The complex structure on the six dimensional sphere

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Abstract

Proof of existence of at least one complex structure on the six-sphere, followed by an explicit computation of its underlying integrable almost complex tensor by the aid of inner automorphisms of the octonions, is exhibited. Both are elementary and self-contained however the size and complexity of the emerging almost complex tensor field on the six-sphere is perplexing.

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1 Introduction

Unlike their real counterparts, complex manifolds are quite rigid objects therefore constructing new examples is often not an easy task. Nevertheless there are two main islands, both within the *Kähler realm*, of the archipelago of complex manifolds comprising tractable cases. One is the category of *projective manifolds*, standing in the focal point of algebraic geometry. The other one is the category of *Stein manifolds* which, on the contrary, can be conveniently studied with the techniques of complex analysis in several variables. Although the in-between *terra incognita* is not so easily accessable, it is still quite populated: its habitants are the numerous irregular neither projective nor Stein—or even not Kähler—manifolds, either compact or non-compact. Examples in two complex dimensions are for example the various non-algebraic tori, Hopf surfaces, Inue surfaces, non-algebraic $K3$ surfaces, etc. (cf., e.g. [1, Chapter VI] for a survey of irregular complex surfaces) while in higher dimensions the picture is not so clear yet. Nevertheless in all dimensions the deviation, at least from the algebraic scenario, is captured in some extent by the concept of the algebraic dimension of a complex manifold i.e. the transcendental degree over $\mathbb{C}$ of its field of global meromorphic functions. Compact complex manifolds whose algebraic and geometric dimensions match (sometimes called *Moishezon manifolds*) are still not far from being algebraic: it is known that after performing finitely many blowing-ups they become projective algebraic. However in general the lower the algebraic dimension of a manifold is the larger its detachment is from the familiar world of complex algebraic manifolds.

An interesting family of higher dimensional non-algebraic examples relevant to us here is based on even dimensional compact Lie groups. *Samelson* discovered [14] that they carry complex structures

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which are surely non-Kähler if the underlying group is simply connected and simple. It is guessed therefore that in fact they are as far from being algebraic as possible: more precisely we *conjecture* that the algebraic dimension of an even dimensional compact, connected, simply connected and simple Lie group as a complex manifold is always zero. It also has been known for some time that if the six dimensional sphere carries a complex structure then this compact complex 3-manifold must be of zero algebraic dimension [2].

In this paper we shall construct explicitly at least one complex structure on the six dimensional sphere. The proof is based on identifying S^6 with an exceptional conjugate orbit in the exceptional compact Lie group G_2, taking its explicit deformation within $G_2^{\mathbb{C}}$ and then restricting a Samelson complex structure to this deformation. The proof to be presented here is elementary and self-contained hence is independent of our former Yang–Mills–Higgs theoretic approach [5]; nevertheless those considerations definitely have been used here as a source of ideas. We just note that meanwhile the treatment in [5] is based on the well-known $SU(3)$-fibration: the *projection* (i.e. a surjective mapping) $\pi : G_2 \to S^6$, our present proof rests on a certain less-known but very remarkable *injection* $f : S^6 \to G_2$. Therefore the two approaches are dual to each other in this sense.

The paper is organized as follows. Sect. 2 consists of a self-contained and elementary proof that S^6 can carry complex structures (cf. Theorem 2.1). Then we make a contact between known results on the Dolbeault cohomologies of a complex structure on G_2 and on S^6: certain Dolbeault cohomology groups of G_2 and of S^6 are both non-trivial and are isomorphic via $S^6 \subset G_2$ considered as a complex-analytic embedding (cf. Lemma 2.1). It also has been known for a long time (cf. e.g. [9, 7]) that if S^6 carries a complex structure then there exist "exotic $\mathbb{C}P^3$'s" i.e. complex manifolds diffeomoprhic to $\mathbb{C}P^3$ as real 6-manifolds however not complex-analytically isomorphic to it. Here we prove another striking consequence: the complex sructure on S^6 implies the existence of "large exotic $\mathbb{C}^3$'s" in a similar sense (cf. Lemma 2.2). By a *large exotic* $\mathbb{C}^3$ we mean a complex manifold which is diffeomorphic to $\mathbb{R}^6$ however is not complex analytically equivalent to the standard $\mathbb{C}^3$ moreover does not admit a complex-analytic embedding into the standard $\mathbb{C}^3$ (the failure of the higher dimensional analogue of the Riemannian mapping theorem implies the existence of an abundance of *small exotic* $\mathbb{C}^3$ *'s* i.e. open complex analytic subsets of the standard $\mathbb{C}^3$ not complex-analytically equivalent to it).

In Sect. 3 we outline the explicit construction of the integrable almost complex tensor field on S^6 underlying its complex structure. The construction is based on identifying the original conjugate orbit of G_2, homeomorphic to S^6, with the subset of *inner automorphisms* within the full automorphism group of the octonions which is G_2 (cf. Lemma 3.1 as well as [3]) and then taking its explicit perturbation within $G_2^{\mathbb{C}}$ (cf. Lemma 3.2). The integrable Samelson almost complex structures then restrict to this perturbation rendering S^6 a complex manifold. The components of the underlying almost complex tensor field considered as local 6×6 matrix functions in principle drop out explicitly from this construction however the result is so unexpectedly complicated that we cannot display it fully here. Nevertheless the steps towards its construction are clearly explained and the curious reader can reproduce the calculations (using a computer is strongly advised) by himself or even go further and bring these matrices into a more digestable form. But already the present form allows to conclude that all complex structures constructed here are equivalent.

Finally Sect. 4 is an appendix and has been added in order to gain a more comprehensive picture. Following [3] we re-prove that the conjugate orbit of G_2 playing the central role here, when regarded as a continuous map $f : S^6 \to G_2$, represents the generator of $\pi_6(G_2) \cong \mathbb{Z}_3$ (cf. Theorem 4.1).

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2 Proof of existence

In this section we present a proof that the six dimensional sphere carries complex structures. The proof is based on identifying S^6 with a geometrically deformed conjugate orbit inside the complexified exceptional compact Lie group $\mathrm{G}_2^{\mathbb{C}}$ and then restricting the Samelson complex structure of G_2 to this deformed orbit. The proof to be presented here is elementary and self-contained hence is independent of our former Yang–Mills–Higgs theoretic approach [5].

Recall [13, 14] that if G is an even dimensional compact real Lie group with real Lie algebra $\mathfrak{g}$ the complex linear subspace $\mathfrak{s}\subset\mathfrak{g}^{\mathbb{C}}:=\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ is a *Samelson subalgebra* if it is a complex Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$ satisfying $\dim_{\mathbb{C}}\mathfrak{s}=\frac{1}{2}\dim_{\mathbb{C}}\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{s}\cap\mathfrak{g}=0$ as real subspaces within $\mathfrak{g}^{\mathbb{C}}$. One can demonstrate that at least one Samelson subalgebra in $\mathfrak{g}^{\mathbb{C}}$ always exists if $\mathfrak{g}$ is the Lie algebra of a *compact* even dimensional Lie group. A choice for a Samelson subalgebra gives rise to a vector space decomposition $\mathfrak{g}^{\mathbb{C}}=\mathfrak{s}\oplus\overline{\mathfrak{s}}$ hence the existence of a real linear isomorphism $\mathrm{Re}:\mathfrak{s}\to\mathfrak{g}$ given by $W\mapsto\mathrm{Re}W$ for all $W\in\mathfrak{s}$ and a real linear map $J_{\mathfrak{s}}:\mathfrak{g}\to\mathfrak{g}$ with $J_{\mathfrak{s}}(\mathrm{Re}W):=-\mathrm{Im}W$ satisfying $J_{\mathfrak{s}}^2=-\mathrm{Id}_{\mathfrak{g}}$. Consequently a choice of a Samelson subalgebra gives rise to a complex vector space $(\mathfrak{g},J_{\mathfrak{s}})$ such that $\mathfrak{g}^{1,0}:=\mathfrak{s}$ is the $+\sqrt{-1}$-eigenspace while $\mathfrak{g}^{0,1}:=\overline{\mathfrak{s}}$ is the $-\sqrt{-1}$-eigenspace of the complexified map $J_{\mathfrak{s}}^{\mathbb{C}}:\mathfrak{g}^{\mathbb{C}}\to\mathfrak{g}^{\mathbb{C}}$. The operator $J_{\mathfrak{s}}=J_{\mathfrak{s},e}$ on $\mathfrak{g}=T_eG$ can be extended to a left-invariant almost complex structure $J_{\mathfrak{s}}$ over the whole G by putting $J_{\mathfrak{s},g}:T_gG\to T_gG$ to be $J_{\mathfrak{s},g}:=L_{g\,*}J_{\mathfrak{s},e}L_{g\,*}^{-1}$. In this way we come up with an almost complex manifold $(G,J_{\mathfrak{s}})$. The Lie algebra property of $\mathfrak{g}^{1,0}$ has not been exploited so far: Taking into account $\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}^{1,0}\oplus\mathfrak{g}^{0,1}$ as well, it additionally tells us that

$$\left[\mathfrak{g}^{1,0},\,\mathfrak{g}^{1,0}\right]^{0,1}=0$$

also holds. The identification $X\mapsto X^{1,0}:=\frac{1}{2}(X-\sqrt{-1}\,J_{\mathfrak{s}}X)$ of real vector fields with $(1,0)$-type complex ones over G maps left-invariant real fields into left-invariant $(1,0)$-type ones and these latter fields can be viewed as $(1,0)$-type complex Lie algebra elements. Observing that $(X,Y)\mapsto[X^{1,0},Y^{1,0}]^{0,1}$ is $C^\infty(G;\mathbb{R})$-bilinear we recognize that the vanishing commutator above actually says that the Nijenhuis tensor of $J_{\mathfrak{s}}$ is zero. Consequently $(G,J_{\mathfrak{s}})$ is integrable to a homogeneous complex manifold $Y_{\mathfrak{s}}$ in light of the Newlander–Nirenberg theorem. Another way to see this is to take the corresponding complex Lie subgroup $S\subset G^{\mathbb{C}}$ of $\mathfrak{s}\subset\mathfrak{g}^{\mathbb{C}}$; then $Y_{\mathfrak{s}}=G^{\mathbb{C}}/S$ as a complex manifold because this quotient as a real manifold is diffeomorphic to G, essentially because $\mathfrak{s}\cap\mathfrak{g}=0$ holds in $\mathfrak{g}^{\mathbb{C}}$, cf. [13, Proposition 2.3].

After these general considerations take the 14 dimensional real compact exceptional Lie group G_2. As we outlined above it can be given the structure of a compact complex 7-manifold; more precisely referring to [13, Example on p. 123] we know[1] that if $\mathfrak{h}\subset\mathfrak{g}_2$ is the Cartan subalgebra with its complexification $\mathfrak{h}^{\mathbb{C}}\subset\mathfrak{g}_2^{\mathbb{C}}$ then for every $u\in P(\mathfrak{h}^{\mathbb{C}})\setminus P(\mathfrak{h})$ i.e. the projectivization of $\mathfrak{h}^{\mathbb{C}}$ with that of its real part removed, there exist Samelson subalgebras $\mathfrak{s}_u\subset\mathfrak{g}_2^{\mathbb{C}}$ with corresponding mutually non-isomorphic compact homogeneous complex 7-manifolds Y_u whose underlying real spaces are all diffeomorphic to G_2. This Lie group has a maximal subgroup isomorphic to $\mathrm{SU}(3)$ and let $\Lambda\in\mathrm{G}_2$ be the generator of its center i.e. $\Lambda\in Z(\mathrm{SU}(3))\subset\mathrm{SU}(3)\subset\mathrm{G}_2$ is the generator, hence $\Lambda\neq e$ but $\Lambda^3=e$. Strongly motivated by [3, 4] we begin with considering the exceptional *conjugate orbit*

$$O(\Lambda):=\left\{g\Lambda g^{-1}\,|\,g\in\mathrm{G}_2\right\}\tag{1}$$

passing through this generator. It is a real submanifold of G_2 diffeomorphic to $\mathrm{G}_2/\mathrm{SU}(3)\cong S^6$. Let $\mathrm{G}_2^{\mathbb{C}}$ be the complexification of G_2. Our real conjugate orbit $O(\Lambda)\subset\mathrm{G}_2$ also complexifies in an obvious way to the complexified conjugate orbit

$$O(\Lambda)^{\mathbb{C}}=\{g\Lambda g^{-1}\,|\,g\in\mathrm{G}_2^{\mathbb{C}}\}\ .$$

[1] But we will also recover this result explicitly in Sect. 3.

By compactness of G_2 there exists a diffeomorphism $\mathrm{G}_2^{\mathbb{C}} \cong T\mathrm{G}_2$ as a real manifold hence we can use the zero section of the tangent bundle to write $\mathrm{G}_2 \subset \mathrm{G}_2^{\mathbb{C}}$. This by restriction gives $O(\Lambda) \subset O(\Lambda)^{\mathbb{C}}$ and the real isomorphism above implies $O(\Lambda)^{\mathbb{C}} \cong TO(\Lambda)$. Note that $O(\Lambda)^{\mathbb{C}}$ is complex-analytically isomorphic to $\mathrm{G}_2^{\mathbb{C}}/\mathrm{SU}(3)^{\mathbb{C}} \cong (S^6)^{\mathbb{C}}$ where $(S^6)^{\mathbb{C}} \subset \mathbb{C}^7$ is the "complex 6-sphere" whose points satisfy $z_1^2+\cdots+z_7^2 = 1$ with $z_1,\dots,z_7 \in \mathbb{C}$. Therefore our embeddings are in accordance with the classical fact that $(S^6)^{\mathbb{C}} \cong TS^6$ as a real manifold and with the existence of an embedding $S^6 \subset (S^6)^{\mathbb{C}}$ by the zero section. To summarize, there exists a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{G}_2 & \subset & \mathrm{G}_2^{\mathbb{C}} & \cong & T\mathrm{G}_2 & \cong & \mathrm{G}_2\times\mathfrak{g}_2 \\
\cup & & \cup & & \cup & & \\
O(\Lambda) & \subset & O(\Lambda)^{\mathbb{C}} & \cong & TO(\Lambda) & & \\
\wr\| & & \wr\| & & \wr\| & & \\
S^6 & \subset & (S^6)^{\mathbb{C}} & \cong & TS^6 & &
\end{array} \tag{2}$$

of compatible embeddings and isomorphisms.

