Percolation in the plane and random walks on expanders, or how long do we have to wait for the exceptional?

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A basic question

Pairwise independence is weaker than full independence. E.g., can have $X_1, X_2, \ldots, X_n$ on $(\Omega, \mathbf{P})$, with values in some V, and a subset $A \subset V$, s.t.

- the events $\{X_i \in A\}$ are pairwise independent, $\mathbf{P}[\,X_i \in A\,] = 1/2$,
- but the joint probability is large: $\mathbf{P}[\,X_1, \ldots, X_n \in A\,] = 1/n$.

Namely, let σ_i, $i = 1, \ldots, k$ be independent uniform ± 1 bits, $n = 2^k$, and $x_S := \prod_{i \in S} \sigma_i \in \{-1, 1\}$, for all $S \subseteq [k]$. Then x_S and x_T are independent for $S \neq T$, but $\mathbf{P}\big[\,x_S = 1 \;\forall\, S \subseteq [k]\,\big] = \mathbf{P}\big[\,\sigma_i = 1 \;\forall\, i \in [k]\,\big] = 2^{-k} = 1/n$.

Can this happen for stationary reversible Markov processes? I.e.,

- a fast pairwise decorrelation $\mathbf{P}[\,X_0, X_t \in A\,] - \mathbf{P}[\,X_0 \in A\,]^2$,
- but a fat exit tail $\mathbf{P}[\,X_s \in A \text{ for all } 0 \leqslant s \leqslant t\,]$?

Say, can the first one be exponential but the second one only polynomial?

A famous example: random walk on expanders

A bounded degree finite graph $G(V,E)$ is an expander if $|\partial S|/|S| \geqslant c > 0$ for all $|S| \leqslant |V|/2$. **With random walks**: $\mathbf{P}[\, X_1 \notin S \mid X_0 \sim \mathrm{Unif}(S)\,] > c$.

With functional analysis: Markov operator $Pf(x) := \mathbf{E}[\, f(X_1) \mid X_0 = x\,]$, self-adjoint on $L^2(V,\pi)$, where π is the stationary measure. Then $(P\mathbf{1}_S, \mathbf{1}_S) \leqslant (1-c)(\mathbf{1}_S, \mathbf{1}_S)$. More generally, if $\pi(\mathrm{supp} f) \leqslant 1-\epsilon$, then

$$(Pf,f) \leqslant (1-\delta_1)(f,f) \quad \text{and} \quad (Pf,Pf) \leqslant (1-\delta_2)(f,f)\,.$$

Equivalently, spectral gap $g := 1-\lambda_2 > 0$, where

$$\lambda_2 := \sup_{f \perp \mathbf{1}} \frac{(Pf,f)}{(f,f)} = \sup_{f \perp \mathbf{1}} \frac{(Pf,Pf)^{1/2}}{(f,f)^{1/2}}\,.$$

Absolute spectral gap $g_* := 1 - \sup\big\{|\lambda| : \lambda \in \mathrm{Spec}(P) \setminus \{1\}\big\}$.

For any $\mathbf{E}_\pi[f] = 0$, we have $\mathbf{E}[\, f(X_0) f(X_t)\,] \leqslant (1-g_*)^t\, \mathbf{E}_\pi[f^2]$.

Theorem (Ajtai, Komlós & Szemerédi 1987). Let $(X_i)_{i=0}^{\infty}$ be a stationary reversible chain with P and π and $\lambda_2 < 1$, and let $\pi(A) \leqslant \beta < 1$. Then there exists $\gamma(\lambda_2, \beta) > 0$ with

$$\mathbf{P}\big[\, X_i \in A \text{ for all } i = 0, 1, \ldots, t \,\big] \leqslant C(1-\gamma)^t \,.$$

Proof. Consider the projection $Q : f \mapsto f\mathbf{1}_A$. Then,

$$\begin{aligned}
\mathbf{P}\big[\, X_i \in A \text{ for } i = 0, 1, \ldots, 2t+1 \,\big] &= \big(Q(PQ)^{2t+1}\mathbf{1}, \mathbf{1}\big) \\
&= \big(P(QP)^t Q\mathbf{1}, (QP)^t Q\mathbf{1}\big)\,, \text{ by self-adjointness of } P \text{ and } Q \\
&\leqslant (1-\delta_1)\,\big((QP)^t Q\mathbf{1}, (QP)^t Q\mathbf{1}\big)\,, \text{ by } \pi(\mathrm{supp}(Qg)) \leqslant \beta \\
&\leqslant (1-\delta_1)\,\big(P(QP)^{t-1} Q\mathbf{1}, P(QP)^{t-1} Q\mathbf{1}\big)\,, \text{ by } Q \text{ being a projection} \\
&\leqslant (1-\delta_1)\,(1-\delta_2)\,\big((QP)^{t-1} Q\mathbf{1}, (QP)^{t-1} Q\mathbf{1}\big)\,, \text{ by } \pi(\mathrm{supp}(Qg)) \leqslant \beta \\
&\leqslant (1-\delta_1)\,(1-\delta_2)^t\,\big(Q\mathbf{1}, Q\mathbf{1}\big)\,, \text{ by iterating previous step} \\
&\leqslant (1-\delta_1)\,(1-\delta_2)^t\,\beta\,,
\end{aligned}$$

