Conformally invariant random processes near their critical point

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CEU PhD seminar & colloquium

Rough outline

What is a conformally invariant scaling limit?

Classical example: 2-dim Brownian motion.

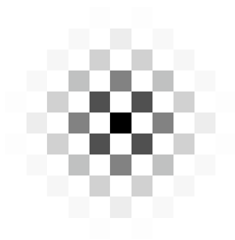
Other models: Self-Avoiding Walks, Loop-Erased Random Walks, Uniform Spanning Trees, Percolation, Ising model

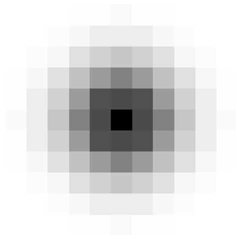
The Fortuin-Kasteleyn random cluster model FK(p, q) is, in some sense, a joint generalization of almost all of these.

Assuming conformal invariance (sometimes proved), the Schramm-Loewner Evolution tells a lot about these models.

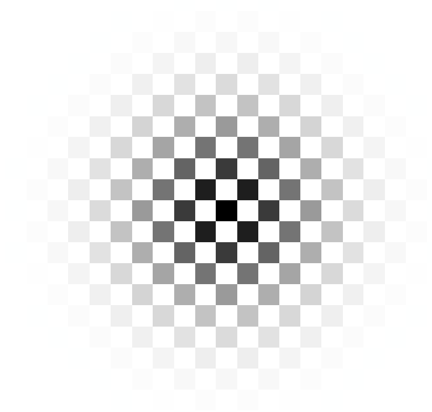
Will look at dynamical and near-critical versions. (Joint works with Christophe Garban and Oded Schramm.)

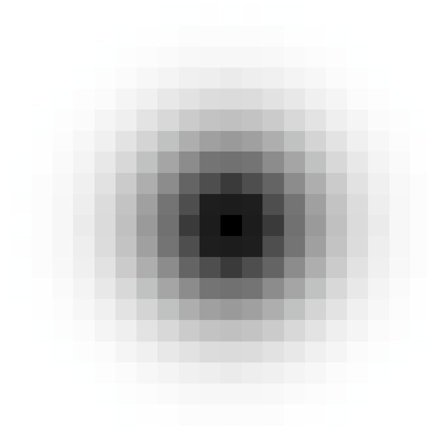
Simple random walk on $\mathbb{Z}^2$, 6 steps



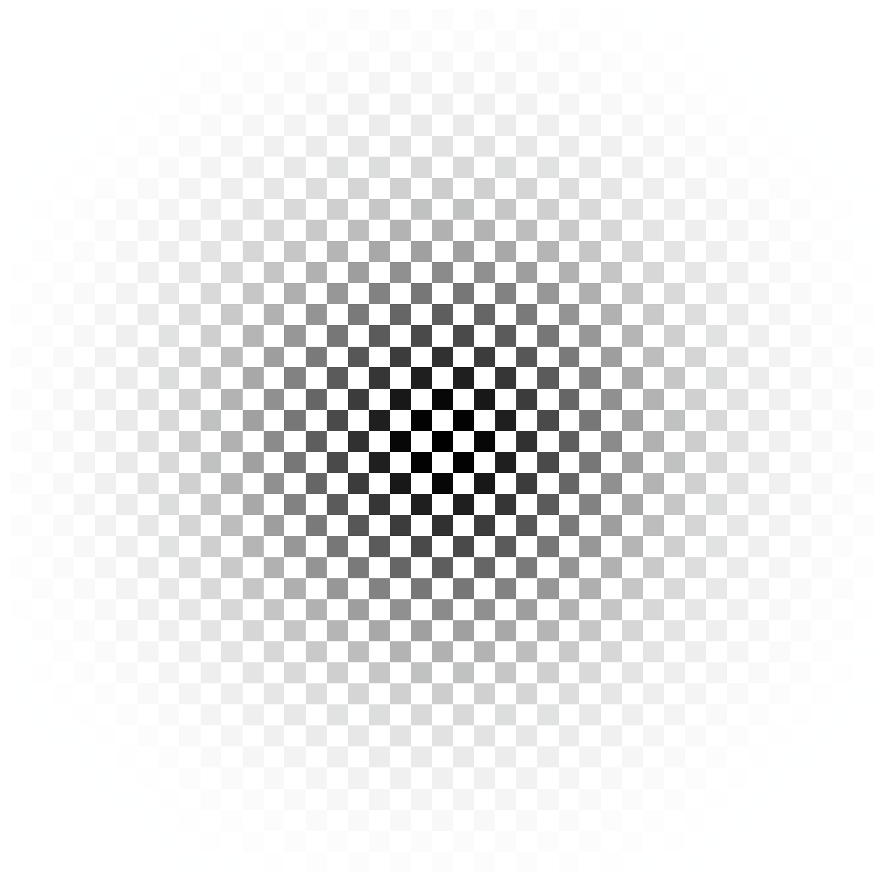


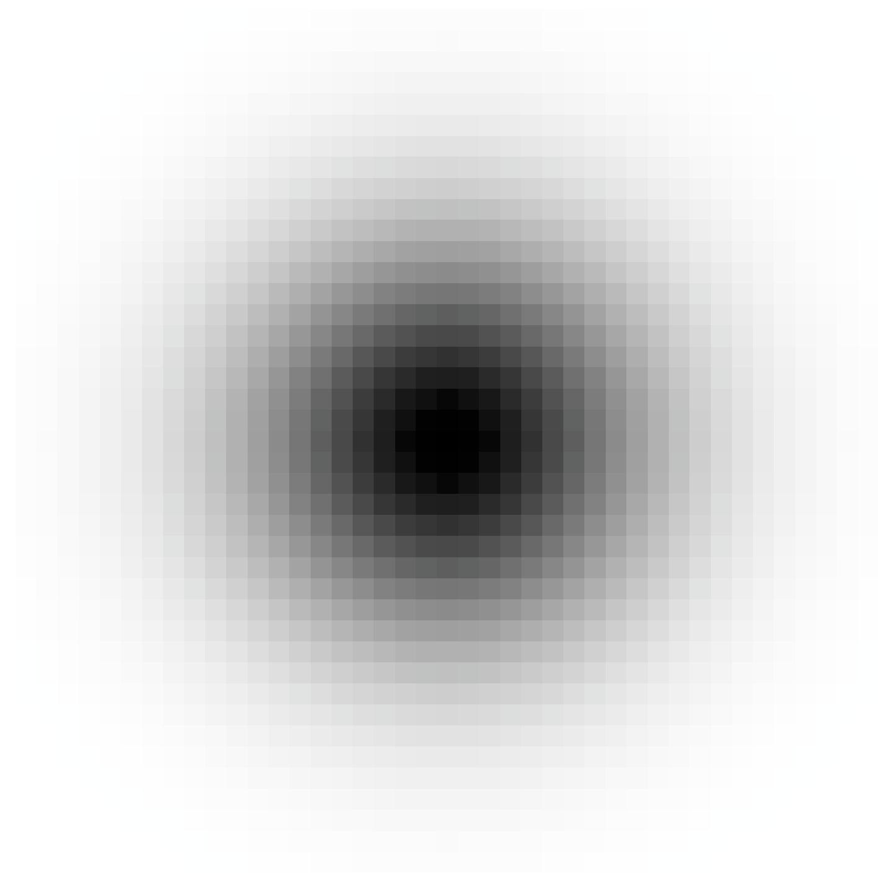
Simple random walk on $\mathbb{Z}^2$, 20 steps





Simple random walk on $\mathbb{Z}^2$, 100 steps





The scaling limit of SRW on $\mathbb{Z}^2$

$X_1, X_2, \ldots$ steps, $S_n = \sum_{i=0}^{n} X_n$ positions in $\mathbb{Z}^2$.

$\mathbf{E}[X_i] = (0,0)$, $\mathrm{Var}[X_i] = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$. CLT: $\frac{S_n}{\sqrt{n/2}} \xrightarrow{d} N(0, \mathrm{Id})$.

