Group-invariant percolation models — August 3, 2015

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The first two pages have almost no probability content, only coarse geometry and graph theory. Then we start percolation.

▷ **Exercise 1.** Consider the standard hexagonal lattice. Show that if you are given a bound $B < \infty$, and can group the hexagons into countries, each being a connected set of at most B hexagons, then it is not possible to have at least 7 neighbours for each country.

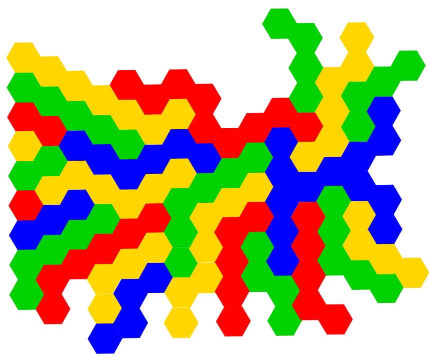

Figure 1: Trying to create at least 7 neighbours for each country.

▷ **Exercise 2.** Recall that being non-amenable means satisfying the strong isoperimetric inequality IP_∞.

(a) Show that a bounded degree tree without leaves is amenable iff there is no bound on the length of "hanging chains", i.e., chains of vertices with degree 2. (Consequently, for trees, $IP_{1+\epsilon}$ implies IP_∞.)

(b) Give an example of a bounded degree tree of exponential volume growth that satisfies no $IP_{1+\epsilon}$, recurrent for the simple random walk on it, and has $p_c = 1$.

▷ **Exercise 3.** Show that a bounded degree graph $G(V, E)$ is nonamenable if and only if it has a wobbling paradoxical decomposition: two injective maps $\alpha, \beta : V \longrightarrow V$ such that $\alpha(V) \sqcup \beta(V) = V$ is a disjoint union, and both maps are at a bounded distance from the identity, or wobbling: $\sup_{x \in V} d(x, \alpha(x)) < \infty$. (Hint: State and use the locally finite infinite bipartite graph version of the Hall marriage theorem, called the Hall-Rado theorem.)

▷ **Exercise 4.** Consider the outer vertex Cheeger constant $h_V := \inf |\partial_V^{\text{out}} S|/|S|$. Show that for any d-regular non-amenable graph G and any $\epsilon > 0$, there exists $K < \infty$ such that we can add edges connecting vertices at distance at most K, such that the new graph G^* will be d^*-regular, no multiple edges, and $h_V(G^*)/d^*$ will be larger than $1 - \epsilon$. (Hint: use the wobbling paradoxical decomposition from the previous exercise. The Mass Transport Principle shows that this proof cannot work in a group-invariant way.)

Question. Is there any group Γ with a sequence of generating sets S_k such that the Cayley graphs $G_k := G(\Gamma, S_k)$, with degree d_k, satisfy $h_V(G_k)/d_k \to 1$?

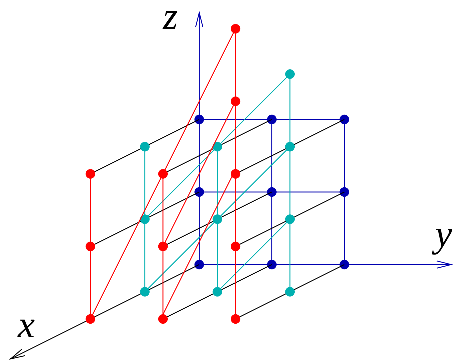

Figure 2: The Cayley graph of the Heisenberg group with generators X, Y, Z.

The **3-dimensional discrete Heisenberg group** is the matrix group

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

If we denote by X, Y, Z the matrices given by the three permutations of the entries $1, 0, 0$ for x, y, z, then $H_3(\mathbb{Z})$ is given by the presentation

$$\langle X, Y, Z \,\big|\, [X, Z] = 1, [Y, Z] = 1, [X, Y] = Z \rangle.$$

▷ **Exercise 5.** Show that the discrete Heisenberg group has 4-dimensional volume growth.

▷ **Exercise 6.**

(a) Show that the Diestel-Leader graph $\mathsf{DL}(k, \ell)$ is amenable iff $k = \ell$.

(b) Show that the Cayley graph of the lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$ with generating set $S = \{\mathsf{R}, \mathsf{Rs}, \mathsf{L}, \mathsf{sL}\}$ is the Diestel-Leader graph $\mathsf{DL}(2, 2)$. How can we obtain $\mathsf{DL}(p, p)$ from $\mathbb{Z}_p \wr \mathbb{Z}$?

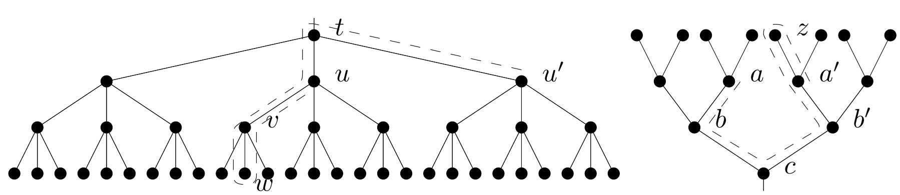

Figure 3: The Diestel-Leader graph $\mathsf{DL}(3, 2)$, with a path: (u, a), (v, b), (w, c), (v, b'), (u, a'), (t, z), (u', a').

▷ **Exercise 7.** Show that amenable transitive graphs are unimodular (that is, they satisfy the Mass Transport Principle).

