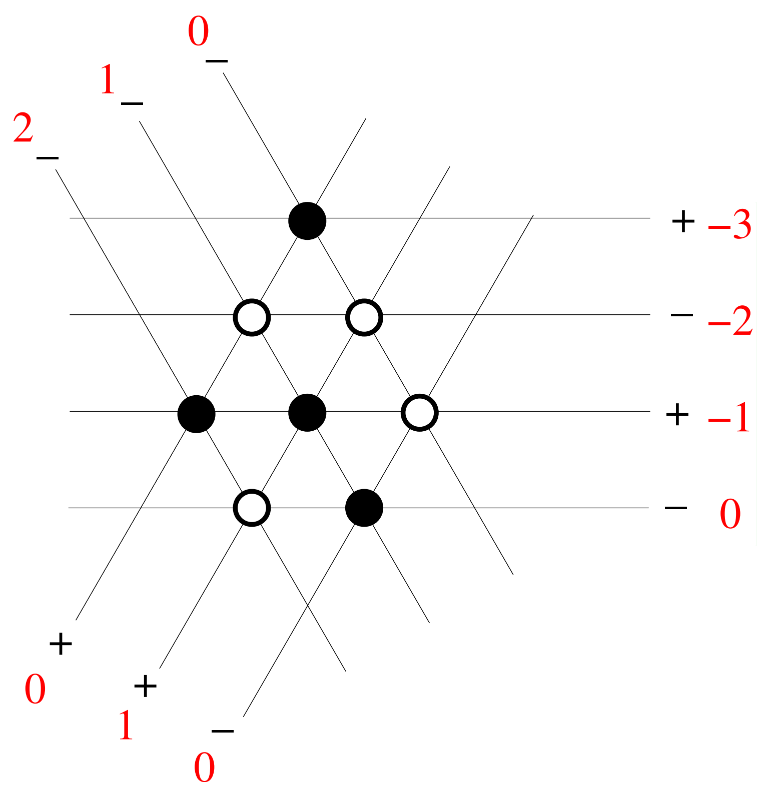
[Winkler, Balister-Bollobás-Stacey, 2000]
Cannot get out for $k \leq 3$,
but yes for $k \geq 4$.



This model, and also Benjamini's **2-wise independent bond percolation** on $\mathbb{Z}^2$, can be reduced to corner — not real universality.

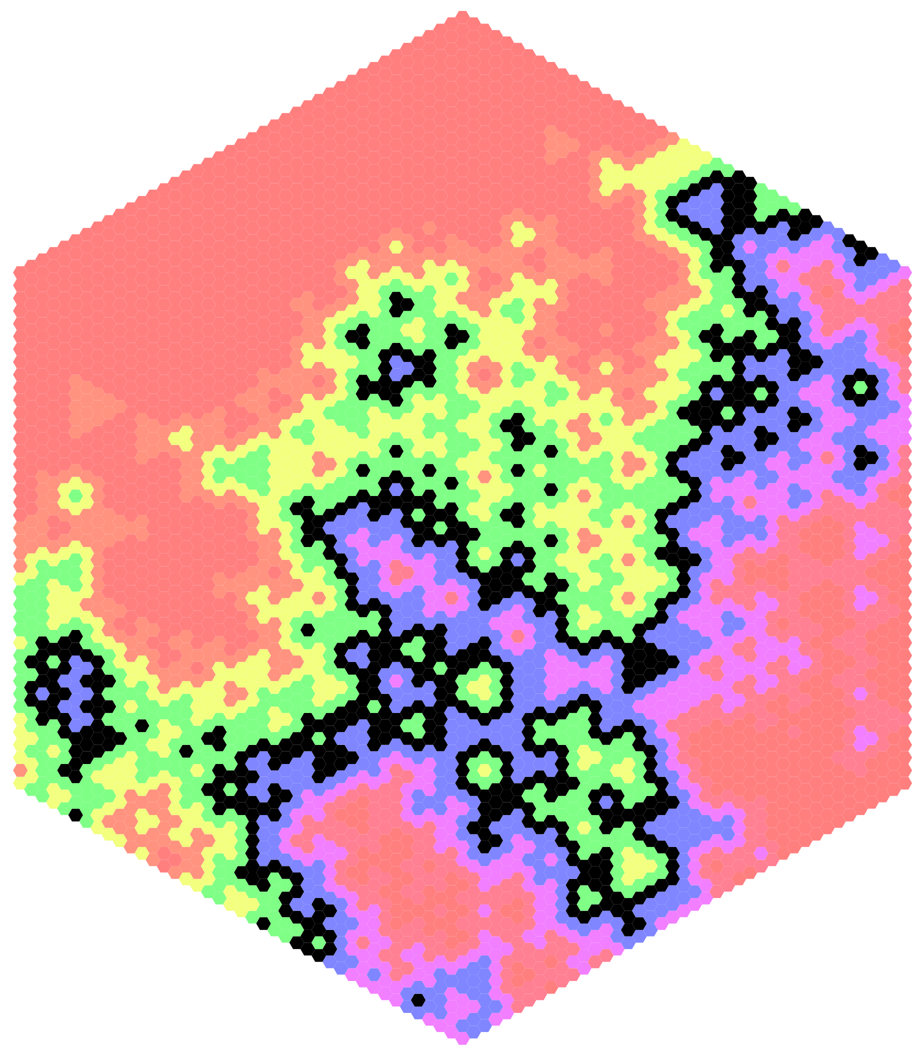
Trixor (even-trixor) [Benjamini, Angel, Schramm]