and done for odd times. For even times, use monotonicity in t. ∎

A general result

Stationary Markov process ω_t, operator T_t. Let $\pi(\mathcal{C}) = \mathbf{P}\big[\,\omega_0 \in \mathcal{C}\,\big] = p$, and let $f = \mathbf{1}_{\mathcal{C}}$. The decay of correlations of f can be quantified by

$$\mathbf{P}\big[\,\omega_0, \omega_t \in \mathcal{C}\,\big] - \mathbf{P}\big[\,\omega_0 \in \mathcal{C}\,\big]^2 = (f, T_t f) - (\mathbf{E} f)^2 \leqslant d(t)\,\mathrm{Var}[f]$$

or $$\mathrm{Var}[T_t f] = (T_t f, T_t f) - (\mathbf{E} f)^2 \leqslant d(2t)\,\mathrm{Var}[f]\,.$$

Same for reversible Markov processes.

Theorem (Hammond, Mossel & P 2011). Under the second condition,

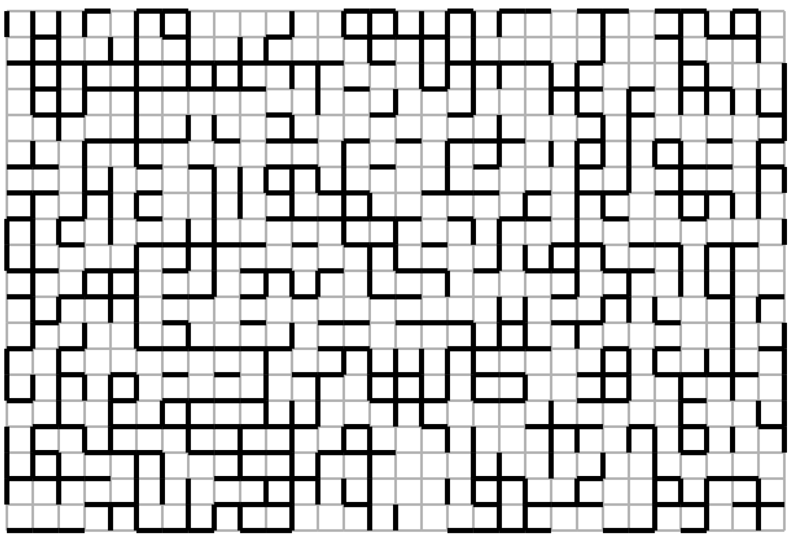
$$\mathbf{P}\Big[\,\omega_s \in \mathcal{C}\ \forall s \in [0,t]\,\Big] \leqslant \begin{cases} t^{-\alpha+o(1)} & \text{if } d(t) = t^{-\alpha+o(1)}, \\ \exp\big(-t^{\frac{\alpha}{1+\alpha}+o(1)}\big) & \text{if } d(t) = \exp(-t^{\alpha+o(1)})\,. \end{cases}$$

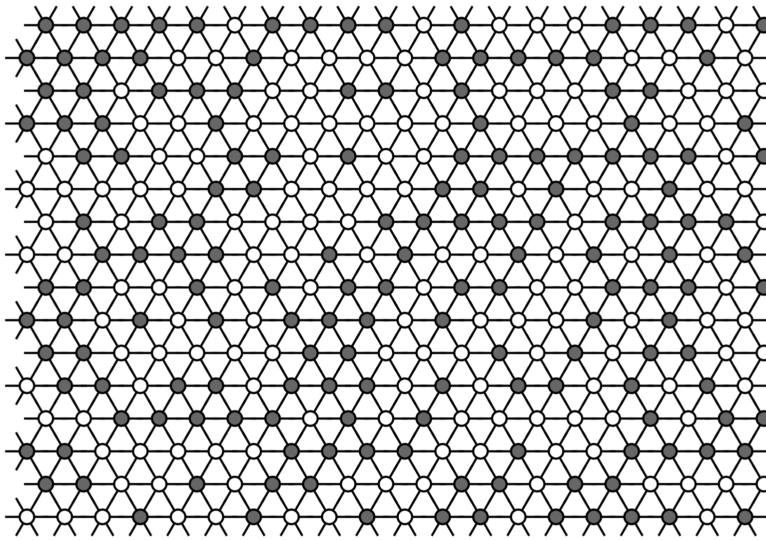
Sharp in the regime of polynomial decay. Open in the exponential case.