▷ **Exercise 8.** Why it is hard to construct large expanders:

(a) If $G' \longrightarrow G$ is a covering map of infinite graphs, then the spectral radii satisfy $\rho(G') \leq \rho(G)$, i.e., the larger graph is more non-amenable. In particular, if G is an infinite k-regular graph, then $\rho(G) \geq \rho(\mathbb{T}_k) = \frac{2\sqrt{k-1}}{k}$.

(b) If $G' \longrightarrow G$ is a covering map of finite graphs, then $\lambda_2(G') \geq \lambda_2(G)$, i.e., the larger graph is a worse expander.

▷ **Exercise 9.** In case you have not seen this, prove that $1/3 \leq p_c(\mathbb{Z}^2, \text{bond}) \leq 2/3$. (Hint: for the lower bound, count self-avoiding open paths. For the upper bound, count closed contours in the dual lattice.)

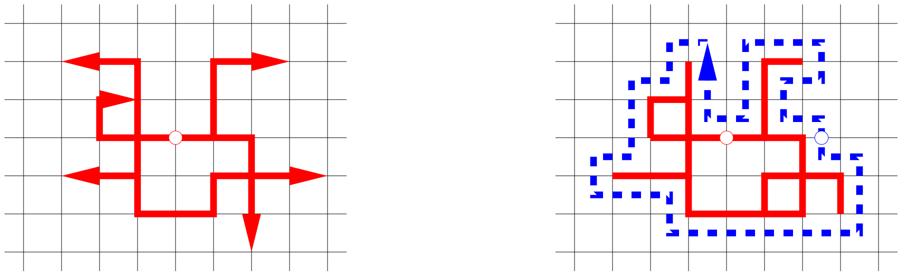

Figure 4: Counting primal self-avoiding paths and dual circuits.

▷ **Exercise 10.** Prove $p_c(\mathbb{T}_d) = 1/(d-1)$, with the First and Second Moment Method applied to Z_n, the number of vertices connected to the root on level n:

(a) For $p < 1/(d-1)$, show that $\mathbf{E}Z_n \to 0$, and conclude that $\theta(p) = 0$.

(b) For $p > 1/(d-1)$, show that there is some $C = C_d < \infty$ such that $\mathbf{E}[\,Z_n^2\,] < C\,(\mathbf{E}Z_n)^2$ for all n. Conclude using Cauchy-Schwarz that $\theta(p) > 0$.

(c) For $p = 1/(d-1)$, note that Z_n is a non-negative martingale. Use the MG Convergence Theorem to show that $Z_n = 0$ eventually, hence $\theta(p) = 0$.

▷ **Exercise 11.** Assume that $\pi : G' \longrightarrow G$ is a topological covering between infinite graphs, or in other words, G is a factor graph of G'. Show that $p_c(G') \leq p_c(G)$.

▷ **Exercise 12.** Prove the Bollobás-Thomason **threshold** theorem: for any sequence monotone events $\mathcal{A} = \mathcal{A}_n$ and any ϵ there is $C_\epsilon < \infty$ such that $\left|p_{1-\epsilon}^{\mathcal{A}}(n) - p_\epsilon^{\mathcal{A}}(n)\right| < C_\epsilon \left(p_\epsilon^{\mathcal{A}}(n) \wedge (1 - p_{1-\epsilon}^{\mathcal{A}}(n))\right)$. (Hint: take many independent copies of low density to get success with good probability at a larger density.)

▷ **Exercise 13** (for Vincent Tassion's course)**.** Prove that having $\phi_p(S) < 1$ for some S implies finite expectation of $\mathscr{C}_o$. (Hint: quite similarly to the proof of exponential decay, write a recursive inequality for $u_n = \max_{x \in B_n(o)} \mathbf{E}_p|\mathscr{C}_x^n|$, where $\mathscr{C}_x^n$ is the cluster of x in the configuration restricted to the ball $B_n(o)$.)

▷ **Exercise 14.**

(a) Give a translation invariant and ergodic percolation on $\mathbb{Z}^2$ with infinitely many ∞ clusters.

(b) Give a translation invariant and ergodic percolation on $\mathbb{Z}^2$ with exactly two ∞ clusters.

▷ **Exercise 15.** As in the lecture, a furcation point of an infinite cluster is a vertex whose removal breaks the cluster into at least 3 infinite components. Show carefully the claim we used in the Burton-Keane theorem: if $\mathscr{C}_\infty$ denotes the union of all the infinite clusters in some percolation on G, and $U \subset V(G)$ is finite, then the size of $\mathscr{C}_\infty \cap \partial_V^{\text{out}} U$ is at least the number of trifurcation points of $\mathscr{C}_\infty$ in U, plus 2.

▷ **Exercise 16.**

(a) In an invariant percolation process on a unimodular transitive graph G, show that almost surely the number of ends of each infinite cluster is 1 or 2 or continuum.

(b) Give an invariant percolation on a non-unimodular transitive graph that has infinite clusters with more than two but finitely many ends.

▷ **Exercise 17.** Consider the graph G with 6 vertices and 7 edges that looks like a figure 8 on a digital display. Consider the uniform measure on the 15 spanning trees of G, denoted by UST, and the uniform measure on the 7 connected subgraphs with 6 edges (one more than a spanning tree), denoted by $\mathsf{UST}+1$. Find an explicit monotone coupling between the two measures (i.e., with $\mathsf{UST} \subset \mathsf{UST}+1$).

Question. Is there such a monotone coupling for every finite graph?