Next we perform a complex deformation of $O(\Lambda) \subset \mathrm{G}_2^{\mathbb{C}}$. Take a fixed Samelson subalgebra $\mathfrak{s}_u \subset \mathfrak{g}_2^{\mathbb{C}}$ with corresponding complex analytic Lie subgroup $S_u \subset \mathrm{G}_2^{\mathbb{C}}$ and likewise take their complex conjugate counterparts $\overline{\mathfrak{s}}_u \subset \mathfrak{g}_2^{\mathbb{C}}$ and $\overline{S}_u \subset \mathrm{G}_2^{\mathbb{C}}$. Consider a new complex analytic submanifold

$$O'(\Lambda)^{\mathbb{C}} := \left\{ s_1\Lambda^2 s_1^{-1} s_2\Lambda^2 s_2^{-1} \,\middle|\, s_1 \in S_u\,,\, s_2 \in \overline{S}_u \right\} \tag{3}$$

inside $\mathrm{G}_2^{\mathbb{C}}$ and its "real part" $O'(\Lambda) \subset O'(\Lambda)^{\mathbb{C}}$ defined by

$$O'(\Lambda) := \left\{ s\Lambda^2 s^{-1}\, \overline{s}\Lambda^2\overline{s}^{-1} \,\middle|\, s \in S_u \right\} \tag{4}$$

and regarded as a *deformed conjugate orbit* through Λ (note that $\Lambda \in O'(\Lambda)$ continues to hold). We assert that $O'(\Lambda)$ is homeomorphic to S^6. To prove this first we observe that $O(\Lambda)^{\mathbb{C}}$ and $O'(\Lambda)^{\mathbb{C}}$ are isotopic in $\mathrm{G}_2^{\mathbb{C}}$. Indeed, the decomposition $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{s}_u \oplus \overline{\mathfrak{s}}_u$ implies $\mathrm{G}_2^{\mathbb{C}} \supset S_u\overline{S}_u$ is open and $\mathrm{G}_2^{\mathbb{C}} \supset S_u \cap \overline{S}_u$ is discrete; thus $S_u\Lambda^2 S_u^{-1} \cap \overline{S}_u\Lambda^2\overline{S}_u^{-1} \subset \mathrm{G}_2^{\mathbb{C}}$ is discrete and since both $S_u \cap \mathrm{SU}(3)^{\mathbb{C}}$ and $\overline{S}_u \cap \mathrm{SU}(3)^{\mathbb{C}}$ are 4 dimensional and Λ^2 commutes with them $\dim_{\mathbb{C}} O'(\Lambda)^{\mathbb{C}} = 2\cdot(7-4) = 6$. Take the 1-parameter subgroups $\sigma_1 : \mathbb{R} \to S_u$ and $\sigma_2 : \mathbb{R} \to \overline{S}_u$ satisfying $\sigma_i(0) = e$ and $\sigma_i(1) = s_i$. Observe that S_u is solvable [13, Proposition 2.4] and simply connected hence $\exp : \mathfrak{s}_u \to S_u$ is a diffeomorphism [17, Theorem 3.18.11] and likewise for $\overline{S}_u$. Consequently σ_i uniquely exist thus for every $n \in \mathbb{N}$ and s_i one can unambigously put $s_i^{1/n} := \sigma_i(\frac{1}{n})$ and $S_u^{1/n} := \{s_1^{1/n} \,|\, s_1 \in S_u\}$ and likewise for $\overline{S}_u^{1/n}$. Furthermore $\mathrm{G}_2^{\mathbb{C}}$ is a complex simple group with trivial center hence $\exp : \mathfrak{g}_2^{\mathbb{C}} \to \mathrm{G}_2^{\mathbb{C}}$ is surjective [12, Corollary 3.4] but not injective. Consequently, recalling Trotter's formula $\exp(X_1+X_2) = \lim\limits_n \left(\exp\frac{X_1}{n}\exp\frac{X_2}{n}\right)^n$ from [17, Corollary 2.12.5], any $g \in \mathrm{G}_2^{\mathbb{C}}$ can be written as $g = \lim\limits_n \left(s_1^{1/n}s_2^{1/n}\right)^n$ that is, $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{s}_u \oplus \overline{\mathfrak{s}}_u$ globalizes to $\mathrm{G}_2^{\mathbb{C}} = \lim\limits_n \left(S_u^{1/n}\ \overline{S}_u^{1/n}\right)^n$ but in a non-unique way. However uniqueness is achieved for the quotient $\mathrm{G}_2^{\mathbb{C}}/\mathrm{SU}(3)^{\mathbb{C}}$. Indeed, the subgroup $\mathrm{SU}(3)^{\mathbb{C}} \subset \mathrm{G}_2^{\mathbb{C}}$ has maximal rank thus the local model $\mathfrak{g}_2^{\mathbb{C}}/\mathfrak{su}(3)^{\mathbb{C}}$ shows that if $a \in \mathrm{G}_2^{\mathbb{C}}$ is close to the unit and $X, Y \in \exp^{-1}(a\mathrm{SU}(3)^{\mathbb{C}})$ then $X + \mathfrak{su}(3)^{\mathbb{C}} = Y + \mathfrak{su}(3)^{\mathbb{C}}$ i.e., $Y - X \in \mathfrak{su}(3)^{\mathbb{C}}$; thus to *any* coset $g\mathrm{SU}(3)^{\mathbb{C}}$ one can *uniquely* find $s_1(S_u \cap \mathrm{SU}(3)^{\mathbb{C}})$ and $s_2(\overline{S}_u \cap \mathrm{SU}(3)^{\mathbb{C}})$ such that $g\mathrm{SU}(3)^{\mathbb{C}} = \lim\limits_n \left(s_1^{1/n}(S_u\cap\mathrm{SU}(3)^{\mathbb{C}})^{1/n}\, s_2^{1/n}(\overline{S}_u\cap\mathrm{SU}(3)^{\mathbb{C}})^{1/n}\right)^n$. In this way the assignment

$$t \longmapsto \lim_n \left(\sigma_1(\tfrac{1}{n})\sigma_2(\tfrac{1-t}{n})\right)^n \Lambda^2 \lim_n \left(\sigma_1(\tfrac{1}{n})\sigma_2(\tfrac{1-t}{n})\right)^{-n} \lim_n \left(\sigma_1(\tfrac{1-t}{n})\sigma_2(\tfrac{1}{n})\right)^n \Lambda^2 \lim_n \left(\sigma_1(\tfrac{1-t}{n})\sigma_2(\tfrac{1}{n})\right)^{-n}$$

is single-valued and $\mathrm{SU}(3)^{\mathbb{C}}$-invariant consequently gives rise to a well-defined isotopy connecting $g\Lambda^2g^{-1}g\Lambda^2g^{-1} = g\Lambda g^{-1} \in O(\Lambda)^{\mathbb{C}}$ at $t = 0$ with $s_1\Lambda^2 s_1^{-1}\, s_2\Lambda^2 s_2^{-1} \in O'(\Lambda)^{\mathbb{C}}$ at $t = 1$ and *vice versa.*

Clearly $\Lambda \in O(\Lambda)^{\mathbb{C}} \cap O'(\Lambda)^{\mathbb{C}}$ is kept fixed during this deformation. Concerning the assertion itself now, first observe that $g = \lim\limits_n \left(s^{1/n}\,\bar{s}^{1/n}\right)^n$ with any $s \in S_u$ is a real element i.e. $g \in \mathrm{G}_2 \subset \mathrm{G}_2^{\mathbb{C}}$. Thus the assertion follows since the homeomorphic image of $O(\Lambda) \subset \mathrm{G}_2$ as in (1) by the isotopy is precisely $O'(\Lambda) \subset \mathrm{G}_2^{\mathbb{C}}$ given by (4); however $O(\Lambda)$ is homeomorphic to S^6 hence so is $O'(\Lambda)$. As a global aspect note that the embedding of S^6 provided by $O(\Lambda) \subset \mathrm{G}_2$ hence by $O'(\Lambda) \subset \mathrm{G}_2^{\mathbb{C}}$ is not null-homotopic (cf. Theorem 4.1) and obviously the square of the corresponding homotopy class is represented by the twin conjugate orbit $O(\Lambda^2) \subset \mathrm{G}_2$ while its third power by $O(\Lambda^3) = e \in \mathrm{G}_2$ i.e. it is already trivial.

Remark. Any $g \in O'(\Lambda) \subset \mathrm{G}_2^{\mathbb{C}}$ admits a unique polar decomposition $g = RU$ where $U \in \mathrm{U}(7) \cap \mathrm{G}_2^{\mathbb{C}} = \mathrm{G}_2$ hence $U \mapsto RU$ identifies the perturbed orbit as defined in (4) with the image of a partially defined smooth section from G_2 into $\mathrm{G}_2^{\mathbb{C}} \cong \mathrm{G}_2 \times \mathfrak{g}_2$. Picking for instance the cell decomposition

$$\mathrm{G}_2 = e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}$$

found in [11] we can assume that this partial section is defined along its 6-skeleton; taking into account the ball content of this cell decomposition and that all the homotopy groups of $\mathfrak{g}_2$ are trivial obstruction theory says that without altering its already defined part we can smoothly extend this partial section to a global one $\sigma_u : \mathrm{G}_2 \to \mathrm{G}_2^{\mathbb{C}}$. By construction $O'(\Lambda) \subset \sigma_u(\mathrm{G}_2)$ and by exploiting the non-uniqueness of the extension we can assume that σ_u is *transversal* to the fibers in the sense that $T_g\sigma_u(\mathrm{G}_2) \cap L_{g*}\mathfrak{s}_u = 0$ within $T_g\sigma_u(\mathrm{G}_2) \otimes_{\mathbb{R}} \mathbb{C} \cong T_g\mathrm{G}_2^{\mathbb{C}}$ over all $g \in \sigma_u(\mathrm{G}_2)$. Transversality then ensures us that the smooth submanifold $\sigma_u(\mathrm{G}_2) \subset \mathrm{G}_2^{\mathbb{C}}$ continues to be integrable *á la* Samelson. This construction is the straightforward generalization of the basic one yielding Y_u which construction, as summarized in (2), corresponds to the case of the trivial section $\tau(U) = U$. The perturbed one given by $\sigma_u(U) = RU$ simply induces an isomorphism of Y_u. In what follows for definiteness Y_u will denote the complex manifold which arises by applying Samelson's construction, as summarized above, to any $\sigma_u(\mathrm{G}_2) \subset \mathrm{G}_2^{\mathbb{C}}$ containing $O'(\Lambda)$.

After these preliminary constructions and considerations we are ready to state

Theorem 2.1. *Take the family $\mathfrak{s}_u \subset \mathfrak{g}_2^{\mathbb{C}}$ with $u \in P(\mathfrak{h}^{\mathbb{C}}) \setminus P(\mathfrak{h})$ of Samelson subalgebras such that the corresponding compact complex 7-manifolds Y_u are all diffeomorphic to G_2.*

Then for every moduli parameter u the complex structure of Y_u induces a complex structure on the deformed conjugate orbit $O'(\Lambda) \subset \mathrm{G}_2^{\mathbb{C}}$ rendering S^6 a compact complex 3-manifold $X_u \subset Y_u$.

Proof. The strategy is simple: we check that the left-translated and intersected Samelson splittings $\left(L_{g*}\mathfrak{s}_u \oplus L_{g*}\bar{\mathfrak{s}}_u\right) \cap T_gO'(\Lambda)^{\mathbb{C}}$ are nice for every $g \in O'(\Lambda)$ hence via $T_gO'(\Lambda)^{\mathbb{C}} = T_gO'(\Lambda) \otimes_{\mathbb{R}} \mathbb{C}$ these splittings induce complex structures on $T_gO'(\Lambda)$ in the standard way which are integrable along $O'(\Lambda)$.

The standard root basis in $\mathfrak{g}_2^{\mathbb{C}}$ looks like[2]

$$\{H_{\pm a,b}, V_{\pm 1}, V_{\pm 2}, V_{\pm 3}, U_{\pm 1}, U_{\pm 2}, U_{\pm 3}\} \tag{5}$$

where a, b are real parameters with $a \neq 0$. It has the following pleasant three properties: the first is that $\{H_{\pm a,b}\}$ span the complexified Cartan subalgebra $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}_2^{\mathbb{C}}$; secondly $\{H_{\pm a,b}, V_{\pm k}\}$ with $k = 1,2,3$ span $\mathfrak{su}(3)^{\mathbb{C}} \subset \mathfrak{g}_2^{\mathbb{C}}$; thirdly $\{H_{+a,b}, V_{+1}, V_{-2}, V_{-3}, U_{+1}, U_{-2}, U_{+3}\}$ span all the family of Samelson subalgebras $\mathfrak{s}_u \subset \mathfrak{g}_2^{\mathbb{C}}$ provided $P(\mathfrak{h}^{\mathbb{C}}) \setminus P(\mathfrak{h}) \cong \mathbb{C}P^1 \setminus \mathbb{R}P^1 \cong \mathbb{C} \setminus \sqrt{-1}\,\mathbb{R} \ni u = a + \sqrt{-1}\,b$ i.e. $a \neq 0$. Moreover $\overline{H}_{+a,b} = -H_{-a,b}$, $\overline{V}_{+k} = V_{-k}$ and $\overline{U}_{+k} = U_{-k}$ hence the remaining basis elements span the complex conjugate subalgebra $\bar{\mathfrak{s}}_u \subset \mathfrak{g}_2^{\mathbb{C}}$. Finally we note that there exist precisely two orthogonal (with respect to the Ad-invariant metric on G_2) complex structures at $u_\pm = \pm\frac{\sqrt{3}}{2} - \frac{\sqrt{-1}}{2}$.

[2] An explicit matrix representation of the members of this basis will be exhibited soon in Sect. 3; therefore the interested reader can check all assertions about this basis by hand.

With respect to this basis consider the usual vector space decomposition $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{su}(3)^{\mathbb{C}} \oplus \mathfrak{m}$. Intersecting it with $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{s}_u \oplus \overline{\mathfrak{s}}_u$ we obtain $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{s}_u \cap \mathfrak{su}(3)^{\mathbb{C}} \oplus \mathfrak{s}_u \cap \mathfrak{m} \oplus \overline{\mathfrak{s}}_u \cap \mathfrak{m} \oplus \overline{\mathfrak{s}}_u \cap \mathfrak{su}(3)^{\mathbb{C}}$ implying a splitting $\mathfrak{m} = \mathfrak{s}_u \cap \mathfrak{m} \oplus \overline{\mathfrak{s}}_u \cap \mathfrak{m}$. It is straightforward that $\mathfrak{m}$ is spanned by $\{U_{\pm k}\}$ with $k = 1,2,3$ therefore $\mathfrak{s}_u \cap \mathfrak{m}$ is spanned by $\{U_{+1}, U_{-2}, U_{+3}\}$ and likewise $\overline{\mathfrak{s}}_u \cap \mathfrak{m}$ by $\{U_{-1}, U_{+2}, U_{-3}\}$. Thus we find that $\dim_{\mathbb{C}}\left(\mathfrak{s}_u \cap \mathfrak{m}\right) = 3 = \dim_{\mathbb{C}}\left(\overline{\mathfrak{s}}_u \cap \mathfrak{m}\right)$ for all moduli parameters u.