Proof is not hard, maybe later. But now the motivation.

Bernoulli(p) bond and site percolation

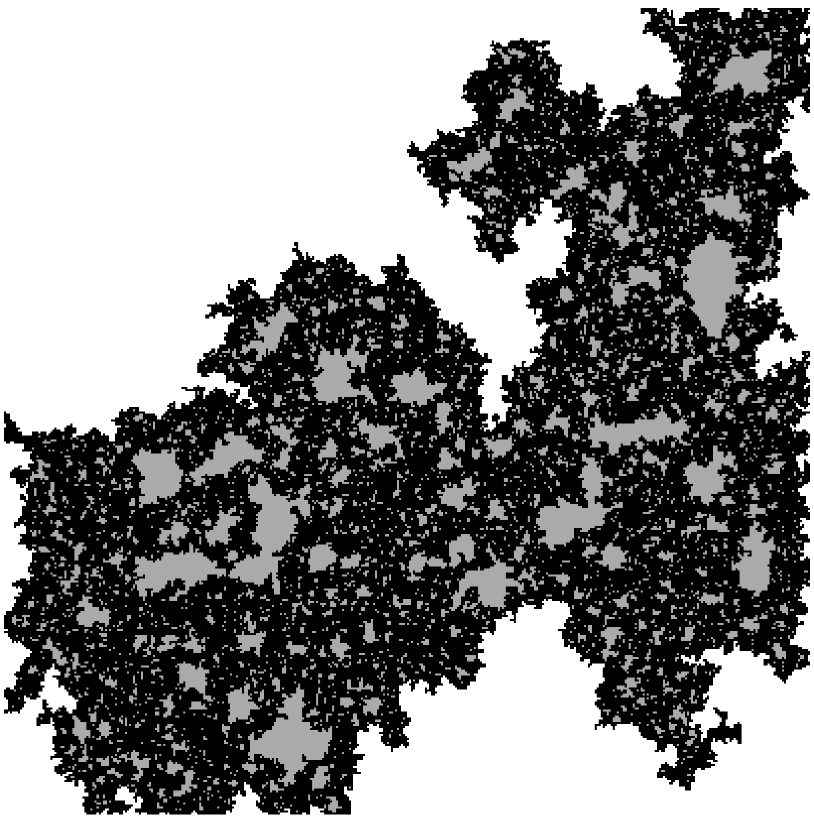
Given an (infinite) graph $G = (V, E)$ and $p \in [0, 1]$. Each site (or bond) is chosen open with probability p, closed with $1 - p$, independently of each other. Consider the open connected clusters. $\theta(p) := \mathbf{P}_p[0 \longleftrightarrow \infty]$.

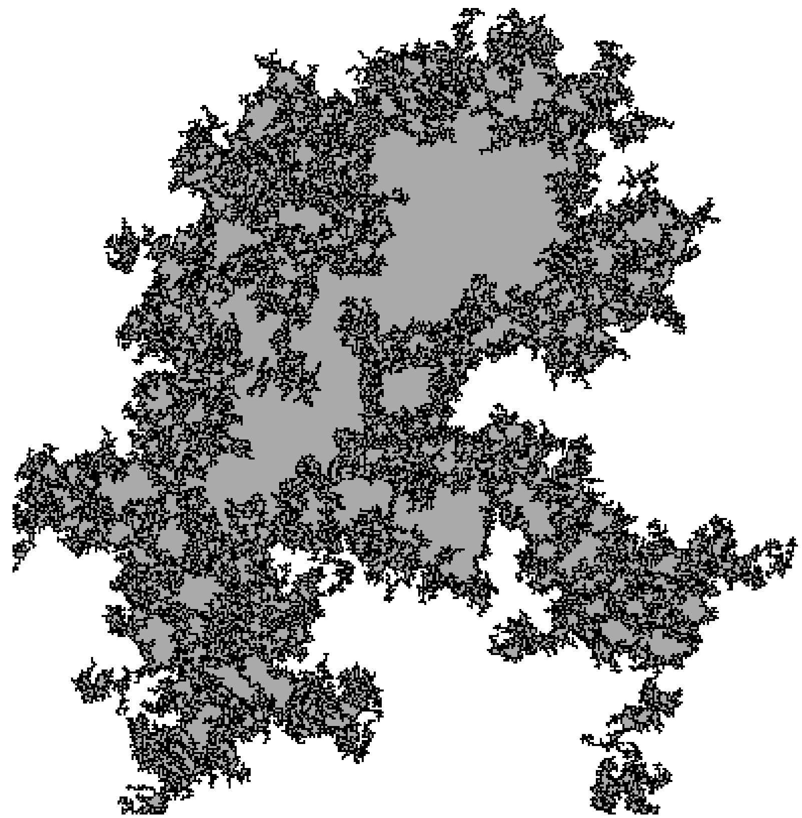




Theorem (Harris 1960 and Kesten 1980).
$p_c(\mathbb{Z}^2, \text{bond}) = p_c(\Delta, \text{site}) = 1/2$, and $\theta(1/2) = 0$.
For $p > 1/2$, there is a.s. one infinite cluster.