Moreover, for $0 < t_1 < t_2 < 1$,

$$\frac{(S_{t_1 n}, S_{t_2 n})}{\sqrt{n/2}} \xrightarrow{d} \Big(N\big(0, t_1 \mathrm{Id}\big),\ N\big(0, t_1 \mathrm{Id}\big) + N\big(0, (t_2 - t_1)\mathrm{Id}\big)\Big).$$

And the limiting path can be proved to be continous. Hence the scaling limit is 2-dimensional Brownian motion:

$$\Big(\frac{S_{tn}}{\sqrt{n/2}} : 0 \leqslant t \leqslant 1\Big) \xrightarrow{d} \big(B_t : 0 \leqslant t \leqslant 1\big).$$

Rotational invariance of $N(0,1)$ implies same for $(B_t)_{0\leqslant t\leqslant 1}$.

Being a scaling limit implies scale invariance:

$$\lambda\, B_t \sim \lambda\, \frac{S_{tn}}{\sqrt{n/2}} = \frac{S_{\lambda^2 tn/\lambda^2}}{\sqrt{n/(2\lambda^2)}} = \frac{S_{\lambda^2 tm}}{\sqrt{m/2}} \sim B_{\lambda^2 t}\,.$$

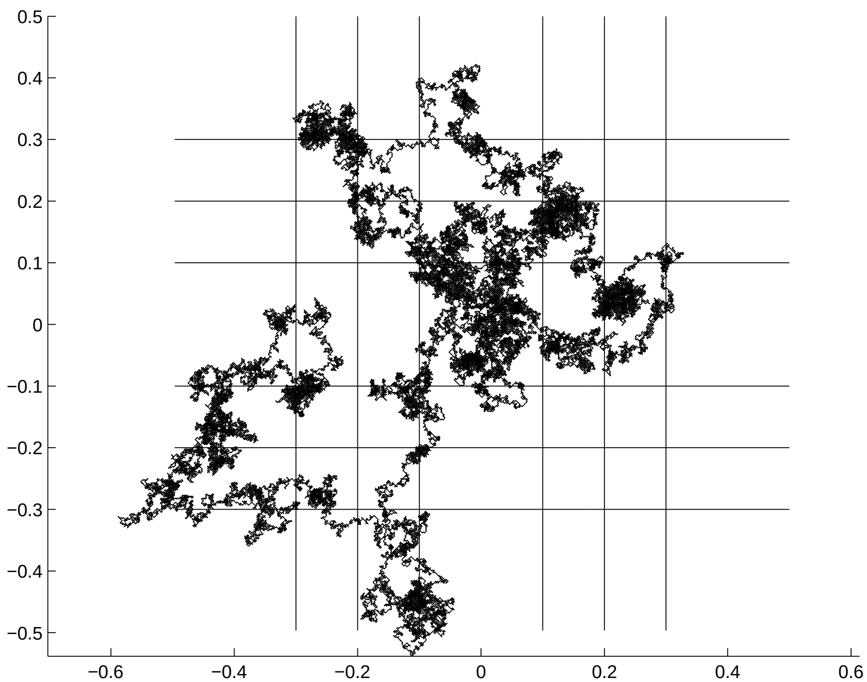
Having independent increments implies that can rotate and scale by different values at different points!

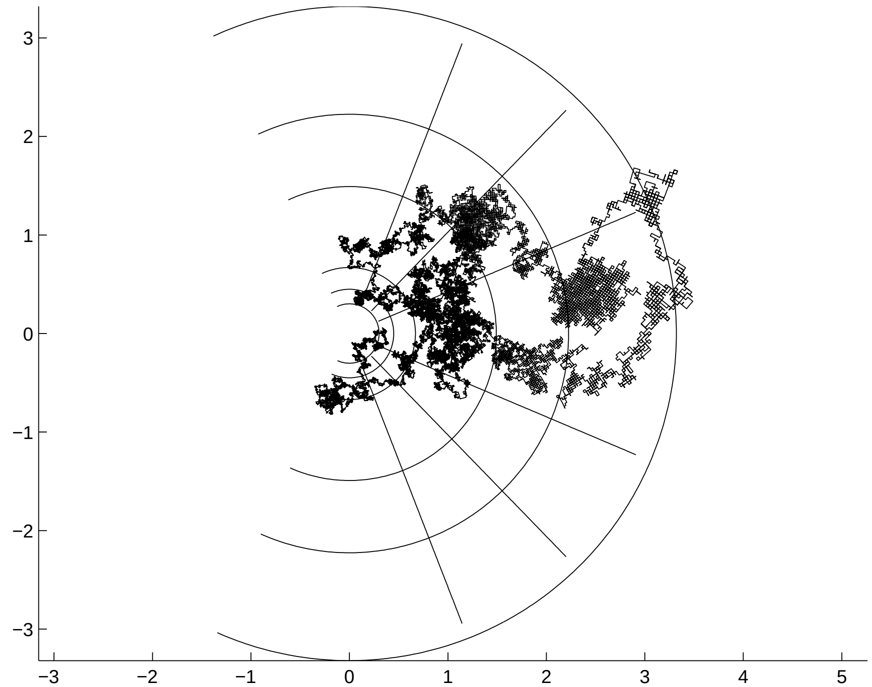
For $D \subseteq \mathbb{C}$, $f : D \longrightarrow \mathbb{C}$ is called conformal if it is holomorphic (complex differentiable), injective, and the derivative is $|f'(z)| \neq 0$ for any $z \in D$.

Multiplying by $f'(z) \in \mathbb{C}$ is best linear approximation to f at z: locally scale by $|f'(z)|$, rotate by $\arg f(z)$.

Hence the trajectory of 2-dim Brownian motion is conformally invariant. Just time will run locally at different speeds. (Lévy 1948)

Simple random walk under an exponential map

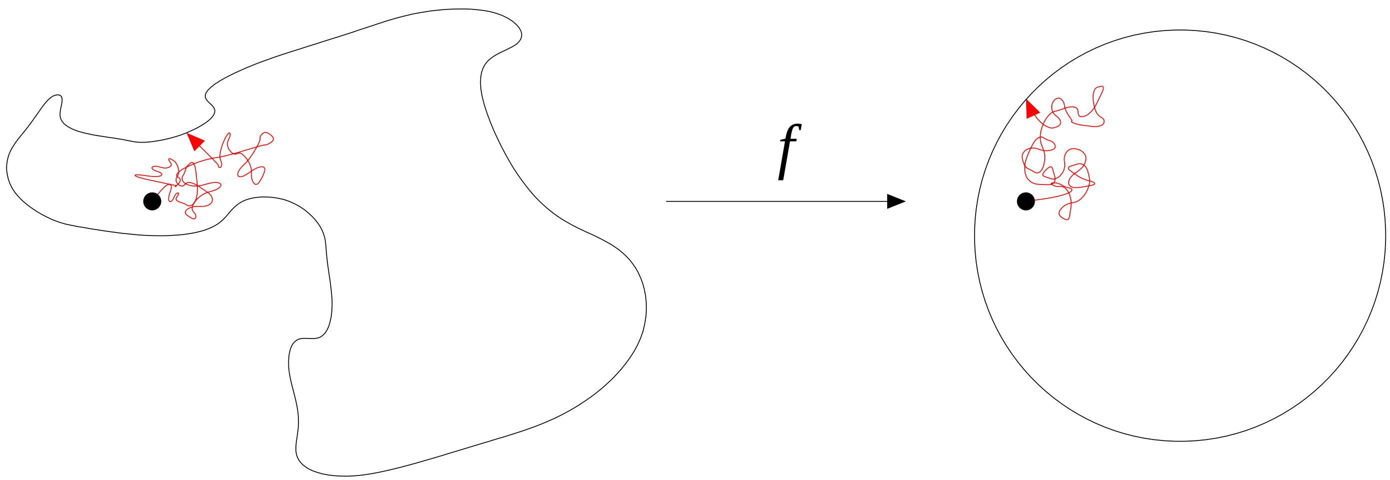




They look the same.