Let $s_1(t)$ be a smooth curve in S_u with the property $s_1(0) = s_1$ and likewise $s_2(t)$ in $\overline{S}_u$ such that $s_2(0) = s_2$. Then a generic curve in $S_u\Lambda^2 S_u^{-1} \subset \mathrm{G}_2^{\mathbb{C}}$ passing through $s_1\Lambda^2 s_1^{-1}$ looks like $s_1(t)\Lambda^2 s_1^{-1}(t)$ and similarly for the curve $s_2(t)\Lambda^2 s_2^{-1}(t)$. Also let $L \in \mathfrak{g}_2^{\mathbb{C}}$ be an element satisfying $\Lambda = \exp L$ hence $\Lambda^2 = \Lambda^{-1} = \exp(-L)$. Then $s_i(t)\Lambda^2 s_i^{-1}(t) = \exp(-s_i(t)Ls_i^{-1}(t))$ for $i = 1,2$. Let us compute the tangent vectors of these curves in the standard way:

$$
\begin{aligned}
L^{-1}_{s_i\Lambda^2 s_i^{-1}*}\left(\frac{\mathrm{d}}{\mathrm{d}t}\exp\left(-s_i(t)Ls_i^{-1}(t)\right)\Big|_{t=0}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(s_iLs_i^{-1})\exp\left(-s_i(t)Ls_i^{-1}(t)\right)\right)\Big|_{t=0}\\
&= \exp(s_iLs_i^{-1})\exp_*(-s_iLs_i^{-1})\frac{\mathrm{d}}{\mathrm{d}t}\left(-s_i(t)Ls_i^{-1}(t)\right)\Big|_{t=0}\\
&= \frac{1-\mathrm{e}^{-\mathrm{ad}_{-s_iLs_i^{-1}}}}{\mathrm{ad}_{-s_iLs_i^{-1}}}\left[R^{-1}_{s_i*}\dot{s}_i\,,\,-s_iLs_i^{-1}\right]\\
&= s_i\left(\frac{\mathrm{e}^{\mathrm{ad}_L}-1}{\mathrm{ad}_L}\,\mathrm{ad}_L\left(L^{-1}_{s_i*}\dot{s}_i\right)\right)s_i^{-1}\\
&= s_i\left(\left(\mathrm{e}^{\mathrm{ad}_L}-1\right)L^{-1}_{s_i*}\dot{s}_i\right)s_i^{-1}\;.
\end{aligned}
$$

Obviously $L^{-1}_{s_1*}\dot{s}_1 \in \mathfrak{s}_u$ and $L^{-1}_{s_2*}\dot{s}_2 \in \overline{\mathfrak{s}}_u$. Moreover $L \in \mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}_2^{\mathbb{C}}$ i.e. it is from the Cartan subalgebra hence ad_L acts diagonally on (5) thus $\left(\mathrm{e}^{\mathrm{ad}_L}-1\right)L^{-1}_{s_1*}\dot{s}_1 \in \mathfrak{s}_u$ and $\left(\mathrm{e}^{\mathrm{ad}_L}-1\right)L^{-1}_{s_2*}\dot{s}_2 \in \overline{\mathfrak{s}}_u$ too. The additional datum that $\exp L \in Z(\mathrm{SU}(3)^{\mathbb{C}})$ therefore $\mathrm{ad}_L(\mathfrak{s}_u \cap \mathfrak{su}(3)^{\mathbb{C}}) = 0 = \mathrm{ad}_L(\overline{\mathfrak{s}}_u \cap \mathfrak{su}(3)^{\mathbb{C}})$ implies that the right hand side in fact belongs to $s_1\left(\mathfrak{s}_u \cap \mathfrak{m}\right)s_1^{-1} \subset \mathfrak{s}_u$ or $s_2\left(\overline{\mathfrak{s}}_u \cap \mathfrak{m}\right)s_2^{-1} \subset \overline{\mathfrak{s}}_u$ respectively. For simplicity write $p_i := s_i\Lambda^2 s_i^{-1}$ moreover $P_1 := S_u\Lambda^2 S_u^{-1}$ and $P_2 := \overline{S}_u\Lambda^2\overline{S}_u^{-1}$. Thus we find that $T_{p_1}P_1 = L_{p_1*}\mathrm{Ad}_{s_1}\left(\mathfrak{s}_u \cap \mathfrak{m}\right)$ and $T_{p_2}P_2 = L_{p_2*}\mathrm{Ad}_{s_2}\left(\overline{\mathfrak{s}}_u \cap \mathfrak{m}\right)$. Recalling that $O'(\Lambda)^{\mathbb{C}} = P_1P_2$ such that $P_1 \cap P_2$ is discrete we can apply Leibniz's rule and then insert these identities to split the tangent space

$$
T_{p_1p_2}O'(\Lambda)^{\mathbb{C}} = R_{p_2*}T_{p_1}P_1 \oplus L_{p_1*}T_{p_2}P_2 = L_{p_1*}R_{p_2*}\mathrm{Ad}_{s_1}(\mathfrak{s}_u \cap \mathfrak{m}) \oplus L_{p_1*}L_{p_2*}\mathrm{Ad}_{s_2}(\overline{\mathfrak{s}}_u \cap \mathfrak{m})\;.
$$

Put $s_1 = s = \overline{s}_2$ hence $p_1 = p = \overline{p}_2$ yielding $T_{p\overline{p}}O'(\Lambda)^{\mathbb{C}} = L_{p*}R_{\overline{p}*}\mathrm{Ad}_s(\mathfrak{s}_u \cap \mathfrak{m}) \oplus L_{p\overline{p}*}\mathrm{Ad}_{\overline{s}}(\overline{\mathfrak{s}}_u \cap \mathfrak{m})$. From (4) we know that $p\overline{p} \in O'(\Lambda)$ is a real point such that the realification of e.g. the complex summand $L_{p\overline{p}*}\mathrm{Ad}_{\overline{s}}(\overline{\mathfrak{s}}_u \cap \mathfrak{m}) \subset T_{p\overline{p}}O'(\Lambda)^{\mathbb{C}}$ is the real tangent space $T_{p\overline{p}}O'(\Lambda)$ whose complexification gives back again $T_{p\overline{p}}O'(\Lambda)^{\mathbb{C}}$. For a complex subspace $V \subsetneqq T_g\mathrm{G}_2^{\mathbb{C}}$ writing its complex conjugate within $T_g\mathrm{G}_2^{\mathbb{C}}$ as $\overline{V}^g := L_{g*}\overline{L^{-1}_{g*}V}$ then with $V := L_{p\overline{p}*}\mathrm{Ad}_{\overline{s}}(\overline{\mathfrak{s}}_u \cap \mathfrak{m})$ we obtain $T_{p\overline{p}}O'(\Lambda)^{\mathbb{C}} = (V^{\mathbb{R}})^{\mathbb{C}} = \overline{V}^{p\overline{p}} \oplus V$. Therefore the asymmetric splitting constructed above in the real case induces a new and symmetric one

$$
T_{p\overline{p}}O'(\Lambda)^{\mathbb{C}} = L_{p\overline{p}*}\overline{L^{-1}_{p\overline{p}*}L_{p\overline{p}*}\mathrm{Ad}_{\overline{s}}(\overline{\mathfrak{s}}_u \cap \mathfrak{m})} \oplus L_{p\overline{p}*}\mathrm{Ad}_{\overline{s}}(\overline{\mathfrak{s}}_u \cap \mathfrak{m}) = L_{p\overline{p}*}\mathrm{Ad}_s(\mathfrak{s}_u \cap \mathfrak{m}) \oplus L_{p\overline{p}*}\mathrm{Ad}_{\overline{s}}(\overline{\mathfrak{s}}_u \cap \mathfrak{m})
$$

moreover readily $L_{p\overline{p}*}\mathrm{Ad}_s(\mathfrak{s}_u \cap \mathfrak{m}) = L_{p\overline{p}*}\mathfrak{s}_u \cap T_{p\overline{p}}O'(\Lambda)^{\mathbb{C}}$ and $L_{p\overline{p}*}\mathrm{Ad}_{\overline{s}}(\overline{\mathfrak{s}}_u \cap \mathfrak{m}) = L_{p\overline{p}*}\overline{\mathfrak{s}}_u \cap T_{p\overline{p}}O'(\Lambda)^{\mathbb{C}}$. Thus summing up all of our findings so far we conclude that

$$
T_gO'(\Lambda)^{\mathbb{C}} = L_{g*}\mathfrak{s}_u \cap T_gO'(\Lambda)^{\mathbb{C}} \oplus L_{g*}\overline{\mathfrak{s}}_u \cap T_gO'(\Lambda)^{\mathbb{C}} \tag{6}
$$

is a left-invariant decomposition with $\dim_{\mathbb{C}}(L_{g*}\mathfrak{s}_u \cap T_gO'(\Lambda)^{\mathbb{C}}) = 3 = \dim_{\mathbb{C}}(L_{g*}\overline{\mathfrak{s}}_u \cap T_gO'(\Lambda)^{\mathbb{C}})$ over all $g \in O'(\Lambda)$ and for all Samelson moduli parameters $u \in P(\mathfrak{h}^{\mathbb{C}}) \setminus P(\mathfrak{h})$.

Thus working over $g \in O'(\Lambda)$ as defined in (4) the Samelson splittings $T_g\mathrm{G}_2^{\mathbb{C}} = L_{g*}\mathfrak{s}_u \oplus L_{g*}\overline{\mathfrak{s}}_u$ induce sub-splittings $T_gO'(\Lambda)^{\mathbb{C}} = Z_{u,g} \oplus \overline{Z}_{u,g}$ where $Z_{u,g} := L_{g*}\mathfrak{s}_u \cap T_gO'(\Lambda)^{\mathbb{C}}$ and $\overline{Z}_{u,g} := L_{g*}\overline{\mathfrak{s}}_u \cap T_gO'(\Lambda)^{\mathbb{C}}$. We know additionally that they are both 3 complex dimensional moreover $T_gO'(\Lambda)^{\mathbb{C}} \cong T_gO'(\Lambda) \otimes_{\mathbb{R}} \mathbb{C}$ and $T_gO'(\Lambda) \cap Z_{u,g} = 0$. Consequently $T_gO'(\Lambda)^{\mathbb{C}} = Z_{u,g} \oplus \overline{Z}_{u,g}$ gives rise to a complex vector space structure $J_{u,g} : T_gO'(\Lambda) \to T_gO'(\Lambda)$ on the underlying real vector space by the general theory. In this way $O'(\Lambda)$ is improved to an almost complex manifold $(O'(\Lambda), J_u)$. Concerning its integrability, let X be a real vector field along $O'(\Lambda)$ and $X^{1,0} = \frac{1}{2}(X - \sqrt{-1}J_uX)$ its corresponding $(1,0)$-type vector field along $(O'(\Lambda), J_u)$. Moreover let $\{Z_1^{1,0}, \dots, Z_7^{1,0}\}$ be a basis over $\mathbb{C}$ of $\mathfrak{g}_2^{1,0} \subset \mathfrak{g}_2^{\mathbb{C}}$ hence satisfying $\left[Z_i^{1,0}, Z_j^{1,0}\right]^{0,1} = 0$ for all $i, j = 1, \dots, 7$ (recall that $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{g}_2^{1,0} \oplus \mathfrak{g}_2^{0,1}$ such that $\mathfrak{g}_2^{1,0} = \mathfrak{s}_u$ and $\mathfrak{g}_2^{0,1} = \overline{\mathfrak{s}}_u$). After identifying these basis elements with pointwise $\mathbb{C}$-linearly independent left-invariant $(1,0)$-type complex vector fields along $\mathrm{G}_2^{\mathbb{C}}$ we can pick $\mathbb{C}$-valued smooth functions $f_1, \dots, f_7$ along $O'(\Lambda)$ (and extend them by zero over the whole $\mathrm{G}_2^{\mathbb{C}}$) to write $X^{1,0} = \sum_i f_iZ_i^{1,0}$. The Nijenhuis bracket of any two real vector fields $X, Y \in C^\infty(O'(\Lambda); TO'(\Lambda))$ then looks like

$$N_{J_u}(X,Y) = \left[X^{1,0}, Y^{1,0}\right]^{0,1} = \left[\sum_{i=1}^{7} f_iZ_i^{1,0}, \sum_{j=1}^{7} g_jZ_j^{1,0}\right]^{0,1} = \sum_{i,j=1}^{7} f_ig_j\left[Z_i^{1,0}, Z_j^{1,0}\right]^{0,1} = 0\,.$$

Thus the Nijenhuis tensor N_{J_u} of $(O'(\Lambda), J_u)$ itself vanishes consequently the almost complex structure is integrable yielding a complex manifold X_u. Finally we observe that X_u is homeomorphic to the 6-sphere. $\square$

Remark. Note that *a priori* the complex manifolds $X_u \subset Y_u$ might be non-isomorphic for different values of the Samelson moduli parameter u. However we will see by the aid of the explicit construction at the end of Sect. 3 that these complex structures do not depend on u. Likewise, repeating everything so far with the apparently different twin conjugate orbit $O(\Lambda^2) \subset \mathrm{G}_2$ we do not obtain new complex manifolds because we will see in Lemma 3.1 that $O(\Lambda^2) = O(\Lambda)$ as subsets of G_2.

Before turning to the explicit construction let us examine certain properties of a complex structure on the six-sphere. On the one hand Gray [6] found in 1997 that if X was a hypothetical complex manifold diffeomorphic to S^6 then $H^{0,1}(X) \cong \mathbb{C}$ and raised the question how to interpret the generator of this cohomology group. In addition Ugarte [16, Corollary 3.3] proved in 2000 that either (i) $H^{1,1}(X) \ncong 0$, or (ii) $H^{1,1}(X) \cong 0$ and $H^{0,2}(X) \ncong 0$. Therefore $H^{0,1}(X_u) \cong \mathbb{C}$ and either $H^{1,1}(X_u) \ncong 0$, or $H^{1,1}(X_u) \cong 0$ and $H^{0,2}(X_u) \ncong 0$. On the other hand Pittie [13, Proposition 4.5] calculated in 1988 the complete Dolbeault cohomology ring of Y_u and in particular demonstrated that $H^{0,1}(Y_u) \cong \mathbb{C}$, $H^{1,1}(Y_u) \cong \mathbb{C}$ and $H^{0,2}(Y_u) \cong 0$ for all moduli parameters $u \in \mathbb{C}P^1 \setminus \mathbb{R}P^1$. Now we make a contact between these pieces of data:

Lemma 2.1. *The complex-analytic embedding $i : X_u \to Y_u$ induces $i^* : H^{0,1}(Y_u) \stackrel{\cong}{\to} H^{0,1}(X_u)$. Moreover $H^{1,1}(X_u) \ncong 0$ and $i^* : H^{0,2}(Y_u) \stackrel{\cong}{\to} H^{0,2}(X_u)$ is an isomorphism too, implying $H^{0,2}(X_u) \cong 0$.*

Proof. According to a result [13, Proposition 4.5] the Dolbeault cohomology ring of Y_u is either the pure exterior algebra $\wedge(x^{0,1}, y^{1,1}, z^{6,5})$ for non-orthognal complex structures $u \in P(\mathfrak{h}^{\mathbb{C}}) \setminus P(\mathfrak{h})$ on G_2 or the mixed one $\wedge(x^{0,1}, z^{2,1}) \otimes \mathbb{C}[y^{1,1}]/((y^{1,1})^6)$ for the two orthogonal ones $u_\pm \in P(\mathfrak{h}^{\mathbb{C}}) \setminus P(\mathfrak{h})$. These will be used below to compute the relevant groups $H^{p,q}(Y_u)$.