Bernoulli$(1/2)$ bond and site percolation



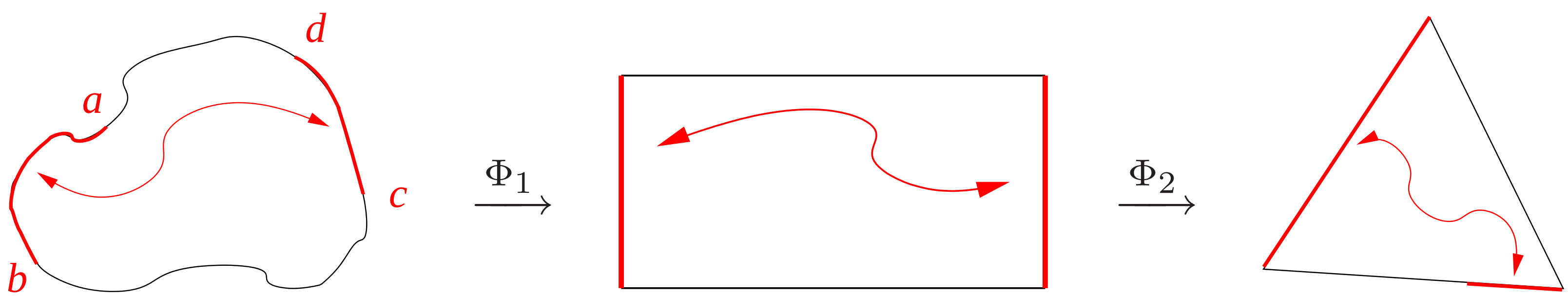


Conformal invariance on Δ

Theorem (Smirnov 2001). For $p = 1/2$ site percolation on Δ_η, and $\mathcal{Q} \subset \mathbb{C}$ a piecewise smooth quad (simply connected domain with four boundary points $\{a, b, c, d\}$),

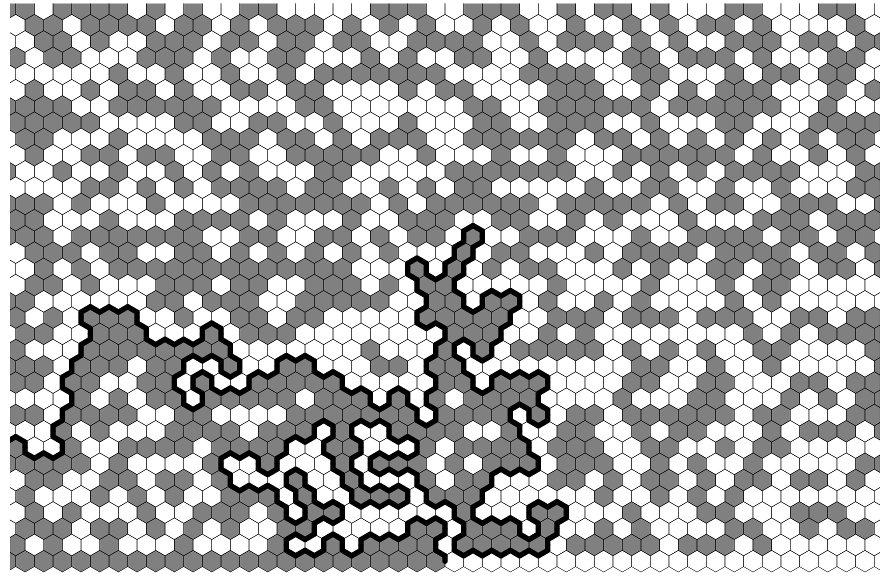
$$\lim_{\eta \to 0} \mathbf{P}\Big[ab \longleftrightarrow cd \text{ inside } \mathcal{Q}\text{, in percolation on } \Delta_\eta \Big]$$

exists, is strictly between 0 and 1, and conformally invariant.



Calls for a continuum scaling limit, encoding macroscopic connectivity, cluster boundaries, etc. Aizenman ‘95, Schramm ‘00, Camia-Newman ‘06, Sheffield ‘09, Schramm-Smirnov ‘10. In physics, correlation functions.

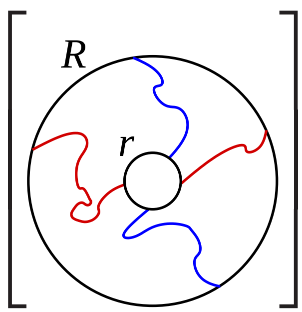
SLE_6 exponents



Given the conformal invariance, the exploration path converges to the Stochastic Loewner Evolution with $\kappa = 6$ (Schramm 2000).