Conformal invariance of trajectory: another formulation

Given $z \in D \subset \mathbb{C}$, the the hitting measure of Brownian motion on ∂D is the harmonic measure ν_z. This is conformally invariant: for $f : D \longrightarrow D'$ conformal, $\nu_{f(z)} = f_*(\nu_z)$.



Riemann mapping theorem: if $D, D' \subsetneq \mathbb{C}$ are simply connected domains, then $\exists\, f : D \longrightarrow D'$ holomorphic bijection (then it is conformal except for a countable set of points).

Conformal invariance of trajectory: another formulation

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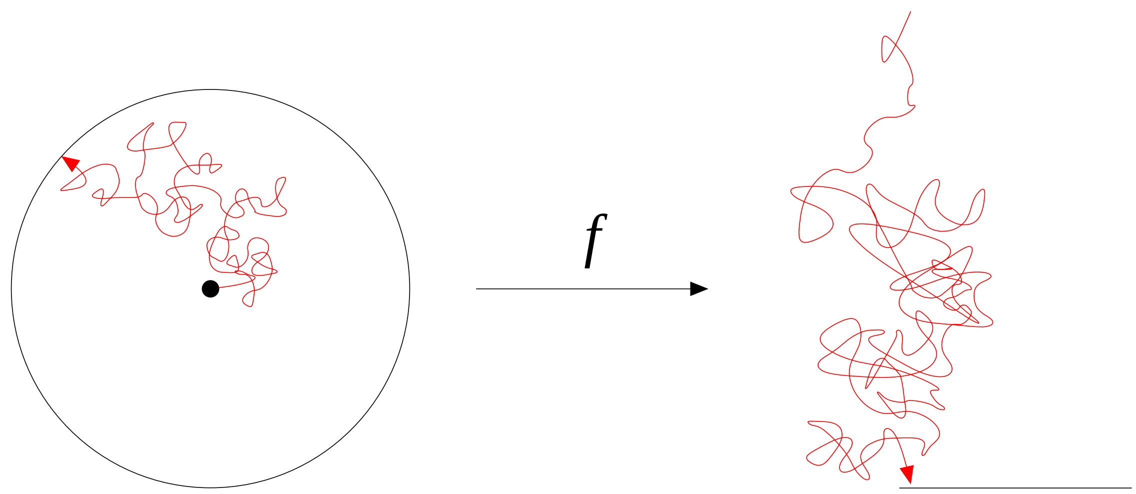
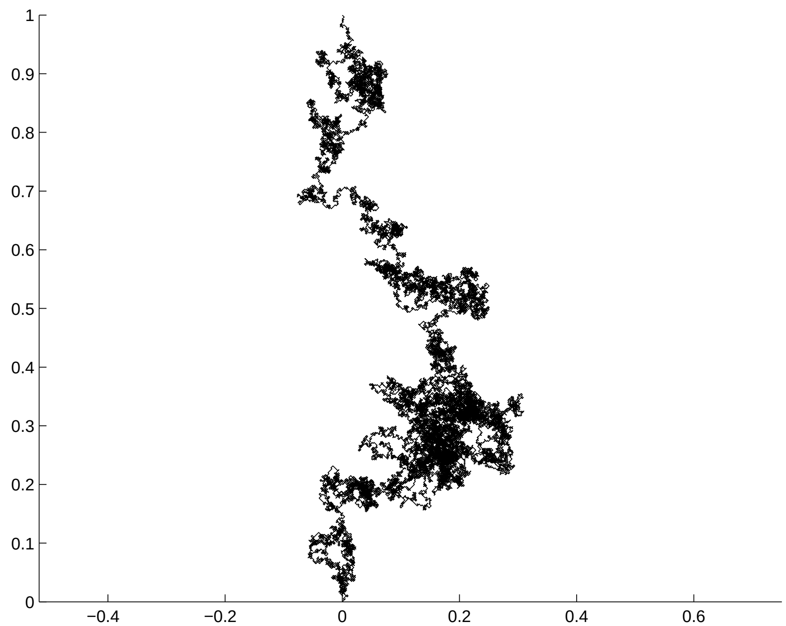


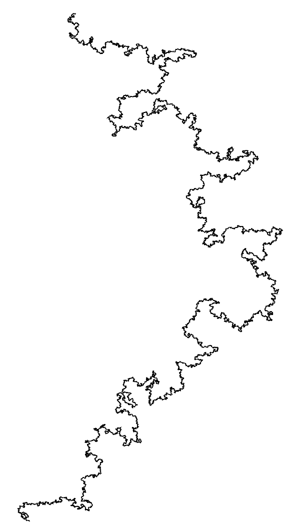
E.g., by $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$, uniform measure on circle is mapped to the arcsine law $\frac{1}{\pi\sqrt{1-x^2}}\,dx$ on $[-1, 1]$. "Electrostatic potential."

$\Longrightarrow$ Having a conformally invariant scaling limit is very useful.

Mandelbrot's 4/3 conjecture

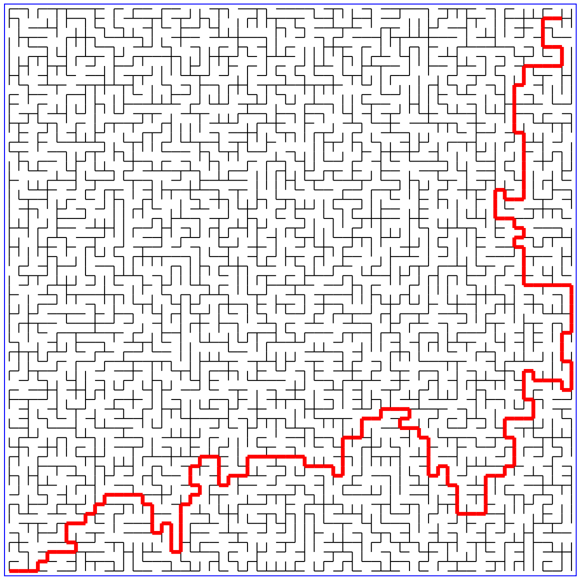
Many things were computed about BM in ancient times; e.g., graph of 1-dim BM has Hausdorff-dimension 3/2, graph of 2-dim BM has zero measure but H-dim 2. But there are harder questions:





Mandelbrot '82 observed visually that Brownian frontier appears to be exactly as wiggly as Self-Avoiding Walk, conjectured to have H-dim 4/3.

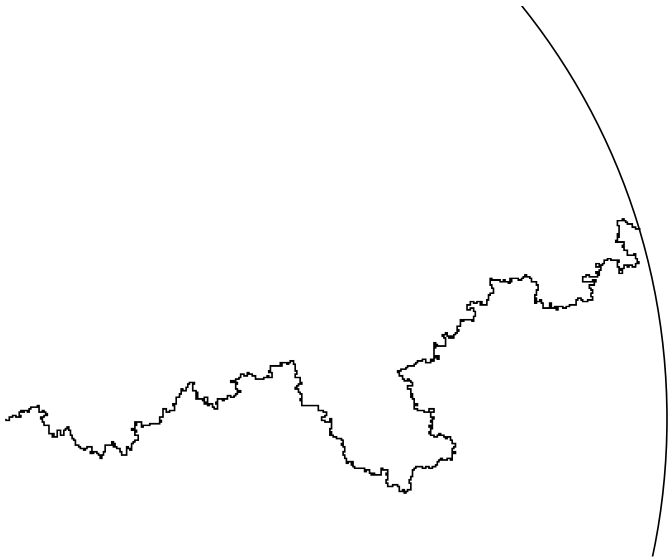
The Uniform Spanning Tree



On a finite graph, take one uniformly from all spanning trees.

Paths inside are loop-erased random walk paths (David Wilson's algorithm '96).

Also related to domino tilings, and Rick Kenyon '00 computed length $\asymp n^{5/4}$.