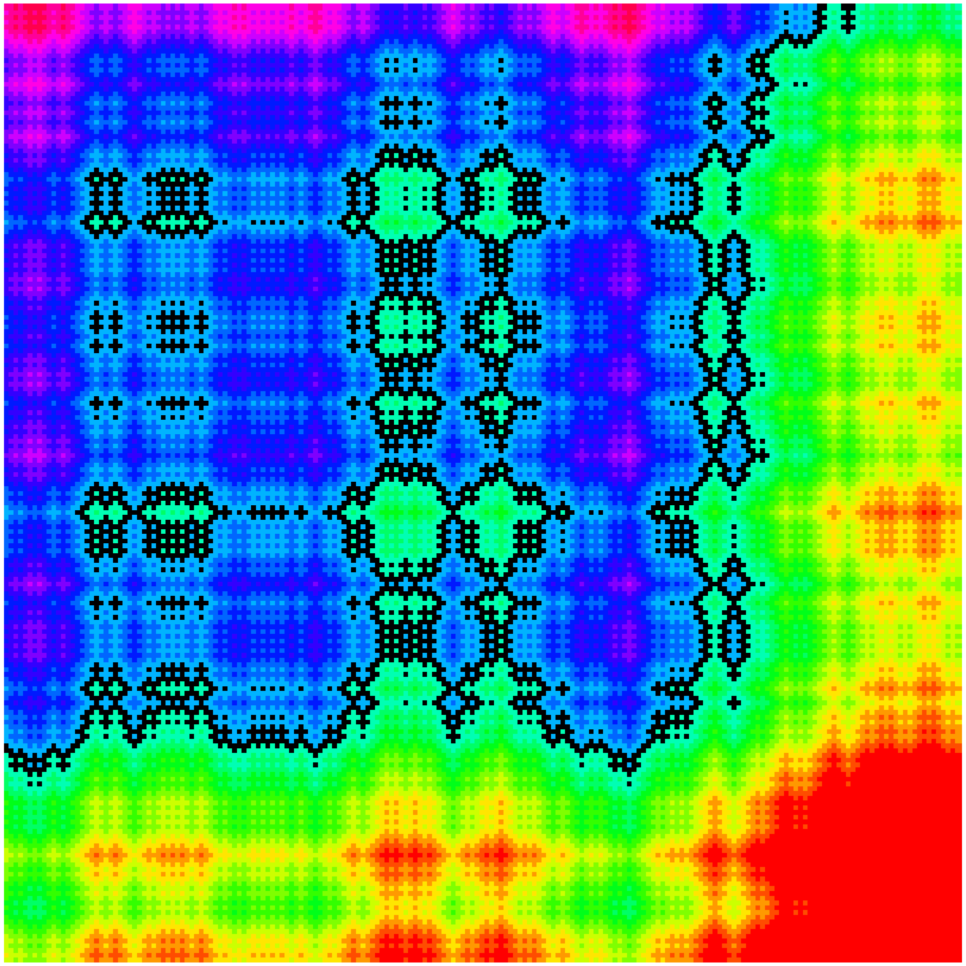
Def 1: Spin of vertex $v = (k, \ell, j)$: $\tau(v) := \xi(k) \cdot \eta(\ell) \cdot \zeta(j)$.