Using the SLE_6 curve, several critical exponents can be computed (Lawler-Schramm-Werner, Smirnov-Werner 2001, plus Kesten 1987), e.g.:

$$\alpha_4(r, R) := \mathbf{P}\left[\ \right] = (r/R)^{5/4+o(1)},$$



$\alpha_1(r, R) = (r/R)^{5/48+o(1)}$, and $\theta(p_c + \epsilon) := \mathbf{P}_{p_c+\epsilon}[0 \longleftrightarrow \infty] = \epsilon^{5/36+o(1)}$.

Dynamical percolation

Triangular lattice Δ_η with mesh η, each site is resampled according to an independent exponential clock.

Question 1: How much time does it take to change macroscopic crossing events? (How noise sensitive are the crossing events? Complexity theory: primitive Boolean functions are quite stable.)

Question 2: On an infinite lattice, are there random times with exceptional behavior, e.g., an infinite cluster? (Dynamical sensitivity?)

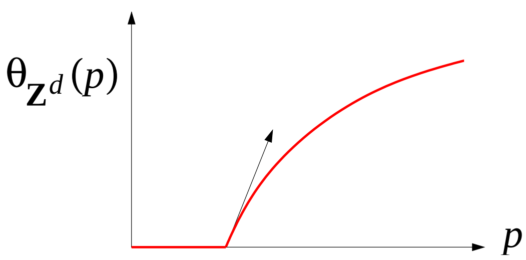
Toy example: Brownian motion on the circle does sometimes hit a fixed point. The set of these exceptional times is a random Cantor set of Lebesgue measure zero (because of Fubini) and Hausdorff-dimension $1/2$.

Question 3: With a well-chosen rate $r(\eta)$ for the clocks (probably coming from Question 1), is there a scaling limit of the process, giving a Markov process on continuum configurations?

Dynamical percolation results

Theorem (Häggström, Peres & Steif 1997).

- No exceptional times when $p \neq p_c$.
- No exceptional times when $p = p_c$ for bond percolation on $\mathbb{Z}^d$, $d \geqslant 19$.



The latter is essentially due to Hara-Slade '90 on the off-critical exponent $\beta = 1$: even switching asymmetrically, $\mathbf{E}\big[$number of ϵ-subintervals of $[0,1]$ with exceptional times$\big] = O(1)$. But the exceptional set is closed without isolated points.

Recall that, on the triangular lattice Δ, we have $\beta = 5/36 = \frac{5/48}{2-5/4} = \frac{\xi_1}{2-\xi_4}$.

Theorem (Garban, P & Schramm 2008).

- There are exceptional times also on $\mathbb{Z}^2$.
- On the triangular grid they have Hausdorff dimension $31/36$.

Lower bound needs decay of correlations in $\mathscr{E}_R = \{t : 0 \overset{\omega_t}{\longleftrightarrow} R\}$:

1. $\mathbf{E}\big[\, f_{\mathcal{Q},\eta}(\omega_0)\, f_{\mathcal{Q},\eta}(\omega_{t\eta^{3/4+o(1)}})\,\big] - \mathbf{E}\big[\, f_{\mathcal{Q},\eta}\,\big]^2 \asymp_{\mathcal{Q}} t^{-2/3}$ as $t \to \infty$, uniformly in mesh η, for the indicator of left-right crossing in the quad $\mathcal{Q}$.

2. $\mathbf{E}\big[\, f_R(\omega_0)\, f_R(\omega_t)\,\big] / \mathbf{E}\big[\, f_R(\omega)\,\big]^2 \asymp t^{-(4/3)\,\xi_1 + o(1)}$, as $t \to 0$, for the indicator of the one-arm event to radius R.

Now, by the Mass Distribution Principle for the measure $\mu_R[a,b] = \int_a^b \mathbf{1}\{0 \overset{\omega_t}{\longleftrightarrow} R\} / \mathbf{P}[\,0 \longleftrightarrow R\,]\, dt$ and some compactness, if

$$\sup_R \int_0^1 \int_0^1 \frac{\mathbf{E}\big[\, f_R(\omega_t)\, f_R(\omega_s)\,\big]}{\mathbf{E}\big[\, f_R(\omega)\,\big]^2\, |t-s|^\gamma}\, dt\, ds < \infty\,,$$

then $\dim(\mathscr{E}) \geqslant \gamma$ a.s. Hence $\dim(\mathscr{E}) \geqslant 1 - \frac{4}{3}\xi_1$.

For $\mathbb{Z}^2$, we have "$\xi_1 + \xi_4 < \xi_5 = 2$", hence $1 - \frac{\xi_1}{2-\xi_4} > 0$, so there exist exceptional times.

Two natural questions on the exceptional set

How do exceptional infinite clusters look like? The first one? A typical one?