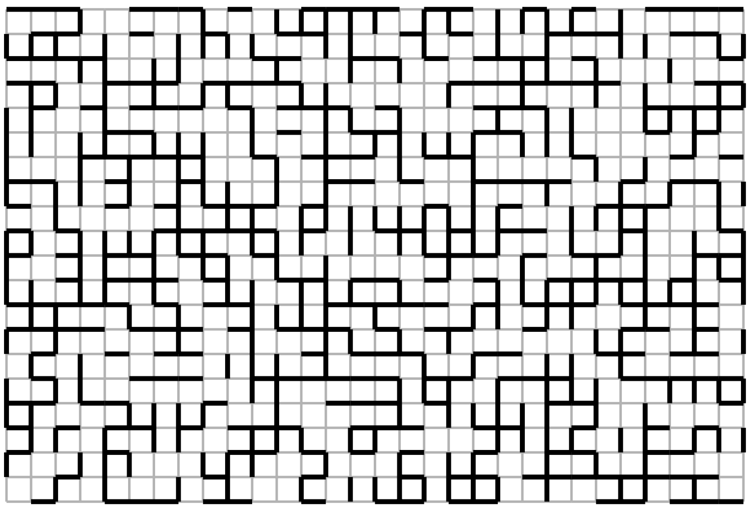
Since we get LERW from SRW, reasonable to think that it has a conformally invariant scaling limit. However, loop-erasure of 2-dim BM is very far from clear!

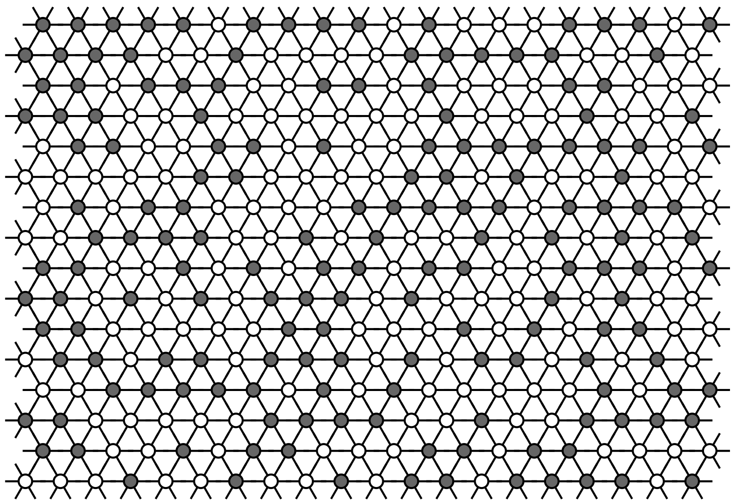
1-dim BM has *no first zero* after $B_0 = 0$. 2-dim BM has *no first loop*; infinitely many loops on all scales. So?

Bernoulli(p) bond and site percolation

Graph $G(V, E)$ and $p \in [0, 1]$. Each site (or bond) is open with probability p, closed with $1 - p$, independently. Consider open connected clusters.

$$p_c(G) := \inf \left\{ p : \mathbf{P}_p[0 \longleftrightarrow \infty] > 0 \right\} = \inf \left\{ p : \mathbf{P}_p[\exists\ \infty \text{ cluster}] = 1 \right\}$$





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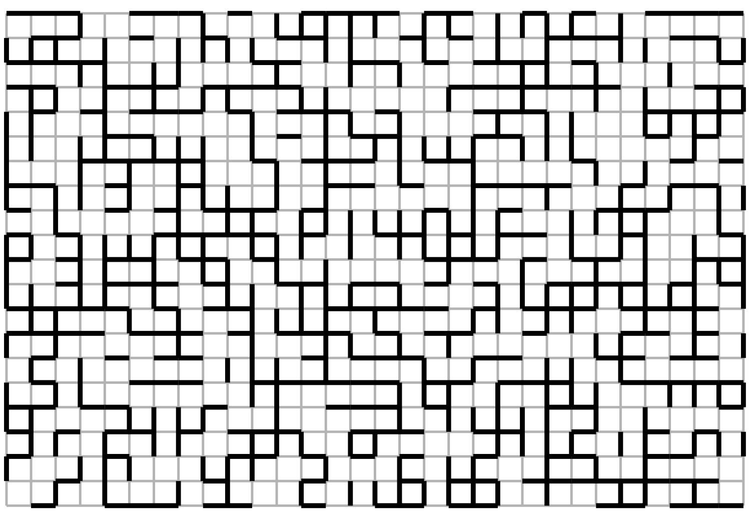

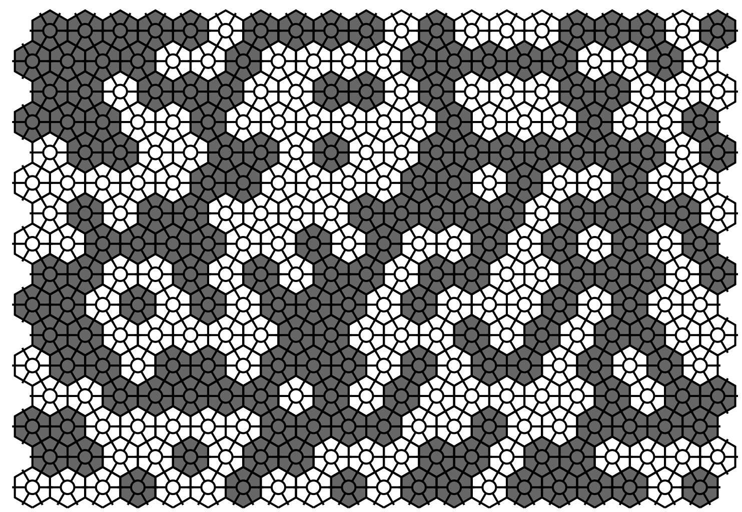

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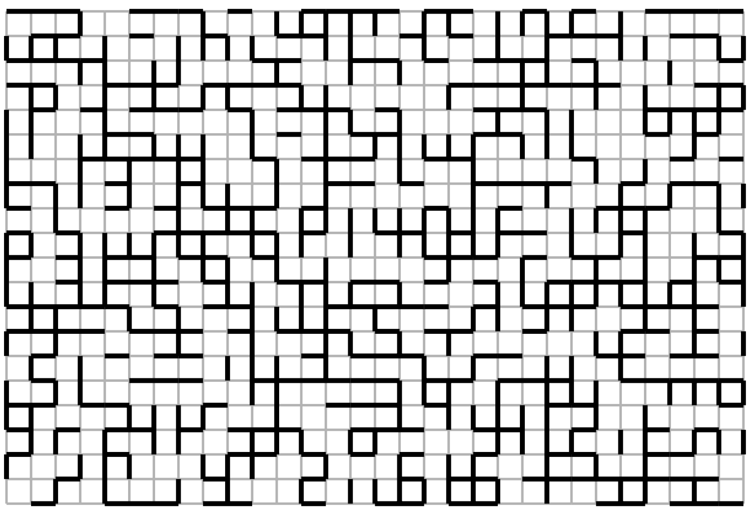