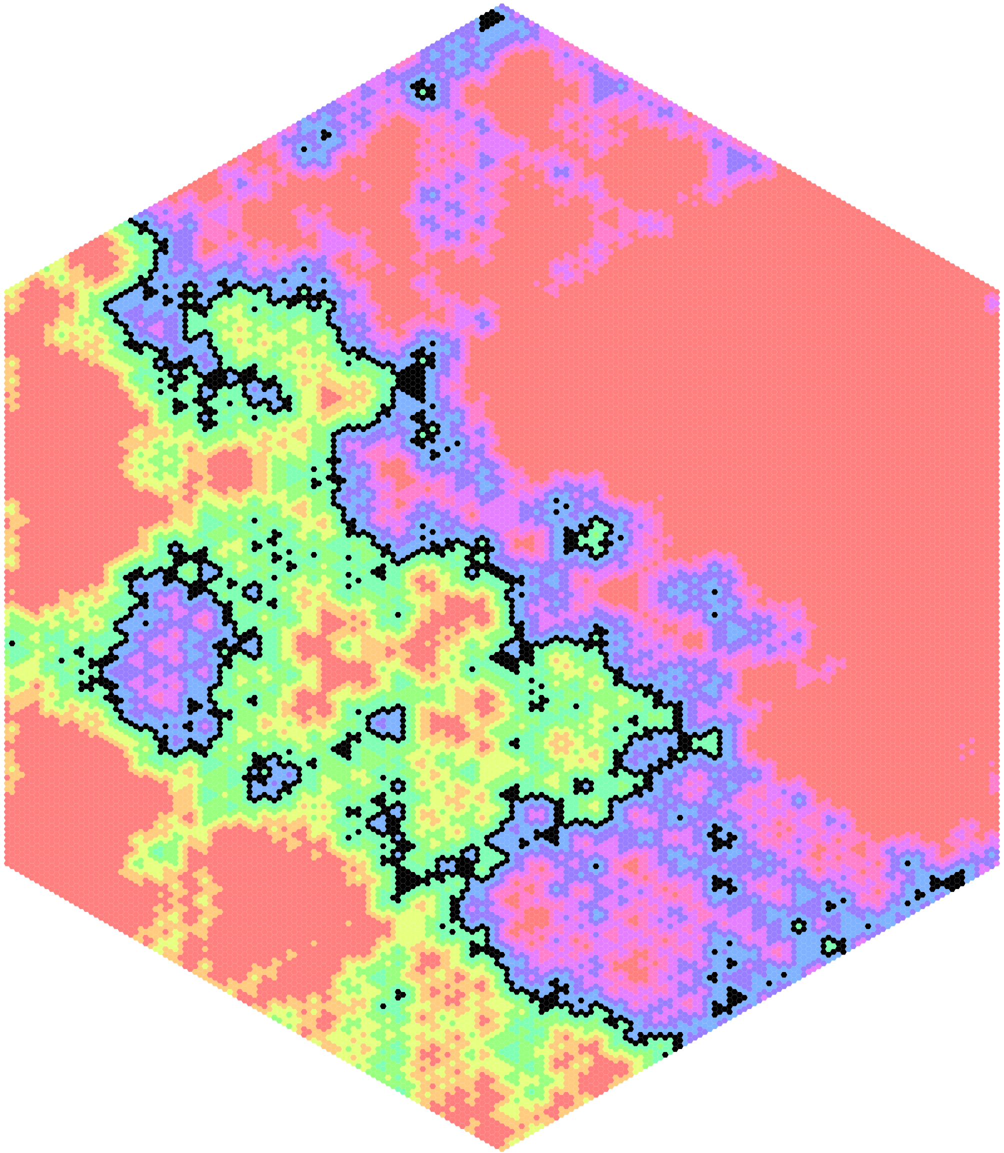


Def 2: Uniform B/W colouring, each vertex having an even number of neighbours of either color.

Def 3: Height function $H(v) := X(k) + Y(\ell) + Z(j)$, with three indept. SRW's.