There is an "infinite critical cluster" in the static world, Kesten's Incipient Infinite Cluster measure (1986): for $H \subset \Delta$ and ω^H configuration in H, the limit $\text{IIC}(\omega^H) = \lim_{R\to\infty} \mathbf{P}[\,\omega^H \,|\, 0 \leftrightarrow R\,]$ exists.

All other natural definitions give the same measure (Járai 2003).

What is the hitting time tail $\mathbf{P}\big[\,\mathscr{E} \cap [0,t] = \emptyset\,\big]$?

To answer the first question, we needed to answer the second one:

Theorem (Hammond, Mossel & P. 2011). The hitting time tail is exponentially small.

Theorem (Hammond, P. & Schramm 2012). The configuration at a "typical" exceptional time has the law of IIC, but the First Exceptional Time Infinite Cluster (FETIC) is thinner.

Local time measure for exceptional times

$$\overline{M}_r(\omega_s) := \frac{\mathbf{1}\{0 \leftrightarrow r\}}{\mathbf{P}[\,0 \leftrightarrow r\,]}, \quad \overline{\mu}_r[a,b] := \int_a^b \overline{M}_r(\omega_s)\,ds, \quad \overline{\mu}[a,b] := \lim_{r\to\infty} \overline{\mu}_r[a,b].$$

This $\overline{M}_r(\omega)$ is a martingale w.r.t. the filtration $\overline{\mathscr{F}}_r$ of the percolation space generated by the variables $\mathbf{1}\{0 \leftrightarrow r\}$. Moreover, $\mathbf{E}\overline{\mu}_r[a,b] = b-a$, and, by the correlation decay, $\sup_r \mathbf{E}\big[\,\overline{\mu}_r[a,b]^2\,\big] < C_1$. So $\lim_r$ exists.

$$M_H(\omega) := \lim_{R\to\infty} \frac{\mathbf{P}[\,0 \leftrightarrow R \mid \omega^H\,]}{\mathbf{P}[\,0 \leftrightarrow R\,]} = \lim_{R\to\infty} \frac{\mathbf{P}[\,\omega^H \mid 0 \leftrightarrow R\,]}{\mathbf{P}[\,\omega^H\,]} = \frac{\mathsf{IIC}(\omega^H)}{\mathbf{P}[\,\omega^H\,]}.$$

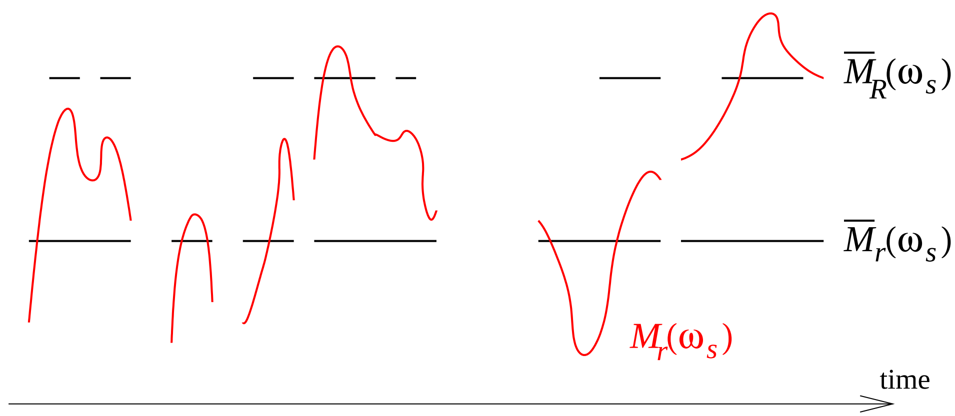
$$M_r(\omega_s) := M_{B_r}(\omega_s), \quad \mu_r[a,b] := \int_a^b M_r(\omega_s)\,ds, \quad \mu[a,b] := \lim_{r\to\infty} \mu_r[a,b].$$

Now $M_r(\omega)$ is a MG w.r.t. the full filtration $\mathscr{F}_r$ generated by $\omega(B_r)$, again $\mathbf{E}\mu_r[a,b] = b-a$, and $M_r(\omega) \leqslant C_2\,\overline{M}_r(\omega)$ because of quasi-multiplicativity:

$$\frac{\mathbf{P}[\,0 \leftrightarrow R \,|\, \omega^{B_r}\,]}{\mathbf{P}[\,0 \leftrightarrow R\,]} \asymp \frac{\mathbf{P}[\,0 \leftrightarrow R \,|\, \omega^{B_r}\,]}{\mathbf{P}[\,0 \leftrightarrow r\,]\mathbf{P}[\,r \leftrightarrow R\,]}$$

$$\leqslant \frac{\mathbf{P}[\,r \leftrightarrow R \,|\, \omega^{B_r}\,]\mathbf{1}\{0 \leftrightarrow r\}}{\mathbf{P}[\,0 \leftrightarrow r\,]\mathbf{P}[\,r \leftrightarrow R\,]} = \frac{\mathbf{1}\{0 \leftrightarrow r\}}{\mathbf{P}[\,0 \leftrightarrow r\,]}.$$