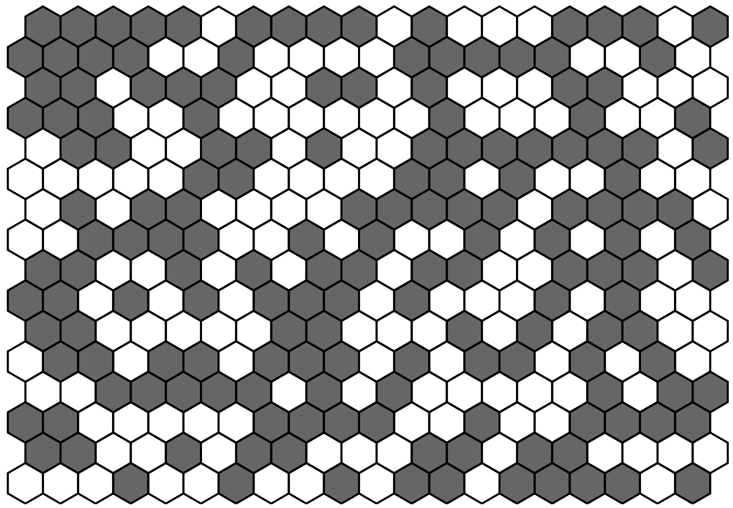

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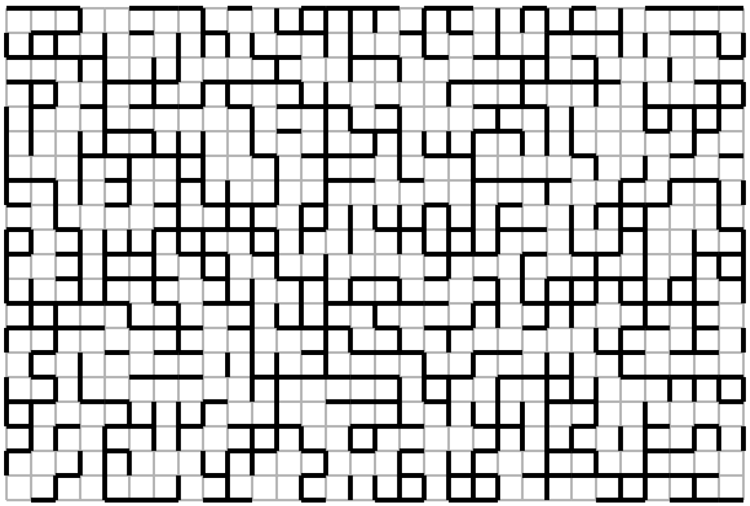
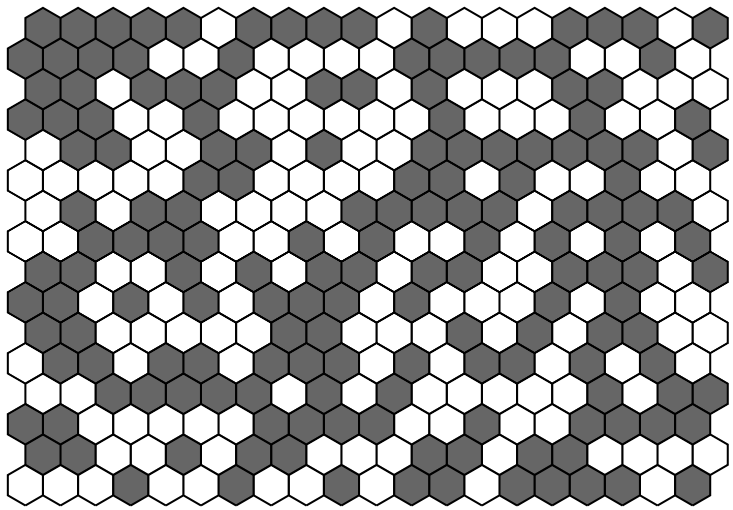




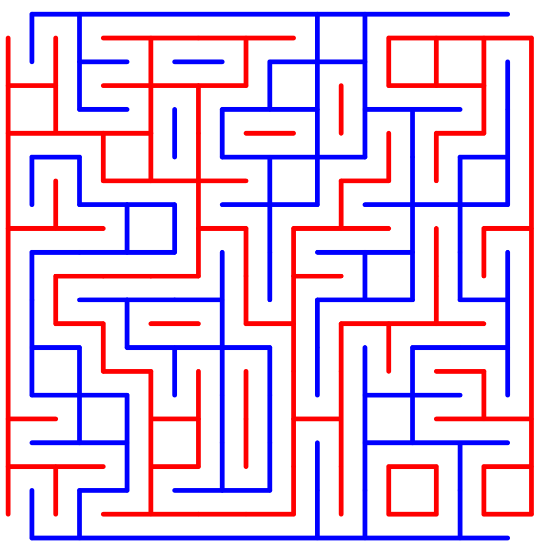
Theorem (Harris 1960 and Kesten 1980).
$p_c(\mathbb{Z}^2, \text{bond}) = p_c(\Delta, \text{site}) = 1/2$, and $\mathbf{P}_{p_c}[0 \longleftrightarrow \partial B_n(0)] = n^{-\Theta(1)}$.
For $p > 1/2$, there is almost surely one infinite cluster.

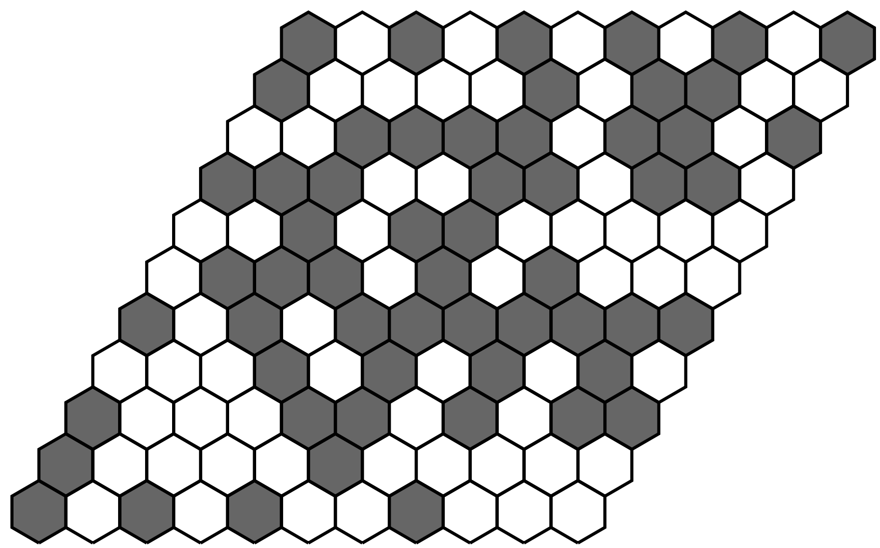
Why is $p_c = 1/2$? Duality!

$\mathbb{Z}^2$ bond percolation at $p = 1/2$: in an $n \times (n+1)$ rectangle, left-right crossing has probability exactly $1/2$, because:

$\mathbf{P}[\,\text{LeftRight}(n, n+1)\,] + \mathbf{P}[\,\text{TopBottom}(n+1, n)\,] = 1$, and they are equal.

For site percolation on Δ, same on an $n \times n$ rhombus.

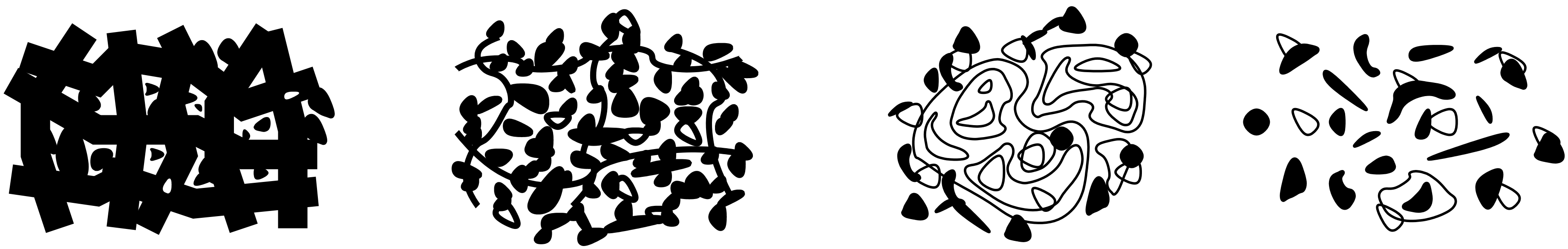




Crossing probabilities and criticality

Theorem (Russo 1978 and Seymour-Welsh 1978). For $p = 1/2$ bond percolation on $\mathbb{Z}^2$ or site percolation on Δ, for $L, n > 0$,