First consider the cohomology in degree 1. Let $0 \neq \varphi \in \Omega^{0,1}(Y_u)$ be a representative of the generator $x^{0,1}$ of $H^{0,1}(Y_u) \cong \mathbb{C}$. By the complex-analytic nature of the embedding, $i^*\varphi = \varphi|_{X_u} \in \Omega^{0,1}(X_u)$ satisfies $\overline{\partial}(\varphi|_{X_u}) = (\overline{\partial}\varphi)|_{X_u} = 0$ hence the restriction $\varphi|_{X_u}$ descends to a representative of a cohomology class in $H^{0,1}(X_u)\cong\mathbb{C}$, too. If $0\neq\psi\in\Omega^{0,1}(X_u)$ is a representative of its generator then there exists $c\in\mathbb{C}$ and smooth $f:X_u\to\mathbb{C}$ with the property $\varphi|_{X_u}=c\psi+\overline{\partial}f$. If $c\neq 0$ then the cohomology class of $\varphi|_{X_u}$ in $H^{0,1}(X_u)$ is non-trivial yielding that i^* is surjective therefore injective as well hence we are done. Assume that $c=0$. This means that $\varphi|_{X_u}=\overline{\partial}f$ hence $i^*:H^{0,1}(Y_u)\to H^{0,1}(X_u)$ is the zero map. We proceed now as follows. We already know that $H^{0,1}(X_u)\cong\mathbb{C}$ and $H^{0,1}(Y_u)\cong\mathbb{C}$ but in addition $H^{0,2}(Y_u)\cong 0$. Putting all of these into the relative Dolbeault cohomology exact sequence of the complex-analytic embedding $i:X_u\to Y_u$ we find

$$\begin{array}{ccccccccc}\dots\longrightarrow & H^{0,1}(Y_u) & \xrightarrow{i^*=0} & H^{0,1}(X_u) & \longrightarrow & H^{0,2}(Y_u\,,X_u) & \longrightarrow & H^{0,2}(Y_u) & \longrightarrow\dots\\ & \wr\| & & \wr\| & & & & \wr\| & \\ & \mathbb{C} & & \mathbb{C} & & & & 0 & \end{array}$$

hence deduce an isomorphism $H^{0,2}(Y_u\,,X_u)\cong\mathbb{C}$. This tells us that there exists a $(0,2)$-form ω on Y_u which is $\overline{\partial}$-closed over Y_u and not $\overline{\partial}$-exact over X_u. But the other isomorphism $H^{0,2}(Y_u)\cong 0$ also says that ω is both $\overline{\partial}$-closed and $\overline{\partial}$-exact over the whole Y_u hence in particular over X_u leading us to a contradiction. Consequently the assumption that $c=0$ was wrong that is, $i^*:H^{0,1}(Y_u)\to H^{0,1}(X_u)$ is surjective or injective hence is an isomorphism as stated.

Next consider the cohomology in degree 2. Let $0\neq\omega\in\Omega^{1,1}(Y_u)$ be a representative of the generator $y^{1,1}$ of $H^{1,1}(Y_u)\cong\mathbb{C}$. By the aid of the cohomology ring structure $\frac{1}{3!}\omega\wedge\omega\wedge\omega$ is not in the trivial class in $H^{3,3}(Y_u)$ at last for the two orthogonal complex structures $u_\pm$; replacing $\omega\mapsto\omega+\partial\overline{\partial}f$ if necessary, where $f:Y_u\to\mathbb{C}$ is a smooth function, we can suppose that $\mathrm{Re}\,\omega$ is a real positive $(1,1)$-form when restricted to a tubular neighbouhood of $X_u\subset Y_u$. Consequently we can regard $\mathrm{Re}\,\omega$ as the associated $(1,1)$-form of a Hermitian metric h_u on X_u. Thus $\mathrm{Vol}(X_u)=\frac{1}{3!}\int_{X_u}\mathrm{Re}\,\omega\wedge\mathrm{Re}\,\omega\wedge\mathrm{Re}\,\omega>0$. Therefore we find $\omega|_{X_u}=\mathrm{Re}\,\omega|_{X_u}+\sqrt{-1}\,\mathrm{Im}\,\omega|_{X_u}$ is not in the trivial class demonstrating $H^{1,1}(X_u)\ncong 0$. Finally, collecting information again from the cohomology ring, the following part of the relative Dolbeault cohomology sequence

$$\begin{array}{ccccccccc}\dots\longrightarrow & H^{0,2}(Y_u) & \xrightarrow{i^*} & H^{0,2}(X_u) & \longrightarrow & H^{0,3}(Y_u\,,X_u) & \longrightarrow & H^{0,3}(Y_u) & \longrightarrow\dots\\ & \wr\| & & & & & & \wr\| & \\ & 0 & & & & & & 0 & \end{array}$$

yields $H^{0,2}(X_u)\cong H^{0,3}(Y_u,X_u)$. But $H^{0,3}(Y_u,X_u)\cong 0$ because $H^{0,3}(Y_u)\cong 0$ as in the considerations in degree 1 above. Therefore $H^{0,2}(X_u)\cong 0$ as stated.[3] □

The six-sphere as a complex manifold is not homogeneous. Consequently blowing it up once in points belonging to different orbits of its automorphism group brings to life spaces which are all diffeomorphic to the complex projective three-space but not complex-analytically isomorphic to each other [7]. LeBrun calls in [9] the existence of such *exotic* $\mathbb{C}P^3$*'s* a "minor disaster". Here we report on a further disaster namely the existence of *large exotic* $\mathbb{C}^3$*'s* in a similar sense:

Lemma 2.2. *Let $x_0\in X_u$ be a point and consider the punctured complex manifold $X_u^\times:=X_u\setminus\{x_0\}$. Then the space $X_u^\times$ is diffeomorphic to $\mathbb{C}^3$ but is not complex-analytically isomorphic to it.*

Proof. Obviously $X_u^\times$ is diffeomorphic to $S^6\setminus\{x_0\}$ i.e. to $\mathbb{R}^6$ like $\mathbb{C}^3$ does. Let $f:X_u^\times\to\mathbb{C}$ be a holomorphic function. By Hartogs' theorem it extends to a holomorphic function $F:X_u\to\mathbb{C}$. However F must be constant [2] consequently f is constant on $X_u^\times$ as well. Since there exists an abundance of non-trivial holomorphic functions on $\mathbb{C}^3$ we conclude that $X_u^\times$ and $\mathbb{C}^3$ are not complex-analytically isomorphic. □

[3] For completeness we record here that the third generators $z^{6,5}$ in the non-orthogonal and $z^{2,1}$ in the orthogonal case of the Dolbeault cohomology ring of Y_u restrict to zero in the induced ring of X_u.

3 Explicit construction

In this section we calculate the integrable almost complex tensors J_u on $O'(\Lambda)$ underlying the complex manifolds X_u of Theorem 2.1. This computation allows one to read off at least that they do not depend on the Samelson parameter u hence the complex structures of Theorem 2.1 are in fact equivalent.

The six-sphere as a complex manifold is not easy to grasp. Since X_u is a compact space it cannot be embedded into the affine complex space $\mathbb{C}^m$ of any dimension; likewise $H^2(X_u;\mathbb{C}) \cong 0$ shows that it is not Kähler–Hodge consequently it does not admit an embedding into any projective complex space $\mathbb{C}P^n$. Therefore its realization as a complex submanifold of some well-known complex manifold fails (we rather would avoid to call the non-Kähler compact 7-manifolds Y_u well-known). Another odd feature is that the algebraic dimension of X_u is zero [2] which means that all global meromorphic functions are constant on it consequently the powerful methods of complex analysis also fail to say anything here. What we nevertheless can try is to seek the integrable almost complex manifold $(O'(\Lambda), J_u)$ underlying X_u. This indeed works but the result is so complicated that we decided not to display it fully here. In spite of this we write down carefully the main steps hence the curious (and computer-aided) reader can easily reproduce the calculations and face their quantitative complexity directly. Finally, before sinking in the heavy details, we raise the question whether or not there exists a better general way to exhibit X_u in a more compact or comprehensible form.

We begin with an explicit construction of the Samelson family J_u with $u \in \mathbb{C} \setminus \sqrt{-1}\,\mathbb{R}$ of all integrable almost complex tensor fields on G_2 as we promised in a footnote of Sect. 2. We also promised in another footnote to write down the root basis (5) of the 14 dimensional $\mathfrak{g}_2^{\mathbb{C}}$ explicitly. So let us start with this. Our representation of the basis is the smallest possible one and is provided by the embedding $\mathfrak{g}_2^{\mathbb{C}} \subset \mathfrak{so}(7)^{\mathbb{C}}$ therefore is in terms of 7×7 complex skew symmetric matrices. The corresponding matrices are orthonormal with respect to the Hermitian (thus *not* Ad-invariant!) scalar product $\langle V, W\rangle^{\mathbb{C}} := \mathrm{tr}(V\overline{W}^T)$ on $\mathfrak{g}_2^{\mathbb{C}}$ and look as follows:

$$H_{\pm a,b} = \frac{1}{2\sqrt{a^2+b^2+b+1}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mp\sqrt{-1} & 0 & 0 & 0 & 0 \\ 0 & \pm\sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a\mp\sqrt{-1}\,b & 0 & 0 \\ 0 & 0 & 0 & a\pm\sqrt{-1}\,b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a\mp\sqrt{-1}(b+1) \\ 0 & 0 & 0 & 0 & 0 & a\pm\sqrt{-1}(b+1) & 0 \end{pmatrix} \quad \begin{matrix} a, b \in \mathbb{R}, \\ a \neq 0 \end{matrix}$$

and one checks that $\{H_{\pm a,b}\}$ span the 2 dimensional complex Cartan subalgebra of $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}_2^{\mathbb{C}}$; moreover

$$V_{\pm 1} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \mp\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & \pm\sqrt{-1} & -1 & 0 & 0 \\ 0 & 1 & \mp\sqrt{-1} & 0 & 0 & 0 & 0 \\ 0 & \pm\sqrt{-1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_{\pm 2} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \pm\sqrt{-1} \\ 0 & 0 & 0 & 0 & 0 & \pm\sqrt{-1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mp\sqrt{-1} & 0 & 0 & 0 & 0 \\ 0 & \mp\sqrt{-1} & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_{\pm 3} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \pm\sqrt{-1} \\ 0 & 0 & 0 & 0 & 0 & \pm\sqrt{-1} & 1 \\ 0 & 0 & 0 & 1 & \mp\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & \mp\sqrt{-1} & -1 & 0 & 0 \end{pmatrix}$$

and one checks that $\{H_{\pm a,b}, V_{\pm 1}, V_{\pm 2}, V_{\pm 3}\}$ span the (maximal) subalgebra $\mathfrak{su}(3)^{\mathbb{C}} \subset \mathfrak{g}_2^{\mathbb{C}}$; and finally

$$U_{\pm 1} = \frac{1}{2\sqrt{6}}\begin{pmatrix} 0 & \mp 2\sqrt{-1} & -2 & 0 & 0 & 0 & 0 \\ \pm 2\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \pm\sqrt{-1} \\ 0 & 0 & 0 & 0 & 0 & \mp\sqrt{-1} & -1 \\ 0 & 0 & 0 & 1 & \pm\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & \mp\sqrt{-1} & 1 & 0 & 0 \end{pmatrix}$$

$$U_{\pm 2} = \frac{1}{2\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & \pm 2\sqrt{-1} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \pm\sqrt{-1} \\ 0 & 0 & 0 & 0 & 0 & \mp\sqrt{-1} & -1 \\ \mp 2\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \pm\sqrt{-1} & 0 & 0 & 0 & 0 \\ 0 & \mp\sqrt{-1} & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{\pm 3} = \frac{1}{2\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 & \mp 2\sqrt{-1} \\ 0 & 0 & 0 & \pm\sqrt{-1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \mp\sqrt{-1} & 0 & 0 \\ 0 & \mp\sqrt{-1} & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \pm\sqrt{-1} & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \pm 2\sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and one checks that $\{H_{+a,b}, V_{+1}, V_{-2}, V_{-3}, U_{+1}, U_{-2}, U_{+3}\}$ span $\mathfrak{s}_u \subset \mathfrak{g}_2^{\mathbb{C}}$ where $u \in \mathbb{C} \setminus \sqrt{-1}\,\mathbb{R}$ is written in the form $u = a + \sqrt{-1}\, b$ with $a \neq 0$; we also find that the remaining elements from the root basis $\{H_{-a,b}, V_{-1}, V_{+2}, V_{+3}, U_{-1}, U_{+2}, U_{-3}\}$ form a basis in the complex conjugate subalgebra $\bar{\mathfrak{s}}_u \subset \mathfrak{g}_2^{\mathbb{C}}$.

