Stochastic Differential Equations

Problem set No 2 — March 6, 2012

▷ **Exercise 1.** Prove directly from the definition of Itô integrals that

(a) $\int_0^t s\,dB_s = tB_t - \int_0^t B_s\,ds$;

(b) $\int_0^t B_s\,dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$;

(c) $\int_0^t B_s^2\,dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s\,ds$. (Hint: $a^2(b-a) = (a-b)^3/3 + (b^3-a^3)/3 - (b-a)^2a$.)

▷ **Exercise 2.**

(a) Show that if $X_t : \Omega \longrightarrow \mathbb{R}^n$ is a martingale w.r.t. a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, then it is also a martingale w.r.t. its *own filtration* $\mathcal{F}_t^X := \sigma\{X_s : s \leq t\}$.

(b) Show that if X_t is a martingale w.r.t. some filtration $\{\mathcal{F}_t\}_{t\geq 0}$, then $\mathbf{E}[\,X_t\,] = \mathbf{E}[\,X_0\,]$ for all $t \geq 0$.

(c) Give an example of a process X_t that satisfies $\mathbf{E}[\,X_t\,] = \mathbf{E}[\,X_0\,]$ for all $t \geq 0$, but is not a martingale w.r.t. its own filtration. (Harder version: $\mathbf{E}[\,X_t \mid X_s\,] = X_s$ for all $t \geq s$.)

▷ **Exercise 3.** Check directly (without Itô integrals) which of the following processes are martingales w.r.t. $\mathcal{F}_t := \sigma\{B_s : s \leq t\}$:

(a) $X_t = B_t + 4t$;

(b) $X_t = B_t^2$;

(c) $X_t = B_t^2 - t$;

(d) $X_t = t^2B_t - 2\int_0^t sB_s\,ds$;

(e) $X_t = B_t^3$;

(f) $X_t = B_t^3 - 3tB_t$;

(g) $X_t = B_1(t)B_2(t)$, where $(B_1(t), B_2(t))$ is a 2-dimensional Brownian motion.

▷ **Exercise 4.** A famous result of Itô (1951) gives the following formula for n times iterated Itô integrals:

$$n! \int_{0\le u_1\le\cdots\le u_n\le t} \ldots \Big(\int\Big(\int dB_{u_1}\Big)\, dB_{u_2}\Big) \ldots dB_{u_n} = t^{n/2} h_n(B_t/\sqrt{t})\,,$$

where h_n is the *Hermite polynomial* of degree n, defined by

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})\,, \qquad n = 0, 1, 2, \ldots$$

(Thus $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$.)

(a) Verify that in each of these n Itô integrals the integrand satisfies the usual requirements.

(b) Verify the formula for $n = 1, 2, 3$.

(c) Deduce part (f) of the previous exercise.

▷ **Exercise 5.** Assume that ξ_t is a stochastic process that satisfies the following:

(i) ξ_t is independent of ξ_s if $t \neq s$;

(ii) it is time-stationary;

(iii) $\mathbf{E}[\,\xi_t\,] = 0$;

(iv) it has continuous paths almost surely.

Show that ξ_t is constant zero almost surely. (Hint: consider $\mathbf{E}\big[\big(\xi_t^{(N)} - \xi_s^{(N)}\big)^2\big]$, where $\xi_t^{(N)} = (-N) \vee (N \wedge \xi_t)$, for $N = 1, 2, \ldots$)

▷ **Exercise 6.** Suppose $f, g \in \mathcal{V}(S, T)$ and that there exist constants C, D such that

$$C + \int_T^S f(t,\omega)\, dB_t(\omega) = D + \int_T^S g(t,\omega)\, dB_t(\omega) \qquad \text{for a.a. } \omega \in \Omega\,.$$

Show that $C = D$ and $f(t,\omega) = g(t,\omega)$ for a.a. $(t,\omega) \in [T, S] \times \Omega$.

▷ **Exercise 7** (Bonus)**.** Let $B : [0,1] \longrightarrow \mathbb{R}$ be one-dimensional Brownian motion with $B_0 = 0$ and let $f : [0,1] \longrightarrow \mathbb{R}$ be a deterministic Hölder(ϵ) function for some $\epsilon > 0$, i.e., there exists $C < \infty$ such that $f(x) - f(y)| < C|x-y|^\epsilon$ for all $x, y \in [0,1]$. Show that the Riemann sums

$$\sum_{i=0}^{n-1} f(i/n)\big(B_{(i+1)/n} - B_{i/n}\big)$$

converge almost surely (not only in L^2) to the Itô integral $\int_0^1 f(t)\, dB_t$, as $n \to \infty$.