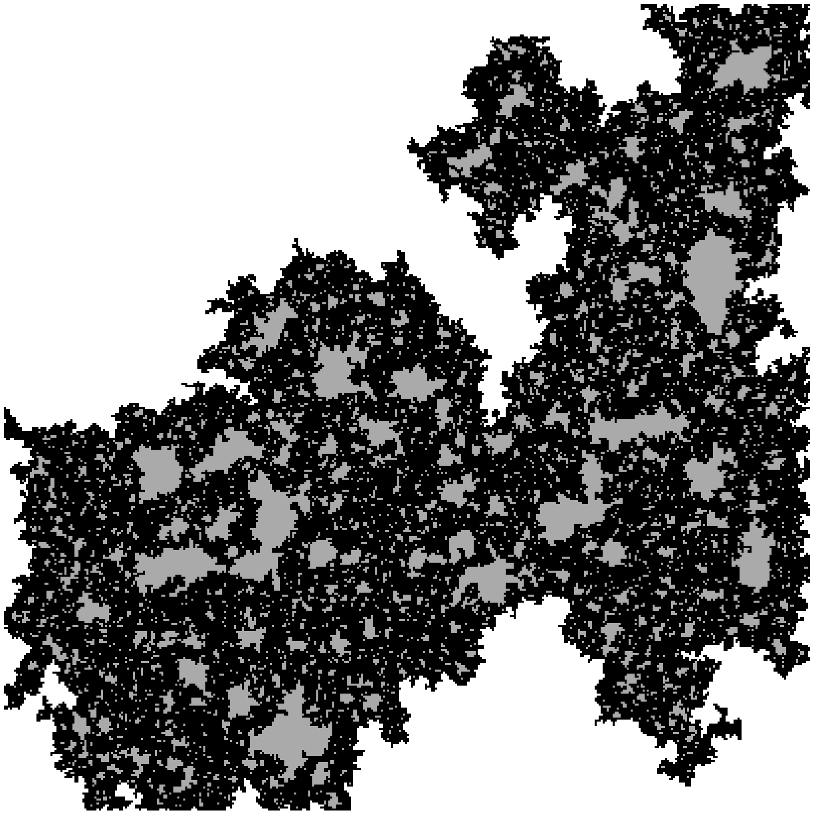
$$0 < a_L < \mathbf{P}[\,\text{left-right crossing in } n \times Ln\,] < b_L < 1.$$

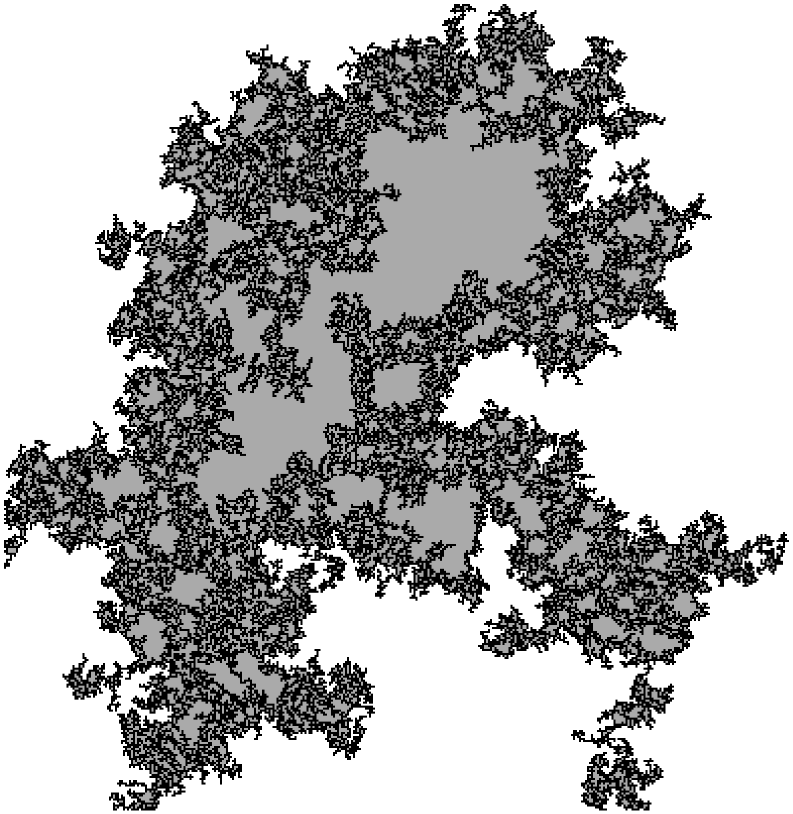


$p \approx 0.9$ $\quad$ $p \approx 0.55$ $\quad$ $p = 0.5$ $\quad$ $p \approx 0.45$

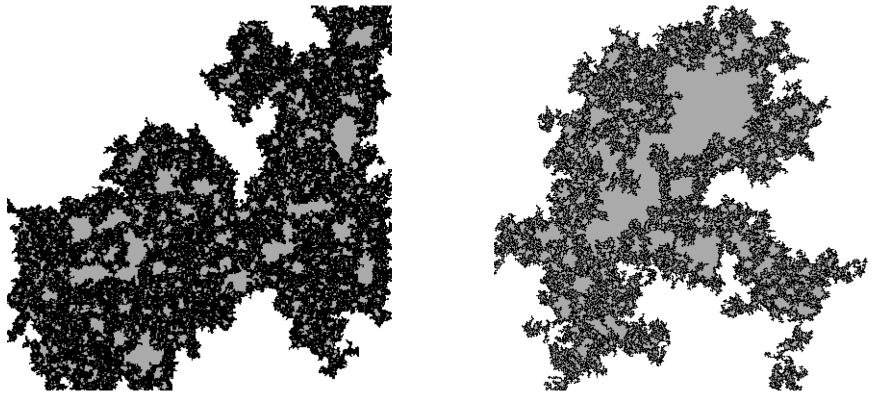
For $p > 1/2$, correlation length $L_\delta(p) := \min\{n : \mathbf{P}_p[\,\mathsf{LR}(n)\,] > 1 - \delta\}$. This is roughly the size of holes in the infinite cluster.

Critical percolation on different lattices





Universality Conjecture



Although p_c depends on the lattice, behavior at p_c should be the same!

E.g., "dimension" of large cluster boundaries should always be 7/4.

Or, $\mathbf{P}_{p_c}[0 \longleftrightarrow \partial B_n] = n^{-5/48+o(1)}$.

Or, off-critical exponent $\mathbf{P}_{p_c+\epsilon}[0 \longleftrightarrow \infty] = \epsilon^{5/36+o(1)}$.

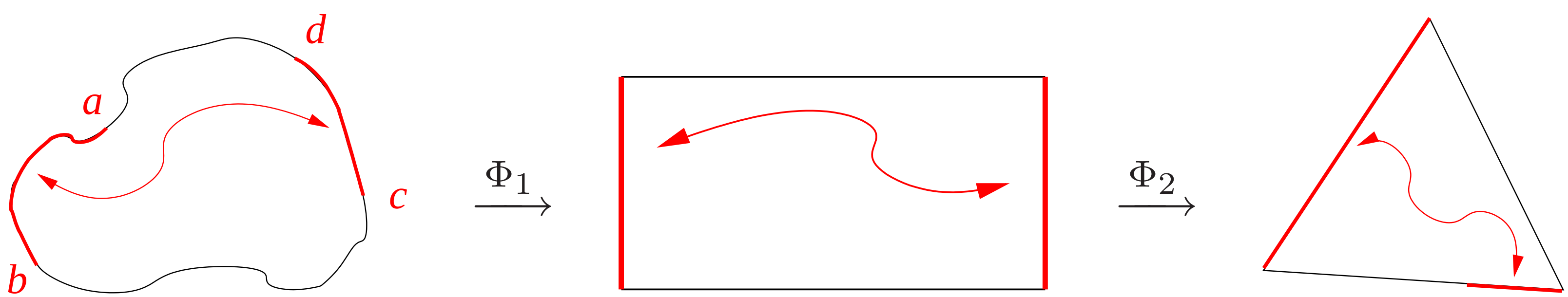
Analogy: Simple random walk on any planar lattice has the same scaling limit: planar Brownian Motion.