Consequently picking $a_i + \sqrt{-1}\, b_i \in \mathbb{C}$ $(i = 0, 1, \dots, 6)$ to write an element $W \in \mathfrak{s}_u$ as

$$\begin{aligned} W \;:=\; & 2\sqrt{a^2+b^2+b+1}\,(a_0 + \sqrt{-1}\, b_0) H_{+a,b} \\ & + 2\sqrt{2}(a_4 + \sqrt{-1}\, b_4) V_{+1} + 2\sqrt{2}(a_5 + \sqrt{-1}\, b_5) V_{-2} + 2\sqrt{2}(a_6 + \sqrt{-1}\, b_6) V_{-3} \\ & + 2\sqrt{6}(a_1 + \sqrt{-1}\, b_1) U_{+1} + 2\sqrt{6}(a_2 + \sqrt{-1}\, b_2) U_{-2} + 2\sqrt{6}(a_3 + \sqrt{-1}\, b_3) U_{+3} \end{aligned}$$

and putting $J_u : \mathfrak{g}_2 \to \mathfrak{g}_2$ to be $J_u(\mathrm{Re}\, W) := -\,\mathrm{Im}\, W$ dictated by the general theory we obtain an $\mathbb{R}$-linear transformation $J_u \in \mathrm{End}\,\mathfrak{g}_2$. Its action on

$$\mathrm{Re}\, W = \begin{pmatrix} 0 & 2b_1 & -2a_1 & -2b_2 & 2a_2 & 2a_3 & 2b_3 \\ -2b_1 & 0 & \frac{1}{2}b_0 & -b_3-a_4 & a_3+b_4 & -a_2-a_5 & -b_2-b_5 \\ 2a_1 & -\frac{1}{2}b_0 & 0 & a_3-b_4 & -a_4+b_3 & b_2-b_5 & -a_2+a_5 \\ 2b_2 & a_4+b_3 & -a_3+b_4 & 0 & -\frac{1}{2}aa_0+\frac{1}{2}bb_0 & -a_1-a_6 & -b_1-b_6 \\ -2a_2 & -a_3-b_4 & a_4-b_3 & \frac{1}{2}aa_0-\frac{1}{2}bb_0 & 0 & b_1-b_6 & -a_1+a_6 \\ -2a_3 & a_2+a_5 & -b_2+b_5 & a_1+a_6 & -b_1+b_6 & 0 & -\frac{1}{2}aa_0+\frac{1}{2}(b+1)b_0 \\ -2b_3 & b_2+b_5 & a_2-a_5 & b_1+b_6 & a_1-a_6 & \frac{1}{2}aa_0-\frac{1}{2}(b+1)b_0 & 0 \end{pmatrix}$$

is by definition

$$-\,\mathrm{Im}\, W = \begin{pmatrix} 0 & 2a_1 & 2b_1 & -2a_2 & -2b_2 & -2b_3 & 2a_3 \\ -2a_1 & 0 & \frac{1}{2}a_0 & -a_3+b_4 & a_4-b_3 & b_2+b_5 & -a_2-a_5 \\ -2b_1 & -\frac{1}{2}a_0 & 0 & -a_4-b_3 & a_3+b_4 & a_2-a_5 & b_2-b_5 \\ 2a_2 & a_3-b_4 & a_4+b_3 & 0 & \frac{1}{2}ba_0+\frac{1}{2}ab_0 & b_1+b_6 & -a_1-a_6 \\ 2b_2 & -a_4+b_3 & -a_3-b_4 & -\frac{1}{2}ba_0-\frac{1}{2}ab_0 & 0 & a_1-a_6 & b_1-b_6 \\ 2b_3 & -b_2-b_5 & -a_2+a_5 & -b_1-b_6 & -a_1+a_6 & 0 & \frac{1}{2}(b+1)a_0+\frac{1}{2}ab_0 \\ -2a_3 & a_2+a_5 & -b_2+b_5 & a_1+a_6 & -b_1+b_6 & -\frac{1}{2}(b+1)a_0-\frac{1}{2}ab_0 & 0 \end{pmatrix}$$

hence we immediately check that $J_u^2 = -\mathrm{Id}_{\mathfrak{g}_2}$. The shape of J_u can be read off from these matrices more explicitly if we introduce the real orthonormal basis

$$\begin{aligned}
H_+ &:= \frac{\sqrt{a^2+b^2+b+1}}{2a}(H_{+a,b}+H_{-a,b})\\
H_- &:= \frac{\sqrt{a^2+b^2+b+1}}{\sqrt{-3}}(H_{+a,b}-H_{-a,b})-\frac{2b+1}{\sqrt{3}}H_+\\
X_{\pm k} &:= \frac{1}{\sqrt{\pm 2}}(U_{+k}\pm U_{-k})\ ,\ k=1,2,3\\
Y_{\pm k} &:= \frac{1}{\sqrt{\pm 2}}(V_{+k}\pm V_{-k})\ ,\ k=1,2,3
\end{aligned}$$

(note that $H_\pm$ are already independent of a,b) on the compact real form $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ of $\mathfrak{g}_2^{\mathbb{C}}$ equipped with the (positive definite) Ad-invariant real scalar product $\langle V,W\rangle := \mathrm{tr}(V\,W^T)$. A straightforward computation verifies that in this basis the action of J_u takes the simple blockdiagonal shape

$$\begin{aligned}
J_uH_+ = -\frac{2b+1}{2a}H_+ + \frac{4a^2+4b^2+4b+1}{2\sqrt{3}\,a}H_- \quad &,\quad J_uH_- = -\frac{\sqrt{3}}{2a}H_+ + \frac{2b+1}{2a}H_-\\
J_uX_{+k} = X_{-k} \quad &,\quad J_uX_{-k} = -X_{+k}\\
J_uY_{+k} = Y_{-k} \quad &,\quad J_uY_{-k} = -Y_{+k}\ .
\end{aligned}\tag{7}$$

One can find precisely two integrable almost complex structures $J_{u_\pm}$ at $a=\pm\frac{\sqrt{3}}{2}$ and $b=-\frac{1}{2}$ which are orthogonal for the aforementioned natural Ad-invariant real scalar product on $\mathfrak{g}_2$ i.e. $\langle J_{u_\pm}V,V\rangle = 0$ and $|J_{u_\pm}V| = |V|$ for all $V\in\mathfrak{g}_2$. After left-translating the J_u's over the whole compact group they give rise to complex structures on G_2 such that the two orthogonal $J_{u_\pm}$ yield two complex structures compatible with the bi-invariant metric on G_2. This completes the construction of all the integrable almost complex tensors J_u on G_2 *à la* Samelson (for further details cf. [13, Example on p. 123]).

Our next task is to construct the deformed conjugate orbit $O'(\Lambda)\subset \mathrm{G}_2^{\mathbb{C}}$. This will be achieved in two technical steps below. First we construct the original conjugate orbit (1) secondly its deformation as defined in (4).

Recall the classical fact that G_2 coincides with the automorphism group of the octonions over the reals; this important fact has not been used so far explicitly. We shall identify $O(\Lambda)\subset \mathrm{G}_2$ with the subset of inner automorphisms of the octonions. Let $\mathbb{O}$ denote the non-associative, unital normed algebra of the octonions (or Cayley numbers) over the reals. In the canonical oriented basis $\{\mathbf{e}_0,\mathbf{e}_1,\dots,\mathbf{e}_7\}$ the $\mathbf{e}_0$ plays the role of the unit hence $\mathrm{Re}\,\mathbb{O} := \mathbb{R}\mathbf{e}_0 \subset \mathbb{O}$ is the real part. To be absolutely unambiguous we communicate our octonionic multiplication convention here:

	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_0$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$
$\mathbf{e}_7$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$

(actually there are many different conventions in use). The basis gives rise to a canonical $\mathbb{R}$-linear isomorphism of oriented spaces $(\mathbb{O},\mathbf{e}_0,\dots,\mathbf{e}_7)\cong\mathbb{R}^8$. We can use the standard scalar product on $\mathbb{R}^8$

to define $\mathrm{Im}\,\mathbb{O} := (\mathrm{Re}\,\mathbb{O})^\perp \subset \mathbb{O}$ and introduce a multiplicative norm $|\cdot|$ on $\mathbb{O}$. Then canonically $(\mathrm{Im}\,\mathbb{O}, \mathbf{e}_1, \dots, \mathbf{e}_7) \cong \mathbb{R}^7$ and in this way we can look at the six-sphere as the set of imaginary octonions of unit length i.e. we will suppose $S^6 \subset \mathrm{Im}\,\mathbb{O}$. If $\mathbf{u}, \mathbf{v} \in \mathbb{O}$ and $\mathbf{v} \neq 0$ then the identity of elasticity convinces us that $(\mathbf{vu})\mathbf{v}^{-1} = \mathbf{v}(\mathbf{uv}^{-1})$ hence it is meaningful to talk about *inner automorphisms* of the octonions. An important result [8] says that $\mathbf{v}$ indeed gives rise to an inner automorphism if and only if $4(\mathrm{Re}\,\mathbf{v})^2 = |\mathbf{v}|^2$ i.e. $3(\mathrm{Re}\,\mathbf{v})^2 = |\,\mathrm{Im}\,\mathbf{v}|^2$ holds. Note that this condition implies $\mathbf{v}^3$ is a non-zero scalar therefore it corresponds to the trivial automorphism of $\mathbb{O}$. Picking an $\mathbf{x} \in S^6$ the non-real octonion $\mathbf{v} := \mathbf{e}_0 + \sqrt{3}\,\mathbf{x}$ satisfying $\mathbf{v}^3 = -8\mathbf{e}_0$ therefore gives an inner automorphism (hence parantheses can be omitted)

$$\mathbf{u} \longmapsto (\mathbf{e}_0 + \sqrt{3}\,\mathbf{x})\mathbf{u}(\mathbf{e}_0 + \sqrt{3}\,\mathbf{x})^{-1} = \frac{1}{4}(\mathbf{e}_0 + \sqrt{3}\,\mathbf{x})\mathbf{u}(\mathbf{e}_0 - \sqrt{3}\,\mathbf{x}) \tag{8}$$

and all inner automorphisms of the octonions are of this form. In this way we get a remarkable map

$$f : S^6 \longrightarrow \mathrm{G}_2 \ . \tag{9}$$

Knowing that $\mathrm{Re}\,\mathbb{O}$ is invariant under all automorphisms and that the corresponding reduced linear map of $\mathrm{Im}\,\mathbb{O}$ is an orientation preserving orthogonal transformation of $\mathbb{R}^7$ we can embed G_2 into $\mathrm{SO}(7)$ as usual. Under the canonical isomorphism $(\mathrm{Im}\,\mathbb{O}, \mathbf{e}_1, \dots, \mathbf{e}_7) \cong \mathbb{R}^7$ put $\mathbf{x} = x_1\mathbf{e}_1 + \dots + x_7\mathbf{e}_7$ and thus the image $f(\mathbf{x})$ of the map (9) at $\mathbf{x}$ takes the impressive shape

$$f\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7 \end{pmatrix} = \frac{\sqrt{3}}{2}\begin{pmatrix}
-\frac{1}{\sqrt{3}}+\sqrt{3}x_1^2 & -x_3+\sqrt{3}x_1x_2 & x_2+\sqrt{3}x_1x_3 & -x_5+\sqrt{3}x_1x_4 & x_4+\sqrt{3}x_1x_5 & x_7+\sqrt{3}x_1x_6 & -x_6+\sqrt{3}x_1x_7\\
x_3+\sqrt{3}x_2x_1 & -\frac{1}{\sqrt{3}}+\sqrt{3}x_2^2 & -x_1+\sqrt{3}x_2x_3 & -x_6+\sqrt{3}x_2x_4 & -x_7+\sqrt{3}x_2x_5 & x_4+\sqrt{3}x_2x_6 & x_5+\sqrt{3}x_2x_7\\
-x_2+\sqrt{3}x_3x_1 & x_1+\sqrt{3}x_3x_2 & -\frac{1}{\sqrt{3}}+\sqrt{3}x_3^2 & -x_7+\sqrt{3}x_3x_4 & x_6+\sqrt{3}x_3x_5 & -x_5+\sqrt{3}x_3x_6 & x_4+\sqrt{3}x_3x_7\\
x_5+\sqrt{3}x_4x_1 & x_6+\sqrt{3}x_4x_2 & x_7+\sqrt{3}x_4x_3 & -\frac{1}{\sqrt{3}}+\sqrt{3}x_4^2 & -x_1+\sqrt{3}x_4x_5 & -x_2+\sqrt{3}x_4x_6 & -x_3+\sqrt{3}x_4x_7\\
-x_4+\sqrt{3}x_5x_1 & x_7+\sqrt{3}x_5x_2 & -x_6+\sqrt{3}x_5x_3 & x_1+\sqrt{3}x_5x_4 & -\frac{1}{\sqrt{3}}+\sqrt{3}x_5^2 & x_3+\sqrt{3}x_5x_6 & -x_2+\sqrt{3}x_5x_7\\
-x_7+\sqrt{3}x_6x_1 & -x_4+\sqrt{3}x_6x_2 & x_5+\sqrt{3}x_6x_3 & x_2+\sqrt{3}x_6x_4 & -x_3+\sqrt{3}x_6x_5 & -\frac{1}{\sqrt{3}}+\sqrt{3}x_6^2 & x_1+\sqrt{3}x_6x_7\\
x_6+\sqrt{3}x_7x_1 & -x_5+\sqrt{3}x_7x_2 & -x_4+\sqrt{3}x_7x_3 & x_3+\sqrt{3}x_7x_4 & x_2+\sqrt{3}x_7x_5 & -x_1+\sqrt{3}x_7x_6 & -\frac{1}{\sqrt{3}}+\sqrt{3}x_7^2
\end{pmatrix}$$

of a 7×7 special orthogonal matrix with $x_1, \dots, x_7 \in \mathbb{R}$ satisfying $x_1^2 + \dots + x_7^2 = 1$. We shall need the derivative $f(\mathbf{x})_* : T_\mathbf{x}S^6 \to T_{f(\mathbf{x})}\mathrm{G}_2$ at $\mathbf{x} \in S^6$ of this map, too. If $\xi \in T_\mathbf{x}S^6$ is a tangent vector then $f(\mathbf{x})_*\xi \in T_{f(\mathbf{x})}\mathrm{G}_2$ is its image. Putting $\xi = \xi_1\mathbf{e}_1 + \dots + \xi_7\mathbf{e}_7$ satisfying $x_1\xi_1 + \dots + x_7\xi_7 = 0$ a long but straightforward computation verifies that

$$f(\mathbf{x})_*\begin{pmatrix} \xi_1\\ \xi_2\\ \xi_3\\ \xi_4\\ \xi_5\\ \xi_6\\ \xi_7 \end{pmatrix} = \frac{3}{2}\begin{pmatrix}
2x_1\xi_1 & -\frac{1}{\sqrt{3}}\xi_3+x_1\xi_2+x_2\xi_1 & \frac{1}{\sqrt{3}}\xi_2+x_1\xi_3+x_3\xi_1 & -\frac{1}{\sqrt{3}}\xi_5+x_1\xi_4+x_4\xi_1 & \frac{1}{\sqrt{3}}\xi_4+x_1\xi_5+x_5\xi_1 & \frac{1}{\sqrt{3}}\xi_7+x_1\xi_6+x_6\xi_1 & -\frac{1}{\sqrt{3}}\xi_6+x_1\xi_7+x_7\xi_1\\
\frac{1}{\sqrt{3}}\xi_3+x_2\xi_1+x_1\xi_2 & 2x_2\xi_2 & -\frac{1}{\sqrt{3}}\xi_1+x_2\xi_3+x_3\xi_2 & -\frac{1}{\sqrt{3}}\xi_6+x_2\xi_4+x_4\xi_2 & -\frac{1}{\sqrt{3}}\xi_7+x_2\xi_5+x_5\xi_2 & \frac{1}{\sqrt{3}}\xi_4+x_2\xi_6+x_6\xi_2 & \frac{1}{\sqrt{3}}\xi_5+x_2\xi_7+x_7\xi_2\\
-\frac{1}{\sqrt{3}}\xi_2+x_3\xi_1+x_1\xi_3 & \frac{1}{\sqrt{3}}\xi_1+x_3\xi_2+x_2\xi_3 & 2x_3\xi_3 & -\frac{1}{\sqrt{3}}\xi_7+x_3\xi_4+x_4\xi_3 & \frac{1}{\sqrt{3}}\xi_6+x_3\xi_5+x_5\xi_3 & -\frac{1}{\sqrt{3}}\xi_5+x_3\xi_6+x_6\xi_3 & \frac{1}{\sqrt{3}}\xi_4+x_3\xi_7+x_7\xi_3\\
\frac{1}{\sqrt{3}}\xi_5+x_4\xi_1+x_1\xi_4 & \frac{1}{\sqrt{3}}\xi_6+x_4\xi_2+x_2\xi_4 & \frac{1}{\sqrt{3}}\xi_7+x_4\xi_3+x_3\xi_4 & 2x_4\xi_4 & -\frac{1}{\sqrt{3}}\xi_1+x_4\xi_5+x_5\xi_4 & -\frac{1}{\sqrt{3}}\xi_2+x_4\xi_6+x_6\xi_4 & -\frac{1}{\sqrt{3}}\xi_3+x_4\xi_7+x_7\xi_4\\
-\frac{1}{\sqrt{3}}\xi_4+x_5\xi_1+x_1\xi_5 & \frac{1}{\sqrt{3}}\xi_7+x_5\xi_2+x_2\xi_5 & -\frac{1}{\sqrt{3}}\xi_6+x_5\xi_3+x_3\xi_5 & \frac{1}{\sqrt{3}}\xi_1+x_5\xi_4+x_4\xi_5 & 2x_5\xi_5 & \frac{1}{\sqrt{3}}\xi_3+x_5\xi_6+x_6\xi_5 & -\frac{1}{\sqrt{3}}\xi_2+x_5\xi_7+x_7\xi_5\\
-\frac{1}{\sqrt{3}}\xi_7+x_6\xi_1+x_1\xi_6 & -\frac{1}{\sqrt{3}}\xi_4+x_6\xi_2+x_2\xi_6 & \frac{1}{\sqrt{3}}\xi_5+x_6\xi_3+x_3\xi_6 & \frac{1}{\sqrt{3}}\xi_2+x_6\xi_4+x_4\xi_6 & -\frac{1}{\sqrt{3}}\xi_3+x_6\xi_5+x_5\xi_6 & 2x_6\xi_6 & \frac{1}{\sqrt{3}}\xi_1+x_6\xi_7+x_7\xi_6\\
\frac{1}{\sqrt{3}}\xi_6+x_7\xi_1+x_1\xi_7 & -\frac{1}{\sqrt{3}}\xi_6+x_7\xi_2+x_2\xi_7 & -\frac{1}{\sqrt{3}}\xi_4+x_7\xi_3+x_3\xi_7 & \frac{1}{\sqrt{3}}\xi_3+x_7\xi_4+x_4\xi_7 & \frac{1}{\sqrt{3}}\xi_2+x_7\xi_5+x_5\xi_7 & -\frac{1}{\sqrt{3}}\xi_1+x_7\xi_6+x_6\xi_7 & 2x_7\xi_7
\end{pmatrix}$$