Hence, both local time measures exist, and are clearly supported inside $\mathscr{E}$.



$$\mathbf{E}\big[\,\overline{M}_R \,\big|\, \mathscr{F}_r\,\big] = \frac{\mathbf{P}[\,0 \leftrightarrow R \,|\, \mathscr{F}_r\,]}{\mathbf{P}[\,0 \leftrightarrow R\,]} \xrightarrow[L^\infty]{\text{a.s.}} M_r\,, \quad \text{for fixed } r \text{ and } R \to \infty\,.$$

Theorem (Hammond, P & Schramm 2012). $\overline{\mu} = \mu$ a.s. At a μ-typical time, the configuration has the distribution of IIC.

Question: is it true that $\operatorname{supp}(\mu) = \mathscr{E}$?

FETIC versus IIC

Mutual singularity should hold, but let's just show that there is some ω^{B_r} such that $\lim_{R\to\infty} \mathsf{FETIC}_R(\omega^{B_r}) \neq \lim_{R\to\infty} \mathsf{IIC}_R(\omega^{B_r})$.

The configuration at a typical switch time for $\{0 \longleftrightarrow R\}$ is size-biased by the number of pivotals. Because of the many pivotals far from the origin, inside B_r this bias becomes negligible as $R \to \infty$, so we still have IIC.

The configuration at FET_R is further size-biased by the length of the non-connection interval ending at the switch time.

For any $\omega = \omega^{B_R}$ satisfying $\{0 \longleftrightarrow R\}$, get $\mathsf{THIN}_r(\omega)$ by thinning inside B_r.

Want to show that the reconnection time $V = V_{r,R}$ started from $\mathsf{THIN}_r(\omega^{B_R})$ is larger in expectation than $N = N_{r,R}$, the one started from the normal ω^{B_R}, uniformly as $R \to \infty$. (While both are very small.)

Because of the thinning, there is some $\epsilon(r) \to 0$ and $g(r) \to \infty$ with

$$\mathbf{P}\big[\, V > g(r) \;\big|\; V > \epsilon(r) \,\big] \;>\; c_1 \,. \tag{1}$$

Also, from stochastic domination, $\mathbf{P}[\, V > \epsilon(r) \,] \;\geqslant\; \mathbf{P}[\, N > \epsilon(r) \,] \,.$ (2)

(1) would be hard, so our thinning is different, and (2) doesn't quite hold.

Write $X^\epsilon = X\,\mathbf{1}_{\{X > \epsilon(r)\}}$. Note that size-biased $\widehat{N}$ times Unif$[0,1]$ is FET.

A size-biasing lemma: $\mathbf{P}\big[\, \widehat{N} > \epsilon(r) \,\big] = \frac{\mathbf{E}[\, N^\epsilon \,]}{\mathbf{E}[\, N \,]} > c_2$ and $\mathbf{E}\big[\, \widehat{N} \,\big] < C_1$ imply

$$\mathbf{E}\big[\, N \;\big|\; N > \epsilon(r) \,\big] \;<\; C_2 \,. \tag{3}$$

From these three,

$$\mathbf{E}[\, V^\epsilon \,] \;\geqslant\; c_1\, g(r)\, \mathbf{P}[\, N > \epsilon(r) \,] \;\geqslant\; c_1\, g(r)\, \frac{\mathbf{E}[\, N^\epsilon \,]}{C_2} \,,$$

hence

$$\mathbf{E}V \;\geqslant\; \mathbf{E}[\, V^\epsilon \,] \;\gg_r\; \mathbf{E}[\, N^\epsilon \,] \;\geqslant\; c_2\, \mathbf{E}N \,.$$

Proof of exponential tail for FET

Dynamical percolation in B_R is just continuous time random walk on the hypercube $\{0,1\}^{B_R}$, with rate 1 clocks on the edges. On $\{0,1\}^n$, discrete time random walk has spectral gap $1/n$, but in continuous time, the gap is uniformly positive, so could try to use [AKSz'87].

Of course, $\mathbf{P}[\,0 \longleftrightarrow R\,]$ is tiny, so we don't want to hit that. But $\mathbf{P}[\,\mathscr{E}_R \cap [0,1] \neq \emptyset\,]$ is uniformly positive!

So, first idea: Markov chain $\{\omega[2t, 2t+1] : t = 0, 1, 2\}$ on a huge state space. This again has a uniform spectral gap. However, it's not reversible!