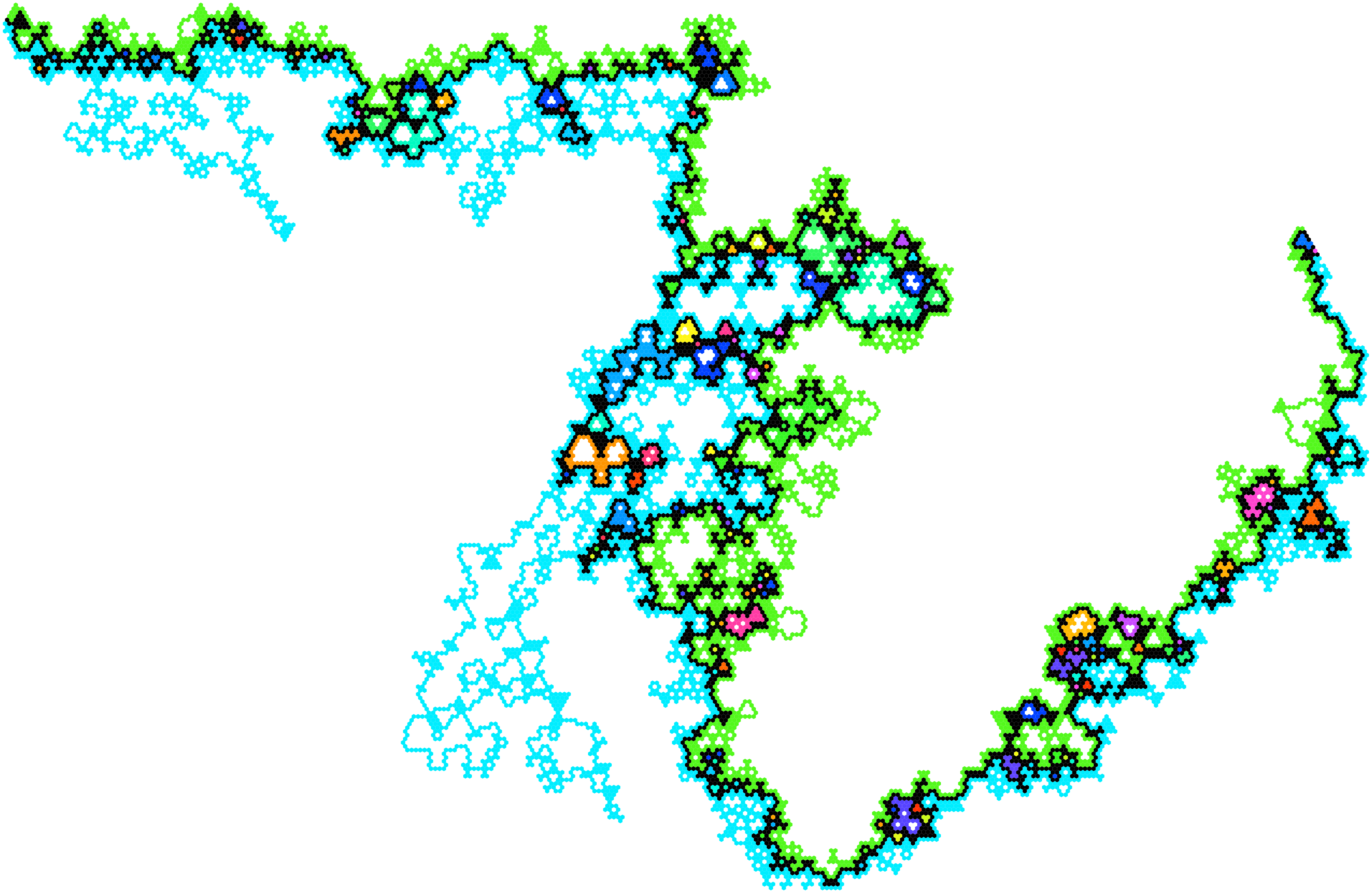




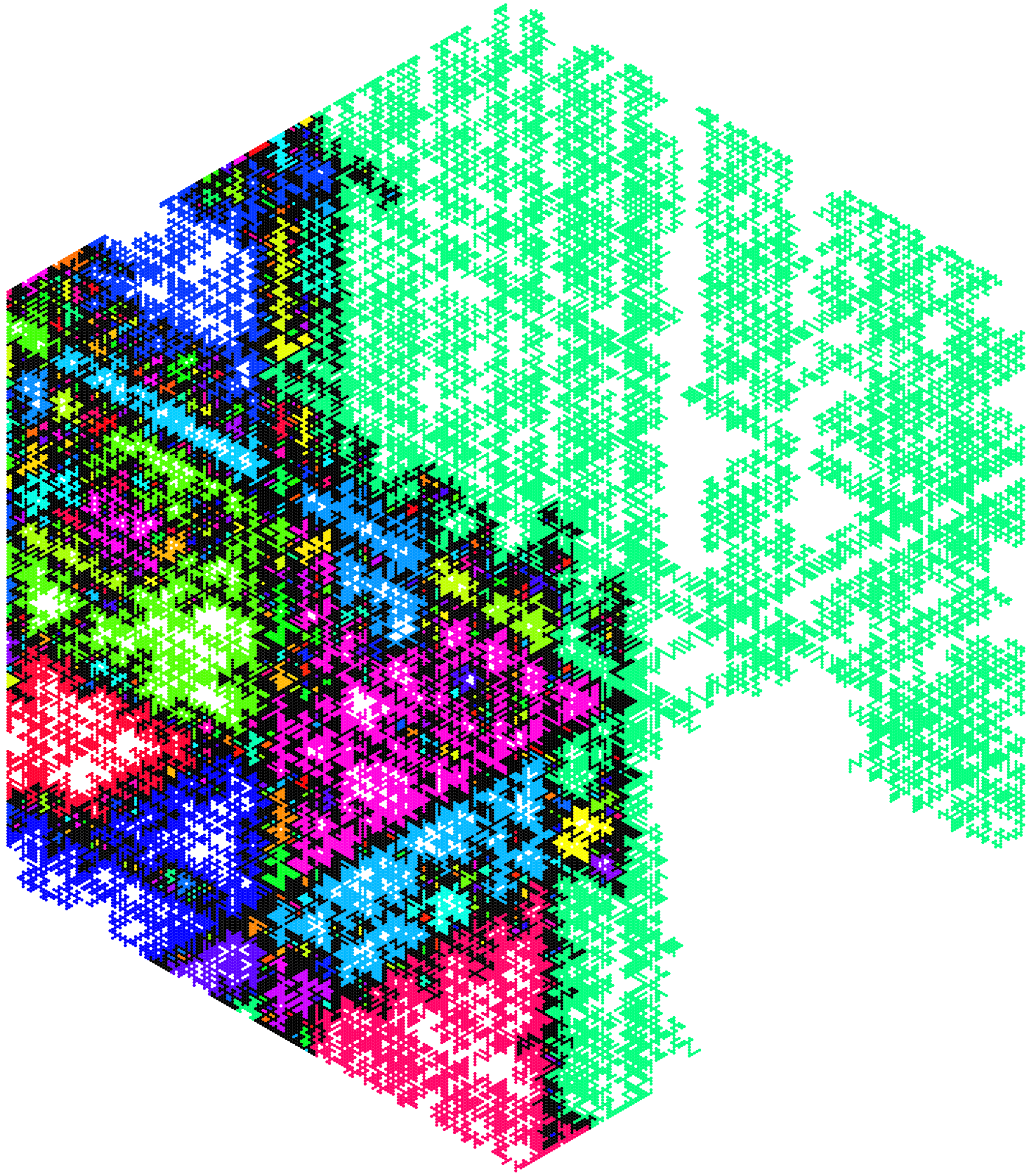


Level curves of "dimension"
$\delta = (\sqrt{17}+1)/4 = 1.28...$ in **corner**,
but seemingly $\delta_3 \in (1.3, 1.35)$ in **trixor**.
Probably $\gamma_3 + \delta_3 = 3/2$, again.