out of which we also obtain (but already will be unable to plot anything from now on) the pullback $L^{-1}_{f(\mathbf{x})*}(f(\mathbf{x})_*\xi) \in T_e\mathrm{G}_2 = \mathfrak{g}_2$. In less fancy notation this is $f(\mathbf{x})^{-1}f(\mathbf{x})_*\xi \in \mathbb{R}(7)$ i.e. the plain matrix product of the group-theoretic inverse $f(\mathbf{x})^{-1}$ and $f(\mathbf{x})_*\xi$ above.

It readily follows from (8) or by a direct computation that $f(\mathbf{x})^3 = 1_{\mathbb{R}^7}$ and $f(\mathbf{x})\mathbf{x} = \mathbf{x}$ i.e. we can visualize $f(\mathbf{x})$ as a degree $\frac{2\pi}{3}$ rotation $R_{\mathbf{x}}$ about the axis through $\mathbf{x} \in \mathbb{R}^7$. Since $\Lambda \in Z(\mathrm{SU}(3)) \cong \mathbb{Z}_3$ satisfies $\Lambda^3 = e \in \mathrm{G}_2$ it is also true that $h^3 = e$ for all $h \in O(\Lambda)$. In fact the two subsets $f(S^6)$ and $O(\Lambda)$ of G_2 are nothing but the same:

Lemma 3.1. (cf. [3, pp. 160-161]) *The conjugate orbit $O(\Lambda) \subset \mathrm{G}_2$ of (1) and the image $f(S^6) \subset \mathrm{G}_2$ of the map (9) coincide as subsets within G_2 i.e.*

$$O(\Lambda) = \left\{ f(\mathbf{x}) \,\middle|\, \mathbf{x} \in S^6 \right\} .$$

Moreover $O(\Lambda) = O(\Lambda^2)$, where $O(\Lambda^2) = \{g\Lambda^2 g^{-1} | g \in \mathrm{G}_2\}$ is the conjugate orbit passing through the square of the generator $\Lambda^2 \in Z(\mathrm{SU}(3)) \subset \mathrm{G}_2$.

Proof. We quickly observe that

$$f\begin{pmatrix} 1\\0\\0\\0\\0\\0\\0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0\\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0\\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2}\\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{10}$$

hence $f(\mathbf{e}_1) = \Lambda \in Z(\mathrm{SU}(3)) \subset \mathrm{G}_2$. Therefore the action $\mathbf{u} \mapsto \Lambda\mathbf{u}$ on $\mathbf{u} \in \mathbb{O}$ arises from the inner automorphism $\mathbf{u} \mapsto (\mathbf{e}_0 + \sqrt{3}\,\mathbf{e}_1)\mathbf{u}(\mathbf{e}_0 + \sqrt{3}\,\mathbf{e}_1)^{-1}$ in (8). Now pick $g \in \mathrm{G}_2$ then the twisted action $\mathbf{u} \mapsto (g\Lambda g^{-1})\mathbf{u}$ looks like

$$\begin{aligned}(g\Lambda g^{-1})\mathbf{u} &= g\left((\mathbf{e}_0 + \sqrt{3}\,\mathbf{e}_1)(g^{-1}\mathbf{u})(\mathbf{e}_0 + \sqrt{3}\,\mathbf{e}_1)^{-1}\right) = -\frac{1}{8}g\Big((\mathbf{e}_0 + \sqrt{3}\,\mathbf{e}_1)(g^{-1}\mathbf{u})(\mathbf{e}_0 + \sqrt{3}\,\mathbf{e}_1)^2\Big)\\ &= -\frac{1}{8}g(\mathbf{e}_0 + \sqrt{3}\,\mathbf{e}_1)\mathbf{u}(g(\mathbf{e}_0 + \sqrt{3}\,\mathbf{e}_1))^2 = -\frac{1}{8}(\mathbf{e}_0 + \sqrt{3}(g\mathbf{e}_1))\mathbf{u}(\mathbf{e}_0 + \sqrt{3}(g\mathbf{e}_1))^2\\ &= (\mathbf{e}_0 + \sqrt{3}(g\mathbf{e}_1))\mathbf{u}(\mathbf{e}_0 + \sqrt{3}(g\mathbf{e}_1))^{-1}\end{aligned}$$

consequently it comes from an inner automorphism by $\mathbf{v} := \mathbf{e}_0 + \sqrt{3}(g\mathbf{e}_1)$. Therefore $f(g\mathbf{e}_1) = g\Lambda g^{-1}$ and taking into account that G_2 acts transitively on $S^6 \subset \mathrm{Im}\,\mathbb{O}$ (with stabilizer subgroup $\mathrm{SU}(3) \subset \mathrm{G}_2$) we conclude that $f(S^6) = O(\Lambda)$.

The fact $f(-\mathbf{x}) = f(\mathbf{x})^T = f(\mathbf{x})^{-1}$ gives the identity $f(\pm\mathbf{x}) = f(\mathbf{x})^{\pm 1}$ for all $\mathbf{x} \in S^6$. Therefore $f(\pm\mathbf{e}_1) = f(\mathbf{e}_1)^{\pm 1} = \Lambda^{\pm 1}$ yielding $f(-\mathbf{e}_1) = \Lambda^{-1} = \Lambda^2$. Taking an element $g \in \mathrm{G}_2$ satisfying $g\mathbf{e}_1 = -\mathbf{e}_1$ (unique up to two-sided multiplication with elements of $\mathrm{SU}(3) \subset \mathrm{G}_2$) we can write $\Lambda^2 = g\Lambda g^{-1}$; hence the two conjugate orbits of Λ and Λ^2 in G_2 are not distinct consequently they must coincide. $\square$

Now we are in a position to construct the deformed conjugate orbit $O'(\Lambda) \subset \mathrm{G}_2^{\mathbb{C}}$. Replace the real vector $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_7\mathbf{e}_7$ satisfying $x_1^2 + \cdots + x_7^2 = 1$ with a complex vector $\mathbf{z} = z_1\mathbf{e}_1 + \cdots + z_7\mathbf{e}_7$ such that $z_1^2 + \cdots + z_7^2 = 1$. Then $\mathbf{w} = \mathbf{e}_0 + \sqrt{3}\,\mathbf{z}$ continues to be invertible and satisfies $3(\mathrm{Re}\,\mathbf{w})^2 = |\mathrm{Im}\,\mathbf{w}|^2$ hence generates an inner automorphism of the complexified octonions $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$. Consequently inserting $\mathbf{z}$ into (9) we obtain the complexified map

$$f^{\mathbb{C}} : (S^6)^{\mathbb{C}} \longrightarrow \mathrm{G}_2^{\mathbb{C}}$$

and by Lemma 3.1 we know that $O(\Lambda)^{\mathbb{C}} = \{f^{\mathbb{C}}(\mathbf{z}) \,|\, \mathbf{z} \in (S^6)^{\mathbb{C}}\}$ and $O(\Lambda)^{\mathbb{C}} = O(\Lambda^2)^{\mathbb{C}}$.

Lemma 3.2. *There exists a bounded domain $\Omega\subset\mathbb{C}^3$ about the origin (whose more precise shape is not important) such that the deformed conjugate orbit $O'(\Lambda)\subset\mathrm{G}_2^{\mathbb{C}}$ as defined in (4) looks like*

$$O'(\Lambda)=\left\{f^{\mathbb{C}}(\mathbf{z})f^{\mathbb{C}}(\overline{\mathbf{z}})\;\middle|\;\begin{array}{l}\mathbf{z}=-\mathbf{e}_1+z_2\mathbf{e}_2-\sqrt{-1}z_2\mathbf{e}_3+z_4\mathbf{e}_4-\sqrt{-1}z_4\mathbf{e}_5+z_6\mathbf{e}_6-\sqrt{-1}z_6\mathbf{e}_7\ ,\\ (z_2,z_4,z_6)\in\overline{\Omega}\end{array}\right\}$$

where $\overline{\mathbf{z}}$ denotes the conjugate of $\mathbf{z}$ as a complex vector in $\mathbb{C}^7$ (and not the conjugate as a complex octonion in $\mathbb{O}\otimes_{\mathbb{R}}\mathbb{C}$) i.e. for every $\mathbf{v}=v_1\mathbf{e}_1+\dots+v_7\mathbf{e}_7$ we define $\overline{\mathbf{v}}:=\overline{v}_1\mathbf{e}_1+\dots+\overline{v}_7\mathbf{e}_7$.

Moreover $O'(\Lambda)$ is homeomorphic to S^6, does not depend on the Samelson parameter $u=a+\sqrt{-1}b$ with $a\neq 0$ and in particular $f^{\mathbb{C}}(-\mathbf{e}_1)f^{\mathbb{C}}(-\mathbf{e}_1)=\Lambda\in O'(\Lambda)$ justifying the notation.

Proof. We have seen that $\Lambda^2=\Lambda^{-1}=f(\mathbf{e}_1)^{-1}=f(-\mathbf{e}_1)=f^{\mathbb{C}}(-\mathbf{e}_1)$ hence knowing that $s_1\in S_u\subset\mathrm{G}_2^{\mathbb{C}}$ is an automorphism we find $f^{\mathbb{C}}(\mathbf{z})=s_1\Lambda^2s_1^{-1}=s_1f^{\mathbb{C}}(-\mathbf{e}_1)s_1^{-1}=f^{\mathbb{C}}(s_1(-\mathbf{e}_1))$ hence $\mathbf{z}=-s_1\mathbf{e}_1$. Consider the unique decomposition $s_1=\lim\limits_n\left(A^{1/n}h_1^{1/n}\right)^n$ where $h_1\in S_u\cap\mathrm{SU}(3)^{\mathbb{C}}$ and using the previously constructed root basis for $\mathfrak{s}_u$ we introduced $A:=\exp\left(\sqrt{-6}\,z_2U_{+1}-\sqrt{-6}\,z_4U_{-2}+\sqrt{6}\,z_6U_{+3}\right)\in S_u$. This latter matrix is relatively easy to compute because fortunately $U_{\pm k}^3=0$ for $k=1,2,3$ yielding

$$A=\begin{pmatrix}
1 & z_2 & -\sqrt{-1}z_2 & z_4 & -\sqrt{-1}z_4 & z_6 & -\sqrt{-1}z_6\\
-z_2 & 1-\frac{1}{2}z_2^2 & \frac{\sqrt{-1}}{2}z_2^2 & \frac{\sqrt{-1}}{2}z_6-\frac{1}{2}z_2z_4 & \frac{1}{2}z_6+\frac{\sqrt{-1}}{2}z_2z_4 & \frac{\sqrt{-1}}{2}z_4-\frac{1}{4}z_2z_6 & \frac{1}{2}z_4+\frac{\sqrt{-1}}{4}z_2z_6\\
\sqrt{-1}z_2 & \frac{\sqrt{-1}}{2}z_2^2 & 1+\frac{1}{2}z_2^2 & \frac{1}{2}z_6+\frac{\sqrt{-1}}{2}z_2z_4 & -\frac{\sqrt{-1}}{2}z_6+\frac{1}{2}z_2z_4 & -\frac{1}{2}z_4+\frac{\sqrt{-1}}{4}z_2z_6 & \frac{\sqrt{-1}}{2}z_4+\frac{1}{4}z_2z_6\\
-z_4 & -\frac{\sqrt{-1}}{2}z_6-\frac{1}{2}z_2z_4 & -\frac{1}{2}z_6+\frac{\sqrt{-1}}{2}z_2z_4 & 1-\frac{1}{2}z_4^2 & \frac{\sqrt{-1}}{2}z_4^2 & -\frac{\sqrt{-1}}{2}z_2-\frac{1}{4}z_4z_6 & -\frac{1}{2}z_2+\frac{\sqrt{-1}}{4}z_4z_6\\
\sqrt{-1}z_4 & -\frac{1}{2}z_6+\frac{\sqrt{-1}}{2}z_2z_4 & \frac{\sqrt{-1}}{2}z_6+\frac{1}{2}z_2z_4 & \frac{\sqrt{-1}}{2}z_4^2 & 1+\frac{1}{2}z_4^2 & \frac{1}{2}z_2+\frac{\sqrt{-1}}{4}z_4z_6 & -\frac{\sqrt{-1}}{2}z_2+\frac{1}{4}z_4z_6\\
-z_6 & -\frac{\sqrt{-1}}{2}z_4-\frac{1}{4}z_2z_6 & \frac{1}{2}z_4+\frac{\sqrt{-1}}{4}z_2z_6 & \frac{\sqrt{-1}}{2}z_2-\frac{1}{4}z_4z_6 & -\frac{1}{2}z_2+\frac{\sqrt{-1}}{4}z_4z_6 & 1-\frac{1}{2}z_6^2 & \frac{\sqrt{-1}}{2}z_6^2\\
\sqrt{-1}z_6 & -\frac{1}{2}z_4+\frac{\sqrt{-1}}{4}z_2z_6 & -\frac{\sqrt{-1}}{2}z_4+\frac{1}{4}z_2z_6 & \frac{1}{2}z_2+\frac{\sqrt{-1}}{4}z_4z_6 & \frac{\sqrt{-1}}{2}z_2+\frac{1}{4}z_4z_6 & \frac{\sqrt{-1}}{2}z_6^2 & 1+\frac{1}{2}z_6^2
\end{pmatrix}$$