Conformal invariance

Theorem (Smirnov '01). For critical site percolation on $\Delta_{1/n}$, if $\mathcal{Q} \subset \mathbb{C}$ is a piecewise smooth quad, then

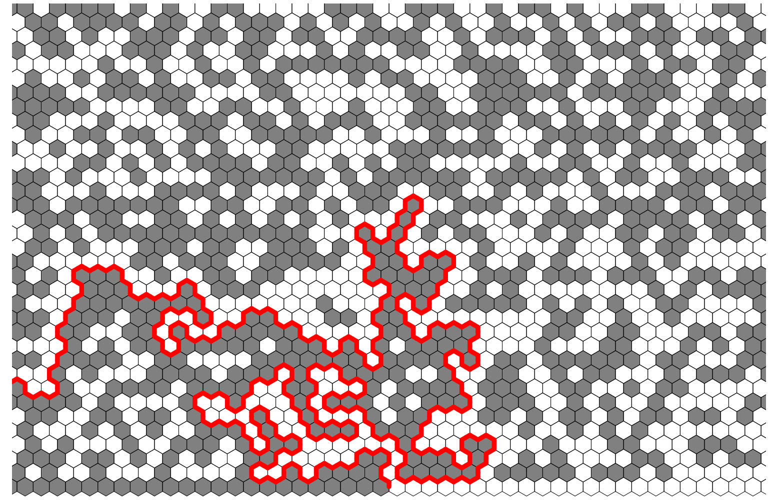
$$\lim_{n\to\infty} \mathbf{P}\Big[ab \longleftrightarrow cd \text{ inside } \mathcal{Q} \cap \Delta_{1/n} \Big]$$

exists, is strictly between 0 and 1, and conformally invariant.



Moreover, there is a continuum scaling limit, encoding macroscopic connectivity structure, cluster boundaries, etc., Schramm '00, Camia-Newman '06, Sheffield '09. In physics, usually just correlation functions.

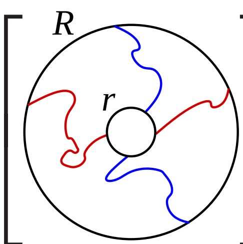
Schramm-Loewner Evolution



Given conformal invariance and spatial Markov property, the exploration path converges to the Stochastic Loewner Evolution with $\kappa = 6$ (Schramm '00).

Using the SLE_6 curve, critical exponents mentioned above can be computed (Lawler-Schramm-Werner, Smirnov-Werner '01, Kesten '87). E.g.:

$$\alpha_4(r, R) := \mathbf{P} \Big[$$



$$\Big] = (r/R)^{5/4+o(1)},$$

Lawler-Schramm-Werner '04: the scaling limit of Loop-Erased Random Walk on nice lattices is SLE_2. The scaling limit of the Peano curve around the Uniform Spanning Tree is SLE_8. Exponents can be computed again.

A unifying model: FK(p, q)

Fortuin-Kasteleyn '69 random cluster model: for $\omega \in \{0,1\}^{E(G)}$,

$$\mathbf{P}_{\mathrm{FK}(p,q)}[\omega] = \frac{1}{Z_{\mathrm{FK}(p,q)}} p^{|\omega|} (1-p)^{|E(G)\setminus\omega|} q^{|\mathrm{clusters}(\omega)|}.$$

$q = 1$: Bernoulli(p) bond percolation. $q \to 0$, then $p \to 0$: UST

$q = 2, 3, \ldots$: sibling of q-Potts; $q = 2$: Ising model of magnetization.

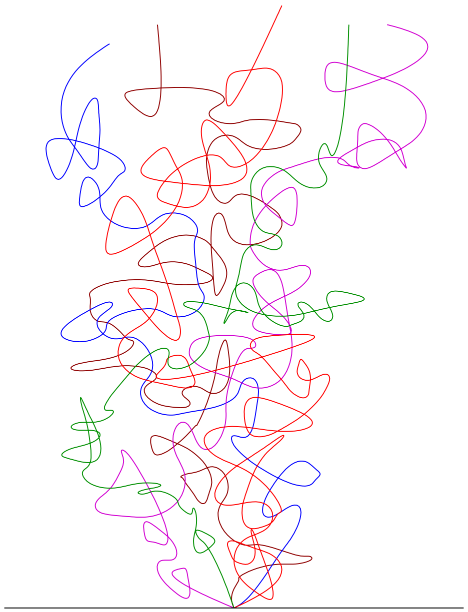
Conjecture. In FK$(p_c(q), q)$ for $0 \leqslant q \leqslant 4$, the scaling limit of the exploration path is SLE$_\kappa$, with $\kappa(q) = 4\pi / \arccos(-\sqrt{q}/2) \in [4, 8]$.
For the corresponding "outer-boundary type" models, we have $16/\kappa$.

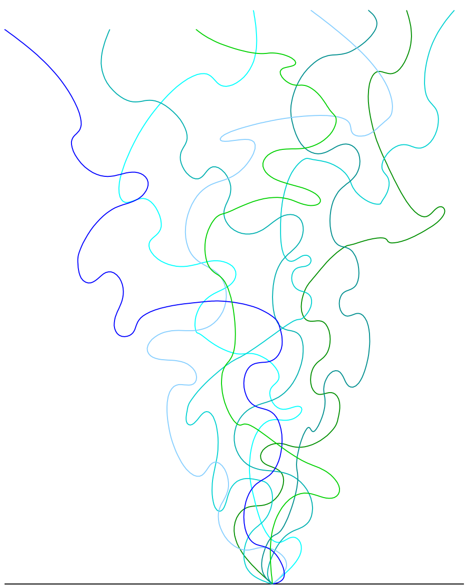
$q = 2$ proved by Smirnov '06, Chelkak-Smirnov '10: SLE$_{16/3}$, SLE$_3$.

Outer boundary of percolation is basically SLE$_{8/3}$, which *should be* the scaling limit of SAW. Just like outer boundary of 2-dim BM.

Domino tilings should have to do with SLE$_4$.

Proof of Mandelbrot's 4/3





Lawler-Schramm-Werner '01: boundary of the union of 5 Brownian excursions is *exactly* the union of 8 $\mathrm{SLE}_{8/3}$'s.

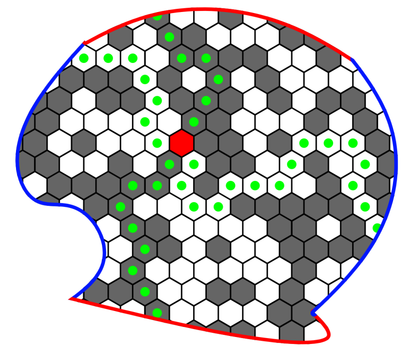
And H-dim of SLE_κ is computable: $1 + \kappa/8$ (Beffara '06).

Perturbations near the critical point

Dynamical percolation: every site is switching between open and closed using iid exponential clocks, keeping critical percolation stationary.

1. *How long does it take to change macroscopic crossings?* Or, how noise sensitive are the crossing events?

A reasonable guess: the expected number of pivotal switches (i.e., changes of the left-right crossing event) should be of order one. Hence time should be $1/\mathbf{E}|\mathrm{Piv}_n| = n^{-3/4+o(1)}$ — very small!



2. On an infinite lattice, are there random exceptional times with an infinite cluster? In other words, which events are dynamically sensitive?

3. In the unit square (or in another conformal rectangle), with mesh $1/n$ and rate $1/\mathbf{E}|\mathrm{Piv}_n|$ for the exponential clocks, is there a scaling limit of the process, giving a Markov process on continuum configurations?

The answers

Theorem (Garban, P & Schramm 2010).