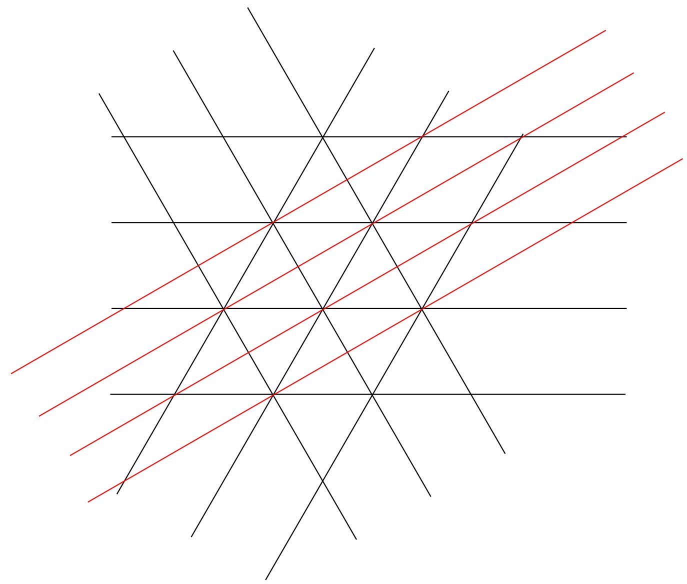
Neighbouring clusters in trixor



Neighbouring clusters in tri-majority



k-xor models

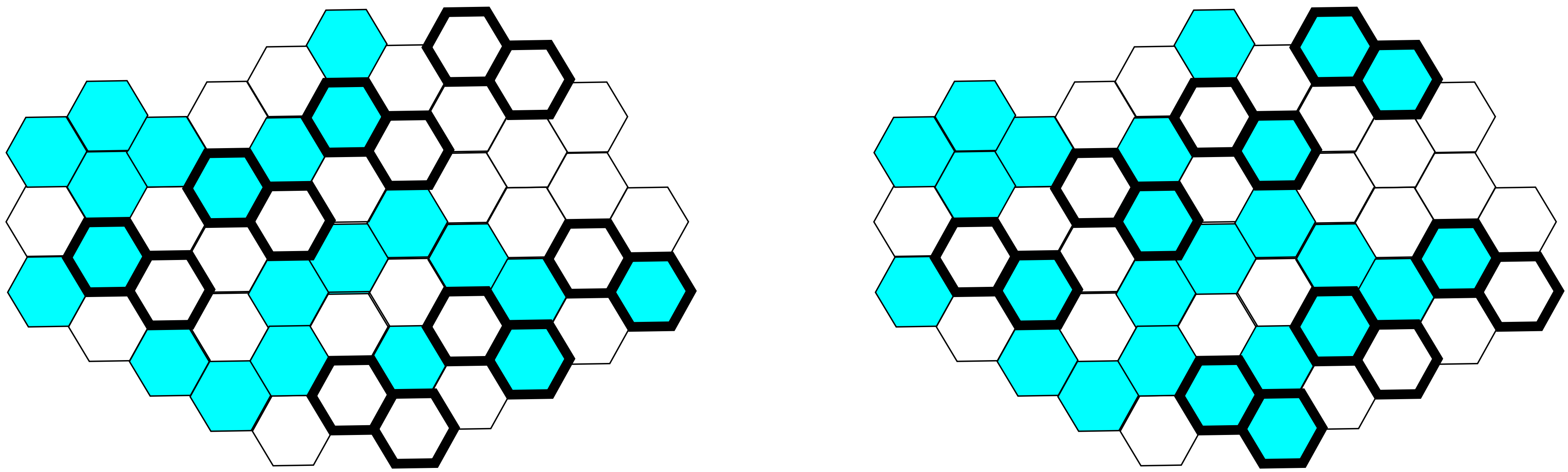


Reasonable conjecture: Exponents γ_k and δ_k, as $k \to \infty$, converge to SLE(6) exponents $5/48$ and $7/4$.