thus it is algebraic. Since h_1 is the stabilizer of $\mathbf{e}_1$ and $\dim_{\mathbb{C}}S_u\Lambda^2S_u^{-1}=3$ as well as A already depends on 3 complex parameters by setting $h_1:=e$ we can replace s_1 with the simple matrix A at the price of parameterizing an open subset of $S_u\Lambda^2S_u^{-1}$ only. Thus the action of this A on $-\mathbf{e}_1$ gives

$$\mathbf{z}=-\mathbf{e}_1+z_2\mathbf{e}_2-\sqrt{-1}z_2\mathbf{e}_3+z_4\mathbf{e}_4-\sqrt{-1}z_4\mathbf{e}_5+z_6\mathbf{e}_6-\sqrt{-1}z_6\mathbf{e}_7$$

thus $\mathbf{z}\in(S^6)^{\mathbb{C}}$ with arbitrary parameters $z_2,z_4,z_6\in\mathbb{C}$. Likewise $f^{\mathbb{C}}(\mathbf{v})\in\overline{S}_u\Lambda^2\overline{S}_u^{-1}$ if

$$\mathbf{v}=-\mathbf{e}_1+v_2\mathbf{e}_2+\sqrt{-1}v_2\mathbf{e}_3+v_4\mathbf{e}_4+\sqrt{-1}v_4\mathbf{e}_5+v_6\mathbf{e}_6+\sqrt{-1}v_6\mathbf{e}_7$$

hence $\mathbf{v}\in(S^6)^{\mathbb{C}}$ with further $v_2,v_4,v_6\in\mathbb{C}$. In this case for the element $s_2\in\overline{S}_u$ taking $-\mathbf{e}_1$ to $\mathbf{v}$ it is enough to put $B:=\exp\left(-\sqrt{-6}\,v_2U_{-1}+\sqrt{-6}\,v_4U_{+2}+\sqrt{6}\,v_6U_{-3}\right)\in\overline{S}_u$ which looks like

$$B=\begin{pmatrix}
1 & v_2 & \sqrt{-1}v_2 & v_4 & \sqrt{-1}v_4 & v_6 & \sqrt{-1}v_6\\
-v_2 & 1-\frac{1}{2}v_2^2 & -\frac{\sqrt{-1}}{2}v_2^2 & -\frac{\sqrt{-1}}{2}v_6-\frac{1}{2}v_2v_4 & \frac{1}{2}v_6-\frac{\sqrt{-1}}{2}v_2v_4 & -\frac{\sqrt{-1}}{2}v_4-\frac{1}{4}v_2v_6 & \frac{1}{2}v_4-\frac{\sqrt{-1}}{4}v_2v_6\\
-\sqrt{-1}v_2 & -\frac{\sqrt{-1}}{2}v_2^2 & 1+\frac{1}{2}v_2^2 & \frac{1}{2}v_6-\frac{\sqrt{-1}}{2}v_2v_4 & \frac{\sqrt{-1}}{2}v_6+\frac{1}{2}v_2v_4 & -\frac{1}{2}v_4-\frac{\sqrt{-1}}{4}v_2v_6 & -\frac{\sqrt{-1}}{2}v_4+\frac{1}{4}v_2v_6\\
-v_4 & \frac{\sqrt{-1}}{2}v_6-\frac{1}{2}v_2v_4 & -\frac{1}{2}v_6-\frac{\sqrt{-1}}{2}v_2v_4 & 1-\frac{1}{2}v_4^2 & -\frac{\sqrt{-1}}{2}v_4^2 & \frac{\sqrt{-1}}{2}v_2-\frac{1}{4}v_4v_6 & -\frac{1}{2}v_2-\frac{\sqrt{-1}}{4}v_4v_6\\
-\sqrt{-1}v_4 & -\frac{1}{2}v_6-\frac{\sqrt{-1}}{2}v_2v_4 & -\frac{\sqrt{-1}}{2}v_6+\frac{1}{2}v_2v_4 & -\frac{\sqrt{-1}}{2}v_4^2 & 1+\frac{1}{2}v_4^2 & \frac{1}{2}v_2-\frac{\sqrt{-1}}{4}v_4v_6 & \frac{\sqrt{-1}}{2}v_2+\frac{1}{4}v_4v_6\\
-v_6 & \frac{\sqrt{-1}}{2}v_4-\frac{1}{4}v_2v_6 & \frac{1}{2}v_4-\frac{\sqrt{-1}}{4}v_2v_6 & -\frac{\sqrt{-1}}{2}v_2-\frac{1}{4}v_4v_6 & -\frac{1}{2}v_2-\frac{\sqrt{-1}}{4}v_4v_6 & 1-\frac{1}{2}v_6^2 & -\frac{\sqrt{-1}}{2}v_6^2\\
-\sqrt{-1}v_6 & -\frac{1}{2}v_4-\frac{\sqrt{-1}}{4}v_2v_6 & \frac{\sqrt{-1}}{2}v_4+\frac{1}{4}v_2v_6 & \frac{1}{2}v_2-\frac{\sqrt{-1}}{4}v_4v_6 & -\frac{\sqrt{-1}}{2}v_2+\frac{1}{4}v_4v_6 & -\frac{\sqrt{-1}}{2}v_6^2 & 1+\frac{1}{2}v_6^2
\end{pmatrix}$$

akin to A. The perturbed complexified orbit $O'(\Lambda)^{\mathbb{C}}$ as defined in (3) is of the form $S_u\Lambda^2S_u^{-1}\overline{S}_u\Lambda^2\overline{S}_u^{-1}$ consequently we obtain an open subset of it which is however obviously closed too; therefore we find that $O'(\Lambda)^{\mathbb{C}}=\{f^{\mathbb{C}}(\mathbf{z})f^{\mathbb{C}}(\mathbf{v})\,|\,\mathbf{z},\mathbf{v}\in(S^6)^{\mathbb{C}}\text{ as above}\}$.

Let us turn now to $O'(\Lambda)$. As defined in (4) the "real part" $O'(\Lambda)$ arises within $O'(\Lambda)^{\mathbb{C}}$ by imposing on the pairs $s_1 \in S_u$ and $s_2 \in \overline{S}_u$ the further reality condition $s_1 = s = \overline{s}_2$. This is because we know that $g = \lim\limits_n \left(s^{1/n}\overline{s}^{1/n}\right)^n$ is a decomposition of a real element; we also know that given $g \in \mathrm{G}_2$ one can find an element $s \in S_u$ such that for every $h \in \mathrm{SU}(3)^{\mathbb{C}}$ there exist $h_1 \in S_u \cap \mathrm{SU}(3)^{\mathbb{C}}$ and $h_2 \in \overline{S}_u \cap \mathrm{SU}(3)^{\mathbb{C}}$ satisfying $gh = \lim\limits_n \left(s^{1/n}h_1^{1/n}\overline{s}^{1/n}h_2^{1/n}\right)^n$. Thus as s runs over S_u the correspondence

$$s\Lambda^2 s^{-1}\,\overline{s}\Lambda^2\overline{s}^{-1} \iff \lim_n \left(s^{1/n}\,\overline{s}^{1/n}\right)^n \Lambda \lim_n \left(s^{1/n}\,\overline{s}^{1/n}\right)^{-n}$$

is a homeomorphism between $O'(\Lambda)$ as defined in (4) and $O(\Lambda)$ as defined in (1) i.e. between $O'(\Lambda)$ and S^6. To find an explicit parameterization of $O'(\Lambda)$ we go on again with observing that every element $g \in \mathrm{G}_2$ can be written non-uniquely in the form $g = \lim\limits_n \left(s^{1/n}\overline{s}^{1/n}\right)^n$ with $s \in S_u$; and in addition, every $s \in S_u$ admits a unique decomposition $s = \lim\limits_n \left(A^{1/n}h^{1/n}\right)^n$ where A is the matrix above and $h \in S_u \cap \mathrm{SU}(3)^{\mathbb{C}}$. Introducing $Z(z_2,z_4,z_6) := \sqrt{-6}\,z_2U_{+1} - \sqrt{-6}\,z_4U_{-2} + \sqrt{6}\,z_6U_{+3} \in \mathfrak{g}_2^{\mathbb{C}}$ hence $\exp Z(z_2,z_4,z_6) = A(z_2,z_4,z_6)$ as so far and likewise $W \in \mathfrak{g}_2^{\mathbb{C}}$ via $\exp W = h$ these Trotter decompositions imply that $g \in \mathrm{G}_2$ can be written as

$$g = \exp\left(Z(z_2,z_4,z_6) + W + \overline{Z}(\overline{z}_2,\overline{z}_4,\overline{z}_6) + \overline{W}\right)\ .$$

Consequently, taking into account the connectedness and compactness of G_2 and that the injective restriction of $\exp : \mathfrak{g}_2 \to \mathrm{G}_2$ is a proper map, since $A(0,0,0) = e$ there exists a smallest bounded connected open subset $\Omega \subset \mathbb{C}^3$ about the origin, the "parameter space", such that every element of G_2 can be obtained out of a matrix $A(z_2,z_4,z_6)$ satisfying $(z_2,z_4,z_6) \in \overline{\Omega}$ and an element h also belonging to the closure of a neighbourhood of $e \in S_u \cap \mathrm{SU}(3)^{\mathbb{C}}$. Since $\dim_{\mathbb{R}} O'(\Lambda) = 6$ and the matrix $A(z_2,z_4,z_6)$ already depends on 6 real variables, we can represent an open subset of $O'(\Lambda)$ simply by matrices of the form $A\Lambda^2A^{-1}\overline{A}\Lambda^2\overline{A}^{-1}$. It is easy to see that this subset is in fact closed too thus coincides with $O'(\Lambda)$. Because $f^{\mathbb{C}}(\mathbf{z}) = A\Lambda^2A^{-1}$ hence $f^{\mathbb{C}}(\overline{\mathbf{z}}) = \overline{f^{\mathbb{C}}(\mathbf{z})} = \overline{A\Lambda^2A^{-1}} = \overline{A}\Lambda^2\overline{A}^{-1}$ we find that the claimed description $O'(\Lambda) = \{f^{\mathbb{C}}(\mathbf{z})f^{\mathbb{C}}(\overline{\mathbf{z}}) \,|\, \text{with } \mathbf{z} \in (S^6)^{\mathbb{C}} \text{ as above with } (z_2,z_4,z_6) \in \overline{\Omega}\}$ follows.

Finally observe that $O'(\Lambda)$ is homeomorphic to S^6 as we already know and it does not depend on $u = a + \sqrt{-1}b$. Moreover $z_2 = z_4 = z_6 = 0$ gives $\mathbf{z} = -\mathbf{e}_1$ and $f^{\mathbb{C}}(-\mathbf{e}_1)f^{\mathbb{C}}(-\mathbf{e}_1) = \Lambda^2\Lambda^2 = \Lambda \in O'(\Lambda)$ as stated. $\square$

Our last steps are then as follows. Put

$$F(\mathbf{z},\overline{\mathbf{z}}) := f^{\mathbb{C}}(\mathbf{z})f^{\mathbb{C}}(\overline{\mathbf{z}})$$

and let $\{X_1,\dots,X_6\}$ be a reasonable $\mathbb{R}$-basis at $T_{F(\mathbf{z},\overline{\mathbf{z}})}O'(\Lambda)$ and extend it smoothly to an $\mathbb{R}$-frame over the punctured space $O'(\Lambda) \setminus \{\text{point}\}$ permitted by the parameterization of Lemma 3.2 i.e. take all $\mathbf{z} = -\mathbf{e}_1 + z_2\mathbf{e}_2 - \sqrt{-1}z_2\mathbf{e}_3 + z_4\mathbf{e}_4 - \sqrt{-1}z_4\mathbf{e}_5 + z_6\mathbf{e}_6 - \sqrt{-1}z_6\mathbf{e}_7$ with $(z_2,z_4,z_6) \in \Omega$ (note that no frame can extend further because $TO'(\Lambda) \cong TS^6$ is not trivial). Then the construction of J_u at $\mathfrak{g}_2$ allows one to compute the action of J_u at $T_{F(\mathbf{z},\overline{\mathbf{z}})}O'(\Lambda)$ by left-invariance i.e. applying the formula $J_uX_i := L_{F(\mathbf{z},\overline{\mathbf{z}})*}J_uL^{-1}_{F(\mathbf{z},\overline{\mathbf{z}})*}X_i$ or in simpler notation

$$J_uX_i := F(\mathbf{z},\overline{\mathbf{z}})J_uF(\mathbf{z},\overline{\mathbf{z}})^{-1}X_i\ .$$

To carry out these computations one has to proceed in principle as follows: first by the aid of the Hermitian scalar product $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathbb{C}}$ on $\mathfrak{g}_2^{\mathbb{C}}$ one computes the matrix coefficients $\left\langle F(\mathbf{z},\overline{\mathbf{z}})^{-1}X_i\,,\,H_\pm\right\rangle^{\mathbb{C}} \in \mathbb{C}$, etc. in order to expand the pullbacks $F(\mathbf{z},\overline{\mathbf{z}})^{-1}X_i \in \mathfrak{g}_2^{\mathbb{C}}$ in the $\mathbb{C}$-linear extension of the $\mathfrak{g}_2$-basis $\{H_\pm, X_{\pm k}, Y_{\pm k}\}$ used in (7); secondly one uses the explicit action (7) of J_u in this basis to obtain $J_u F(\mathbf{z},\overline{\mathbf{z}})^{-1}X_i \in \mathfrak{g}_2^{\mathbb{C}}$; and thirdly by multiplying this matrix with $F(\mathbf{z},\overline{\mathbf{z}})$ from the left one transfers the result from $\mathfrak{g}_2^{\mathbb{C}}$ back to $T_{F(\mathbf{z},\overline{\mathbf{z}})}\mathrm{G}_2^{\mathbb{C}}$. Writing

$$J_{ij} := \left\langle J_u F(\mathbf{z},\overline{\mathbf{z}})^{-1}X_i \, , \, F(\mathbf{z},\overline{\mathbf{z}})^{-1}X_j \right\rangle^{\mathbb{C}}$$

one tries to check that $J_u X_i - \sum\limits_{j=1}^{6} J_{ij}X_j = 0$ implying $F(\mathbf{z},\overline{\mathbf{z}})J_u F(\mathbf{z},\overline{\mathbf{z}})^{-1}X_i \in T_{F(\mathbf{z},\overline{\mathbf{z}})}O'(\Lambda) \subset T_{F(\mathbf{z},\overline{\mathbf{z}})}\mathrm{G}_2^{\mathbb{C}}$ i.e. the Samelson almost complex structure indeed restricts to $O'(\Lambda)\setminus\{\text{point}\}$. Moreover introducing the singular metric

$$g_{ik} := \left\langle F(\mathbf{z},\overline{\mathbf{z}})^{-1}X_i \, , \, F(\mathbf{z},\overline{\mathbf{z}})^{-1}X_k \right\rangle^{\mathbb{C}}$$

and its inverse g^{ik} along $O'(\Lambda)\setminus\{\text{point}\}$ one tries to check that the matrix coefficients

$$J^i_k := \sum_{j=1}^{6} g^{ij}J_{jk}$$

of the true $(1,1)$-tensor J_u are already smooth as $(z_2,z_4,z_6) \to \partial\overline{\Omega}$ yielding a well-defined J_u over the whole $O'(\Lambda)$ as desired. However it is a *mission impossible* to perform and plot these brute force computations here.