So, another trick: $L^2(\Omega, \mathbf{P})$ is the space of trajectories $\{\omega_t : t \in \mathbb{R}\}$, on it the event $A_t := \{\mathscr{E}_R \cap [t, t+1] = \emptyset\}$ for any $t \in \mathbb{R}$, then the projection $Q_t f := f \mathbf{1}_{A_t}$ is still self-adjoint and $\mathbf{P}[\,\mathrm{supp}(Q_t g)\,] \leqslant \beta < 1$ for any g. On the other hand, for $g_i(\omega) := \mathbf{E}[\,f_i(\omega[0,1]) \mid \omega_0 = \omega\,]$, we have $\mathbf{E}\big[\,f_1(\omega[0,1])\, f_2(\omega[t,t+1])\,\big] = \mathbf{E}_\pi[g_1\, T_t g_2]$, hence the spectral gap of T_t can be used.

What about the tail of left-right connection?

As mentioned before, $\mathbf{E}\big[\, f_{\mathcal{Q},\eta}(\omega_0)\, f_{\mathcal{Q},\eta}(\omega_{t\eta^{3/4+o(1)}})\,\big] - \mathbf{E}[\, f_{\mathcal{Q},\eta}\,]^2 \asymp_{\mathcal{Q}} t^{-2/3}$ as $t \to \infty$, uniformly in mesh η, hence natural to rescale time like this.

In fact, there exists a scaling limit of dynamical percolation [Garban, P. & Schramm 2012], so one can either talk about the rescaled finite chains, "uniformly in η", or about the scaling limit process.

Earlier theorem [HMP'11] gives $\mathbf{P}\big[\, f_{\mathcal{Q}}(\omega_s) = 1\ \forall\, s \in [0,t]\,\big] \leqslant t^{-2/3+o(1)}$.

In fact, by cutting $\mathcal{Q}$ vertically into L slabs: $\leqslant t^{-2L/3+o(1)}$ for any L, superpolynomial decay.

Exponential lower bound is easy from dynamical FKG inequality.

Conjecture. $\mathbf{P}\big[\, f_{\mathcal{Q}}(\omega_s) = 1$ for all $s \in [0,t]\,\big] = \exp(-t^{2/3+o(1)})$.

Supported by a *very* non-rigorous renormalization argument.

Proof of Correlation decay $\implies$ exit time tail [HMP'11]

Let $p < \lambda < 1$. Consider $A_s := \Big\{\omega \in S : \mathbf{P}[\,\omega_s \in \mathcal{C} \mid \omega_0 = \omega\,] < \lambda\Big\}$, the set of not very good hiding places.

Fix large k, let $\tau = t/k$. Check $\omega_s \in \mathcal{C}$ at $s = j\tau$, for $j = 0, \ldots, k$. Let ℓ be the last of these times when $\omega_{\ell\tau} \in A_\tau^c \cap \mathcal{C}$. Then

$$\begin{aligned}\mathbf{P}\Big[\,\omega_s \in \mathcal{C}\ \forall s \in [0,t]\Big] &\leqslant \lambda^k + \sum_{\ell=0}^{k} \lambda^{(k-\ell-1)\vee 0}\,\mathbf{P}\big[\,\omega_{\ell\tau} \in A_\tau^c\,\big] \\ &\leqslant \lambda^k + \frac{2-\lambda}{1-\lambda}\,\mathbf{P}[\,A_\tau^c\,]\,.\end{aligned}$$

On the other hand, if s is large, then one expects $\mathbf{P}\big[\,A_s^c\,\big]$ to be small. Indeed,

$$\lambda\,\mathbf{P}\big[\,A_s^c\,\big] \;\leqslant\; \mathbf{E}\big[\,f(\omega_s)\;\big|\;A_s^c\,\big]\,\mathbf{P}\big[\,A_s^c\,\big] = \mathbf{E}\big[\,\mathbf{1}_{A_s^c}\,T_s f\,\big]$$

$$= \mathbf{E}\big[\,\mathbf{1}_{A_s^c} p\,\big] + \mathbf{E}\Big[\,\mathbf{1}_{A_s^c}\,(T_s f - \mathbf{E} f)\,\Big]\,.$$

Rearranging and using Cauchy-Schwarz,

$$(\lambda - p)\,\mathbf{P}\big[\,A_s^c\,\big] \;\leqslant\; \mathbf{E}\Big[\,\mathbf{1}_{A_s^c}\,(T_s f - \mathbf{E} f)\,\Big] \leqslant \|\mathbf{1}_{A_s^c}\|_2\,\|T_s f - \mathbf{E} f\|_2\,,$$

hence $\quad (\lambda - p)\,\mathbf{P}\big[\,A_s^c\,\big]^{1/2} \leqslant \|T_s f - \mathbf{E} f\|_2 = \mathrm{Var}[T_s f]^{1/2}\,.$

Thus

$$\mathbf{P}\big[\,A_\tau^c\,\big] \leqslant \frac{p - p^2}{(\lambda - p)^2}\,d(2t/k)\,,$$

and can optimize the sum of two terms over k.