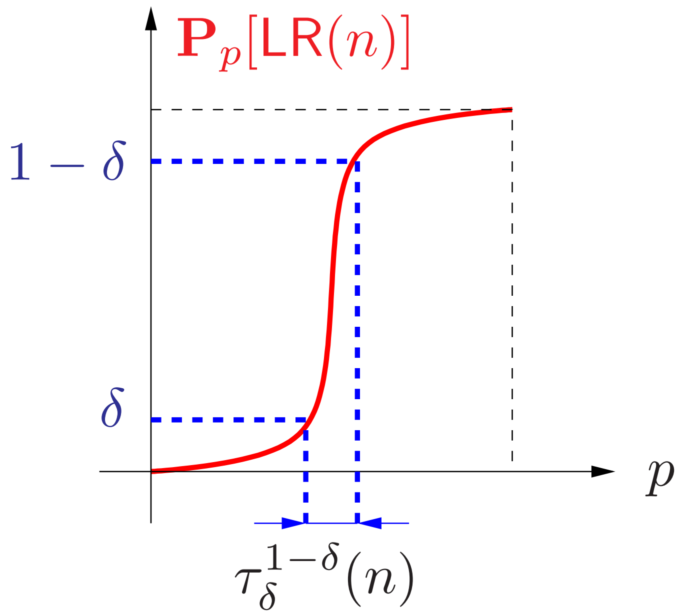
- At time $\gg 1/\mathbf{E}|\mathrm{Piv}_n|$, crossing events completely decorrelate.
- There are exceptional times on $\mathbb{Z}^2$.
- On the triangular grid they have Hausdorff dimension $31/36$.
- On the triangular grid, there are exceptional times with an infinite white **and** an infinite black cluster simultaneously. $(1/9 \leqslant \dim \leqslant 2/3)$

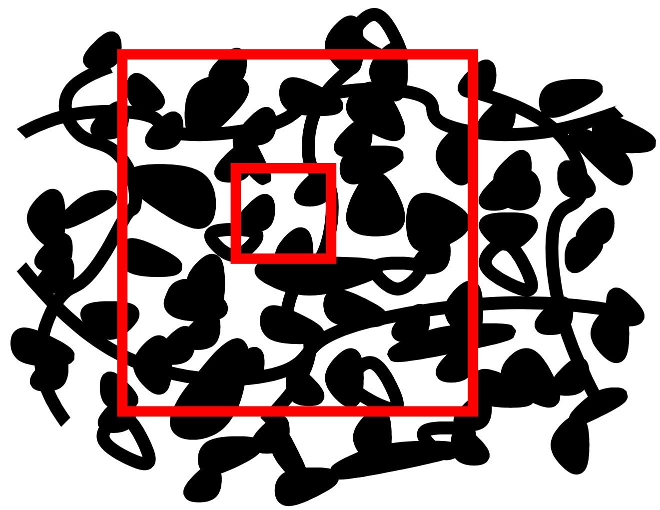
Proof uses discrete Fourier analysis. (Finding the decomposition of crossing event indicators into eigenfunctions of the dynamics.)

Theorem (Garban, P & Schramm 2013). Dynamical and near-critical percolation with the right rescaling have Markovian, conformally covariant scaling limits.

The near-critical regime

Recall the correlation length $L_\delta(p) := \min\{n : \mathbf{P}_p\big[\,\mathsf{LR}(n)\,\big] > 1-\delta\}$.





$L_\delta\big(p_c + \tau_{1/2}^{1-\delta}(n)\big) = n$

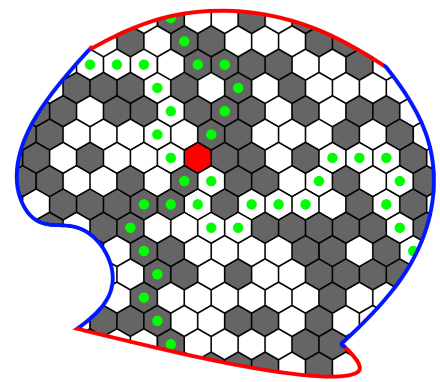
Kesten '87: Near-critical window for percolation is given by number of pivotal points at criticality: $\tau(n) = n^{-3/4+o(1)} \approx 1/\mathbf{E}_{p_c}|\mathsf{Piv}_n|$.

Duminil-Copin, Garban & Pete '11: In Ising-FK, this is NOT the case. Still, we can find $\tau(n) = n^{-1+o(1)}$ using conformal invariance techniques.

The near-critical ensemble in percolation

Standard coupling: to each site (or bond) $x \in G$, assign $V(x)$ i.i.d. Unif$[0,1]$, and let x be **open at level** p if $V(x) \leqslant p$.

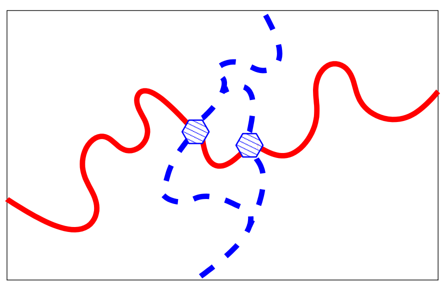
In $\mathcal{Q} \cap \Delta_{1/n}$, when raising p from p_c, when does it become well-connected?



A site is pivotal in ω if flipping it changes the existence of a left-right crossing. Equivalent to having alternating 4 arms. For nice quads, there are not many pivotals close to $\partial\mathcal{Q}$, hence

$$\mathbf{E}_{p_c}|\mathrm{Piv}_n| \asymp n^2\,\alpha_4(n) = n^{3/4+o(1)} \text{ on } \Delta_{1/n}.$$

If $p - p_c \gg n^{-3/4+o(1)}$, we have opened many critical pivotals, hence already supercritical. But maybe many new pivotals appeared on the way, hence there is a pivotal switch earlier?



New pivotals do appear. But will they be switched as p is raised?

Stability by Kesten (1987): multi-arm probabilities stay comparable inside this regime, hence changes are not faster, and this $n^{-3/4+o(1)}$ is indeed the critical window.

And then the near-critical scaling relation:

$$\begin{aligned}\mathbf{P}_p\big[\,0 \leftrightarrow \infty\,\big] &\asymp \mathbf{P}_p\big[\,0 \leftrightarrow L(p)\,\big] \asymp \mathbf{P}_{1/2}\big[\,0 \leftrightarrow L(p)\,\big] \\ &\asymp \big((p-1/2)^{-4/3+o(1)}\big)^{-5/48+o(1)} = (p-1/2)^{5/36+o(1)}.\end{aligned}$$

The near-critical ensemble in FK(p, q)

Want a monotone coupling as p varies, i.e., random $Z \in [0,1]^{E(G)}$ labeling such that $Z_{\leqslant p} \subset E(G)$ is FK(p,q). Desirably Markov in p.

Harder than in percolation. Grimmett '95 showed its existence: defined a Markov chain Z_t on labelings with the right stationary measure. (Works only for $q \geqslant 1$.)

Another difference from percolation: from specific heat computation in the Ising model, density of edges in $Z_{\leqslant p_c+\epsilon} \setminus Z_{\leqslant p_c}$ is not $\asymp \epsilon$, but $\epsilon \log(1/\epsilon)$ for $q = 2$, and polynomial blowup for $q > 2$.

Onsager vs pivotals

From Onsager '44 and other Ising results: correlation length $\epsilon^{-1+o(1)}$, with a related but different definition, using correlation decay. I.e., $\tau(n) = n^{-1+o(1)}$ should be the window. But DC&G computed $\mathbf{E}|\mathsf{Piv}_n| = n^{13/24+o(1)}$, too few! And specific heat doesn't help enough.

Hence, correlation length is not given by amount of pivotals at criticality. Stability in near-critical window fails, the changes are faster. How come?

Conclusion: *Any* monotone coupling must be very strange: when raising p in the monotone coupling, open bonds do not arrive in a uniform, Poissonian way, but with self-organization, to create more pivotals and build long connections. Would contradict Markov property in p, unless there are clouds of open bonds appearing together.

We don't understand geometry of clouds, but at least can see directly in Grimmett's coupling that clouds do happen. Intuitively: good to open many edges together, without lowering number of clusters.