Exotica or the failure of the strong cosmic censorship in four dimensions

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The meaning and physical formulation of the strong cosmic censor conjecture (**SCCC**)

We have the strong conviction that in the *classical* physical world at least, every physical event (possibly except the Big Bang) has a cause which is another and preceding physical event. R. Penrose' original aim in the 1960-1970's with formulating the strong cosmic censor conjecture (or hypothesis) was to protect this concept of causality in general gravitational situations. Mathematically speaking space-times having this property are *globally hyperbolic* therefore

SCCC. *A physically relevant, generic space-time is globally hyperbolic.*

Remark

By a *space-time* we will always mean an *inextendible*, oriented and time-oriented Lorentzian 4-manifold (M, g).

(i) Without the "genericity" assumption the **SCCC** is trivially not true, almost all basic solutions (AdS, Taub–NUT, Gödel, Kerr–Newman, etc.) are counterexamples. But problem: what is "genericity"?

(ii) Conventional approach: using the initial value formulation "genericity" can be defined rigorously. Within this framework during the past five decades several partial results appeared and gave a support for the validity of the **SCCC** in certain classes of space-times (Gowdy space-times, space-times with compact Cauchy horizon, scalar field space-times, etc.).

A conceptional problem with the initial value approach to the **SCCC**

(i) In this approach “genericity” of three dimensional initial data are defined rigorously and considered. However, when thinking about the **SCCC**, certainly one should be able to talk about the “genericity” of the four dimensional space-time itself.

(ii) In light of discoveries in low dimensional differential topology during the 1980’s typical smooth 4-manifolds have unexpected properties (called exotica) not detectable from a 3-manifold perspective. Therefore one has to be caucious when applies the genuinely 3 dimensional initial value paradigm, investigated during the 1950-60’s, to genuinely 4 dimensional space-time problems.

The idea of a 4 dimensional counterexample

Definition

Let (M, g) be a maximal extension of the maximal Cauchy development $(D(S), g|_{D(S)})$ of an initial data set (S, h, k). The (continuous) Lorentzian manifold (M', g') is a perturbation of (M, g) relative to (S, h, k) if

(i) M' has the structure

$$M' := \text{the connected component of } M \setminus \mathscr{H} \text{ containing } S$$

where, for a connected open subset $S \subset U \subseteq M$ containing the initial surface, $\emptyset \subsetneqq \mathscr{H} \subsetneqq \partial U$ is a closed subset in the boundary of U;

(ii) (M', g') does not admit further extensions and $(S, h') \subset (M', g')$ with $h' := g'|_S$ is a spacelike complete sub-3-manifold.

Definition

Let (M, g) be a maximal extension of the maximal Cauchy development $(D(S), g|_{D(S)})$ of an initial data set (S, h, k). Then (M, g) is a robust counterexample to the **SCCC** if it is very stably non-globally hyperbolic i.e., all of its perturbations (M', g') relative to (S, h, k) are not globally hyperbolic.

Remark

(i) Perturbations of the whole 4 dimensional space-time are considered;

(ii) The concept of a robust counterexample is logically stronger than the "generic" one.

The Gompf–Taubes family of large exotic $\mathbb{R}^4$'s

Strongly motivated by the Bernal–Sánchez smooth splitting theorem (2003) and the Chernov–Nemirovski smooth censorship theorem (2013) we consider a family of fake $\mathbb{R}^4$'s:

Theorem (Gompf 1993, Taubes 1987)

There exists a pair (R^4, K) consisting of a differentiable 4-manifold R^4 homeomorphic but not diffeomorphic to the standard $\mathbb{R}^4$ and a compact oriented smooth 4-manifold $K \subset R^4$ such that

(i) *R^4 cannot be smoothly embedded into the standard $\mathbb{R}^4$ i.e., $R^4 \nsubseteq \mathbb{R}^4$ but it can be smoothly embedded as a proper open subset into the complex projective plane i.e., $R^4 \subsetneq \mathbb{C}P^2$;*

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(ii) *Take a homeomorphism $f : \mathbb{R}^4 \to R^4$, let $0 \in B^4_t \subset \mathbb{R}^4$ be the standard open 4-ball of radius $t \in \mathbb{R}^+$ centered at the origin and put $R^4_t := f(B^4_t)$ and $R^4_{+\infty} := R^4$. Then*

$$\left\{ R^4_t \;\middle|\; r \leqq t \leqq +\infty \text{ such that } 0 < r < +\infty \text{ satisfies } K \subset R^4_r \right\}$$

is an uncountable family of nondiffeomorphic exotic $\mathbb{R}^4$'s none of them admitting a smooth embedding into $\mathbb{R}^4$ i.e., $R^4_t \nsubseteqq \mathbb{R}^4$ for all $r \leqq t \leqq +\infty$. $\diamond$

Theorem (Etesi 2015)

The space R^4 carries a smooth Lorentzian Ricci-flat metric g. Moreover there exists an open (i.e., non-compact without boundary) contractible spacelike and complete sub-3-manifold $(S,h) \subset (R^4, g)$ in it such that $h = g|_S$. The Ricci-flat Lorentzian manifold (R^4, g) might be timelike and (or) null geodesically incomplete.

Idea of the proof.

(i) Apply Penrose' non-linear graviton construction (i.e., twistor theory) for the embedding $R^4 \subset \mathbb{C}P^2$ to obtain a hyper-Kähler metric g_1 on R^4;

(ii) Use Wick rotation (in a precise way) to get a Ricci-flat Lorentzian metric g on R^4. $\diamond$

Lemma (Etesi 2015)

Consider the pair (R^4, K) and the Ricci-flat Lorentzian manifold (R^4, g) with its open contractible spacelike and complete sub-3-manifold $(S, h) \subset (R^4, g)$. Let (S, h, k) be the initial data set inside (R^4, g) induced by (S, h) and let (M', g') be a perturbation of (R^4, g) relative to (S, h, k).
Assume that $K \subset M'$ holds. Then (M', g') is not globally hyperbolic.

Idea of the proof. Assume (M', g') was globally hyperbolic. Then $M' \cong \mathbb{R}^4$. Moreover there exists $0 < r \leqq t_0 \leqq +\infty$ such that

$$\begin{array}{ccccccccc} K & \subsetneqq & R^4_{t_0} & \subseteqq & M' & \subseteqq & R^4_{+\infty} & = & R^4 \\ & & & & \wr\| & & & & \\ & & & & \mathbb{R}^4 & & & & \end{array}$$

but we know $R^4_{t_0} \nsubseteqq \mathbb{R}^4$ and this is a contradiction. $\diamond$

Although (R^4, g) is not a robust counterexample as we defined, it is still very reasonable to consider the Ricci-flat Lorentzian space (R^4, g) as a generic counterexample to the **SCCC**.

Remark

The resolution of the apparent conflict between the initial value approach (providing affirmative answers for the **SCCC**) and this more global one (providing results against the **SCCC**) is that the initial value approach probes only the tubular neighbourhood of 3 dimensional spacelike submanifolds in the ambient 4 dimensional space-time hence cannot detect exotica at all!

For further details please check:

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http://www.math.bme.hu/~etesi/censor3.pdf

Thank you!