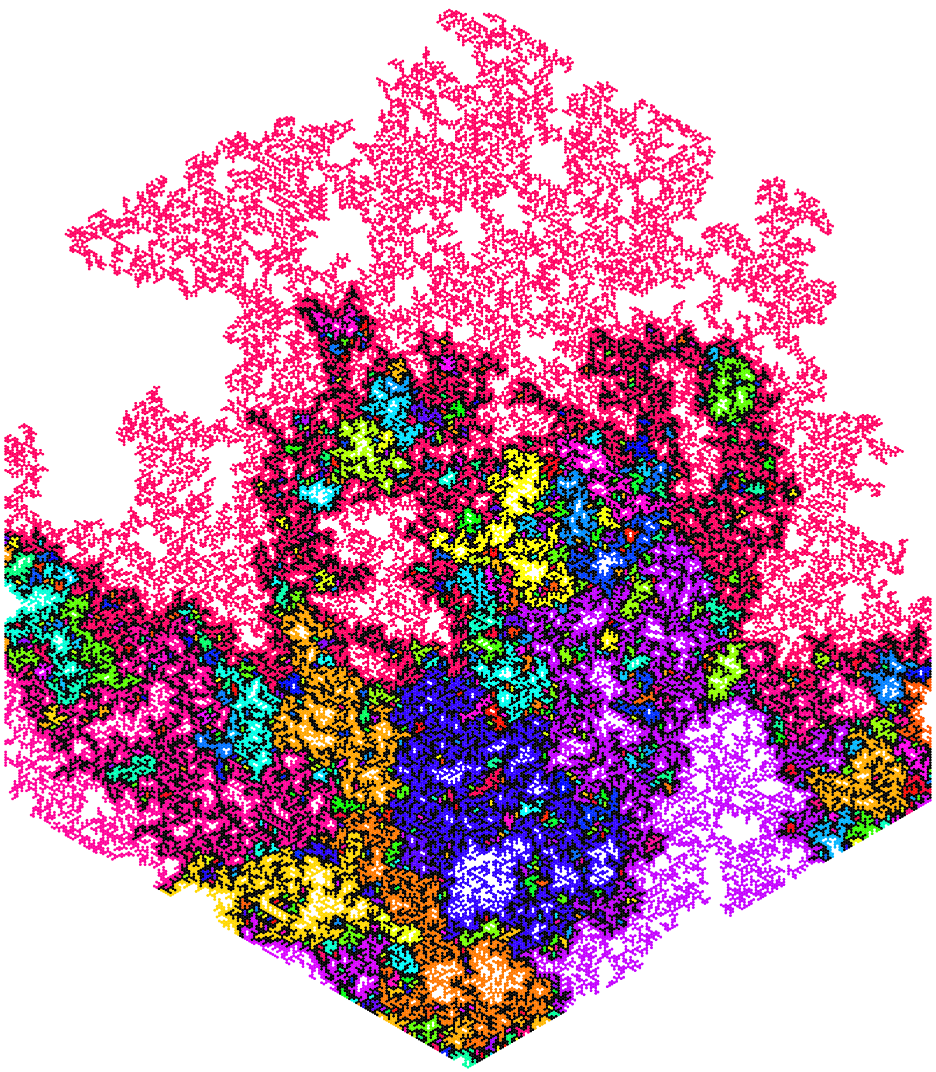
Amazing reality: Already for $k = 4$, quaxor seems to have SLE(6) scaling limit! (With a suitable embedding.)

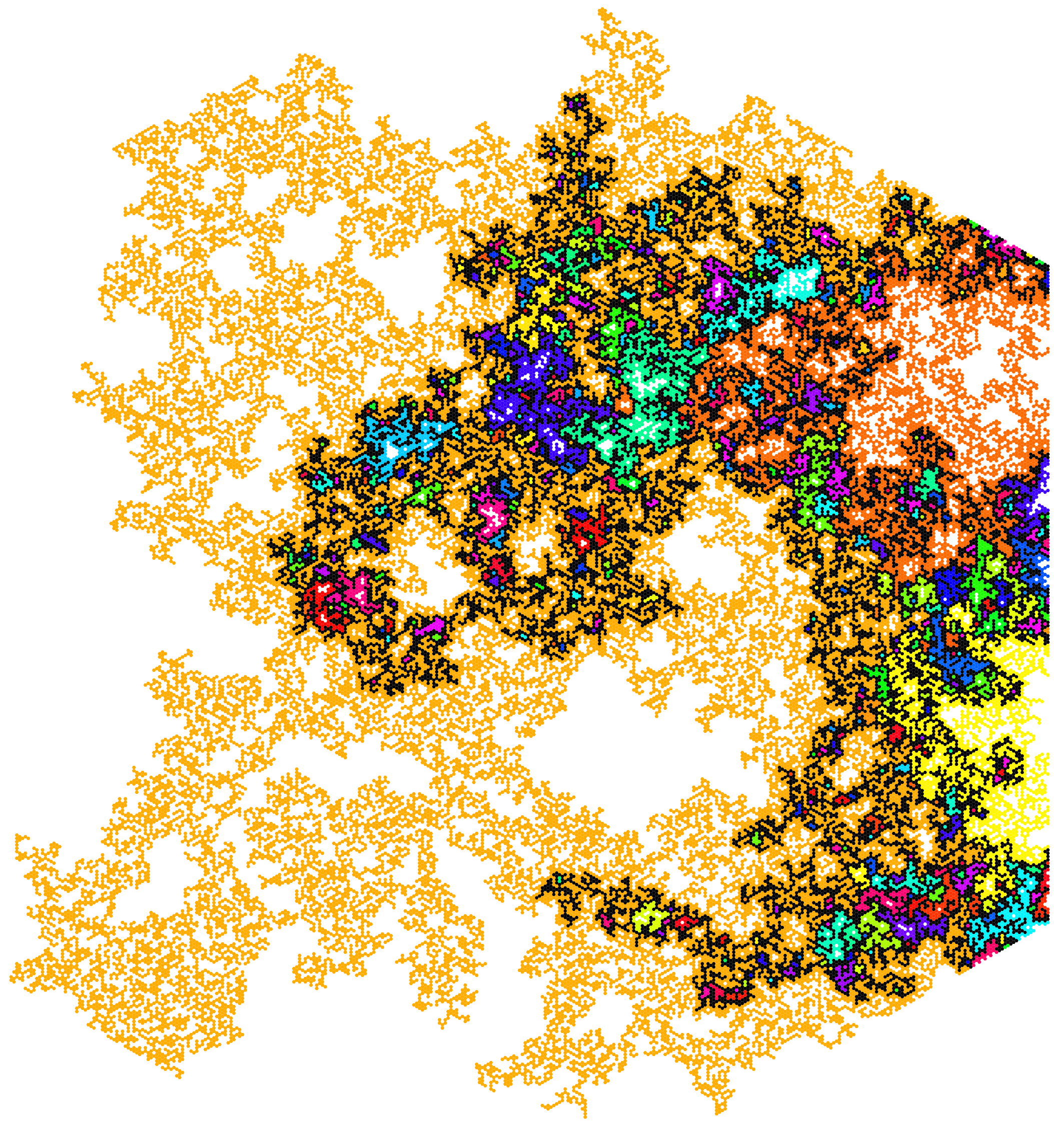
Obvious difficulty: No height function any more.

From even- to odd-trixor, deterministically [Omer Angel]



Neighbouring clusters in odd-trixor and ordinary percolation





Open problems

- Quaxor and odd-trixor: Finite clusters only. Scaling to SLE(6) (=conf. inv.+locality).

- Corner seems noise- and dynamically stable, unlike ordinary percolation [Benjamini-Kalai-Schramm, Schramm-Steif] or 2-dim SRW [Hoffman]. Quaxor and odd-trixor?

- Nodal lines of random Gaussian plane waves? [Bogomolny-Schmit 2002]

- For p-biased corner, $\mathbb{P}_p$(contour of origin is infinite) $= (p - 1/2)^{\beta + o(1)}$? In Bernoulli, $5/48 = \gamma = 3/4 \cdot \beta$, where $3/4$ governs noise-sensitivity [Garban-P-Schramm].

- Compute the exponents γ_3, δ_3 for trixor.

- Interpolation between Additive Brownian Motion and Gaussian Free Field?

- Scaling of large corner cycles to the [Dalang-Mountford] Jordan-curve?

- $\{\xi(n)\} \in \{\pm 1\}^{\mathbb{Z}}$ is hospitable if, for $\{X_j\}_{j=0}^{\infty}$ SRW on $\mathbb{Z}$ and $S_k := \sum_{j=0}^{k-1} \xi(X_j)$, $(X_k, S_k) = (0,0)$ inf. often a.s. Otherwise, hostile. E.g., periodic sequence with same number of ± 1's is hospitable, while $\xi(n) :=$ sgn(n) is hostile. I.i.d. $\mathbb{P}(\xi(n) = 1) = 1/2$ is hostile a.s. [Campanino-Petritis]: Is hospitality invariant under finite permutations?