The Samelson complex structure and its integrability shows up more clearly if we rather pass to complexification and exploit some technical observations made during the proof of Theorem 2.1. The obvious tangent vectors $\left\{\frac{\partial F(\mathbf{z},\overline{\mathbf{z}})}{\partial z_2}, \frac{\partial F(\mathbf{z},\overline{\mathbf{z}})}{\partial \overline{z}_2}, \dots, \frac{\partial F(\mathbf{z},\overline{\mathbf{z}})}{\partial z_6}, \frac{\partial F(\mathbf{z},\overline{\mathbf{z}})}{\partial \overline{z}_6}\right\}$ comprise a $\mathbb{C}$-basis in $T_{F(\mathbf{z},\overline{\mathbf{z}})}O'(\Lambda)^{\mathbb{C}}$. Consequently, writing $L^{-1}_{F(\mathbf{z},\overline{\mathbf{z}})*}\frac{\partial F(\mathbf{z},\overline{\mathbf{z}})}{\partial z_{2k}}$ etc., simply as a matrix product $F(\mathbf{z},\overline{\mathbf{z}})^{-1}\frac{\partial F(\mathbf{z},\overline{\mathbf{z}})}{\partial z_{2k}}$ etc., from now on, $\left\{F(\mathbf{z},\overline{\mathbf{z}})^{-1}\frac{\partial F(\mathbf{z},\overline{\mathbf{z}})}{\partial z_2}, \dots, F(\mathbf{z},\overline{\mathbf{z}})^{-1}\frac{\partial F(\mathbf{z},\overline{\mathbf{z}})}{\partial \overline{z}_6}\right\}$ is a $\mathbb{C}$-basis in $L^{-1}_{F(\mathbf{z},\overline{\mathbf{z}})*}T_{F(\mathbf{z},\overline{\mathbf{z}})}O'(\Lambda)^{\mathbb{C}} \subset \mathfrak{g}_2^{\mathbb{C}}$. It has an asymmetric shape

$$\left\{ f^{\mathbb{C}}(\overline{\mathbf{z}})^{-1}f^{\mathbb{C}}(\mathbf{z})^{-1}\frac{\partial f^{\mathbb{C}}(\mathbf{z})}{\partial z_2}f^{\mathbb{C}}(\overline{\mathbf{z}}) \, , \, f^{\mathbb{C}}(\overline{\mathbf{z}})^{-1}\frac{\partial f^{\mathbb{C}}(\overline{\mathbf{z}})}{\partial \overline{z}_2}, \dots, f^{\mathbb{C}}(\overline{\mathbf{z}})^{-1}\frac{\partial f^{\mathbb{C}}(\overline{\mathbf{z}})}{\partial \overline{z}_6} \right\} .$$

Instead of this straightforward basis consider the not only symmetric but even simpler collection

$$\left\{ f^{\mathbb{C}}(\mathbf{z})^{-1}\frac{\partial f^{\mathbb{C}}(\mathbf{z})}{\partial z_2} \, , \, f^{\mathbb{C}}(\overline{\mathbf{z}})^{-1}\frac{\partial f^{\mathbb{C}}(\overline{\mathbf{z}})}{\partial \overline{z}_2}, \dots, f^{\mathbb{C}}(\mathbf{z})^{-1}\frac{\partial f^{\mathbb{C}}(\mathbf{z})}{\partial z_6} \, , \, f^{\mathbb{C}}(\overline{\mathbf{z}})^{-1}\frac{\partial f^{\mathbb{C}}(\overline{\mathbf{z}})}{\partial \overline{z}_6} \right\}$$

within $\mathfrak{g}_2^{\mathbb{C}}$ (but yet the full size of these matrices can be guessed from the shape of the real derivative $f(\mathbf{x})^{-1}f(\mathbf{x})_*\xi$ computed above). Note that actually $\left\{ f^{\mathbb{C}}(\mathbf{z})^{-1}\frac{\partial f^{\mathbb{C}}(\mathbf{z})}{\partial z_2} \, , \, f^{\mathbb{C}}(\mathbf{z})^{-1}\frac{\partial f^{\mathbb{C}}(\mathbf{z})}{\partial z_4} \, , \, f^{\mathbb{C}}(\mathbf{z})^{-1}\frac{\partial f^{\mathbb{C}}(\mathbf{z})}{\partial z_6} \right\}$ gives a $\mathbb{C}$-basis in $L^{-1}_{f^{\mathbb{C}}(\mathbf{z})*}T_{f^{\mathbb{C}}(\mathbf{z})}S_u\Lambda^2 S_u^{-1} \subset \mathfrak{s}_u$ and the rest forms a $\mathbb{C}$-basis in $L^{-1}_{f^{\mathbb{C}}(\overline{\mathbf{z}})*}T_{f^{\mathbb{C}}(\overline{\mathbf{z}})}\overline{S}_u\Lambda^2\overline{S}_u^{-1} \subset \overline{\mathfrak{s}}_u$. Consequently, since by recalling (6) we know that a splitting

$$L^{-1}_{F(\mathbf{z},\overline{\mathbf{z}})*}T_{F(\mathbf{z},\overline{\mathbf{z}})}O'(\Lambda)^{\mathbb{C}} = L^{-1}_{f^{\mathbb{C}}(\mathbf{z})*}T_{f^{\mathbb{C}}(\mathbf{z})}S_u\Lambda^2 S_u^{-1} \oplus L^{-1}_{f^{\mathbb{C}}(\overline{\mathbf{z}})*}T_{f^{\mathbb{C}}(\overline{\mathbf{z}})}\overline{S}_u\Lambda^2\overline{S}_u^{-1}$$

holds, in fact this symmetric collection of matrices constitutes a $\mathbb{C}$-basis in $L^{-1}_{F(\mathbf{z},\overline{\mathbf{z}})*}T_{F(\mathbf{z},\overline{\mathbf{z}})}O'(\Lambda)^{\mathbb{C}}$ too. Define $Z_{2k-1} := F(\mathbf{z},\overline{\mathbf{z}})f^{\mathbb{C}}(\mathbf{z})^{-1}\frac{\partial f^{\mathbb{C}}(\mathbf{z})}{\partial z_{2k}}$ and $Z_{2k} := F(\mathbf{z},\overline{\mathbf{z}})f^{\mathbb{C}}(\overline{\mathbf{z}})^{-1}\frac{\partial f^{\mathbb{C}}(\overline{\mathbf{z}})}{\partial \overline{z}_{2k}}$ for $k = 1,2,3$; then $\{Z_1,\dots,Z_6\}$ gives rise to a smooth $\mathbb{C}$-frame field if $(z_2,z_4,z_6) \in \Omega$ that is over the punctured space $O'(\Lambda)\setminus\{\text{point}\}$.

From this point we proceed as before. Introducing again $J^{\mathbb{C}}_{ij} := \left\langle J^{\mathbb{C}}_u F(\mathbf{z},\overline{\mathbf{z}})^{-1}Z_i\, ,\, F(\mathbf{z},\overline{\mathbf{z}})^{-1}Z_j\right\rangle^{\mathbb{C}}$ we simply find that

$$J^{\mathbb{C}}_{ij} = \pm\sqrt{-1}\, g_{ij}$$

where we write again $g_{kl} := \left\langle F(\mathbf{z},\overline{\mathbf{z}})^{-1}Z_k\, ,\, F(\mathbf{z},\overline{\mathbf{z}})^{-1}Z_l\right\rangle^{\mathbb{C}}$ and g^{kl} for the components of a complex singular metric and its inverse along $O'(\Lambda)\setminus\{\text{point}\}$. This is because from the observations on the frame made above we know that $F(\mathbf{z},\overline{\mathbf{z}})^{-1}Z_k \in \mathfrak{s}_u = \mathfrak{g}_2^{1,0}$ for $k=1,3,5$ and likewise $F(\mathbf{z},\overline{\mathbf{z}})^{-1}Z_k \in \overline{\mathfrak{s}}_u = \mathfrak{g}_2^{0,1}$ for $k=2,4,6$ where $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{g}_2^{1,0}\oplus\mathfrak{g}_2^{0,1}$ is the $\pm\sqrt{-1}$-eigenspace decomposition of the complexified Lie algebra with respect to the complex linear extension $J^{\mathbb{C}}_u$ of the Samelson complex structure. Consequently by left-invariance of the Samelson complex structure the matrix elements $J^{\mathbb{C}}{}^i_k = \sum\limits_{j=1}^{6} g^{ij}J^{\mathbb{C}}_{jk}$ of the true $(1,1)$-tensor $J^{\mathbb{C}}_u$ simply look like

$$J^{\mathbb{C}}{}^i_k = \pm\sqrt{-1}\,\delta^i_k$$

hence it readily follows that despite the degeneration of the frame $\{Z_1,\dots,Z_6\}$ the $(1,1)$-tensor $J^{\mathbb{C}}_u$ itself remains smooth as $(z_2,z_4,z_6)\to\partial\overline{\Omega}$ hence we obtain a well-defined $J^{\mathbb{C}}_u$ over the whole $O'(\Lambda)$.

Reality of $O'(\Lambda)\subset O'(\Lambda)^{\mathbb{C}}$ implies that $F(\mathbf{z},\overline{\mathbf{z}})\overline{F(\mathbf{z},\overline{\mathbf{z}})^{-1}Z}_{2k-1} = F(\mathbf{z},\overline{\mathbf{z}})F(\mathbf{z},\overline{\mathbf{z}})^{-1}Z_{2k}$ for $k=1,2,3$ in $T_{F(\mathbf{z},\overline{\mathbf{z}})}O'(\Lambda)^{\mathbb{C}} = T_{F(\mathbf{z},\overline{\mathbf{z}})}O'(\Lambda)\otimes_{\mathbb{R}}\mathbb{C}$. Consequently we can introduce an $\mathbb{R}$-frame over the punctured space $O'(\Lambda)\setminus\{\text{point}\}$ canonically induced by the $\mathbb{C}$-frame $\{Z_1,\dots,Z_6\}$ i.e. $X_{2k-1} := \frac{1}{2}(Z_{2k-1}+Z_{2k})$ and similarly $X_{2k} := \frac{1}{2\sqrt{-1}}(Z_{2k-1}-Z_{2k})$ for all $k=1,2,3$. In the resulting $\mathbb{R}$-frame $\{X_1,\dots,X_6\}$ the real almost complex tensor field therefore takes the very simple form

$$J_u|_{O'(\Lambda)\setminus\{\text{point}\}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1 & 0\end{pmatrix}$$

demonstrating its smooth extendibility over the whole $O'(\Lambda)\cong S^6$. Concerning its integrability however some care is needed because the frame $\{X_1,\dots,X_6\}$ is not obviously torsion-free hence the constancy of J_u in this frame might not prove anything. Let us therefore compute the Nijenhuis tensor. Observe that the $(0,1)$-parts $X^{1,0}_j = \frac{1}{2}(X_j-\sqrt{-1}J_uX_j)$ look simply $X^{1,0}_{2k-1} = \frac{1}{2}Z_{2k-1}$ and $X^{1,0}_{2k} = \frac{1}{2\sqrt{-1}}Z_{2k-1}$ for $k=1,2,3$ consequently $F(\mathbf{z},\overline{\mathbf{z}})^{-1}X^{1,0}_j\in\mathfrak{g}_2^{1,0}$ for all $j=1,\dots,6$. Thus

$$N_{J_u}(X_i,X_j) = \left[X^{1,0}_i\, ,\, X^{1,0}_j\right]^{0,1} = 0$$

because $\left[\mathfrak{g}_2^{1,0}\, ,\mathfrak{g}_2^{1,0}\right]^{0,1} = 0$. Thus $N_{J_u}=0$ guaranteeing integrability over $O'(\Lambda)\setminus\{\text{point}\}$ hence over the whole $O'(\Lambda)$.

Note that in this second picture all the computational difficulties and geometric subtleties concerning J_u have been compressed into the highly non-trivial frame field $\{X_1,\dots,X_6\}$ in which J_u looks simply constant. For instance, since as exhibited in Lemma 3.2 the space $O'(\Lambda)$ hence the frame $\{X_1,\dots,X_6\}$ along $O'(\Lambda)\setminus\{\text{point}\}$ are independent of the Samelson moduli parameter $u=a+\sqrt{-1}b$ with $a\neq 0$, we see that the constructed complex manifold X_u in Theorem 2.1 is *unique* i.e. independent of the Samelson moduli parameter $u\in\mathbb{C}\setminus\sqrt{-1}\mathbb{R}$.

With this observation we conclude the struggle with the explicit construction of the complex structure on the six-sphere.

4 Appendix: inner automorphisms and $\pi_6(\mathrm{G}_2) \cong \mathbb{Z}_3$

To close we take a look of the conjugate orbit $O(\Lambda) \subset \mathrm{G}_2$ in (1) from a topological viewpoint. As we have seen in Lemma 3.1 it can be identified with the image of the map $f: S^6 \to \mathrm{G}_2$ in (9); we demonstrate that in this form the conjugate orbit represents the generator of the sixth homotopy group of the automorphism group of the octonions. Consequently this homotopy group is non-trivial and is generated by inner automorphisms (thus $O'(\Lambda) \subset \mathrm{G}_2^{\mathbb{C}}$ in (4) is also homotopically non-trivial). We acknowledge that this group has been known for a long time [10] and even our proof is essentially the same as the nice geometric one in [3].

Theorem 4.1. (cf. [3, 10]) *There exists an isomorphism $\pi_6(\mathrm{G}_2) \cong \mathbb{Z}_3$. Moreover the map (9) constructed out of the collection of rotations induced by inner automorphisms (8) of the octonions, is a representative of the generator of this homotopy group.*

Proof. Recall that $\mathrm{Spin}(7) \subset \mathrm{Cliff}_0(\mathbb{R}^7) \cong \mathrm{Cliff}(\mathbb{R}^6) \cong \mathbb{R}(8)$ hence the unique spin representation of $\mathrm{SO}(7)$ acts on $\mathbb{R}^8$. This gives rise to an embedding $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. The projection $P: \mathrm{SO}(8) \to S^7$ sending a matrix onto its (let us say) first column restricts to $\mathrm{Spin}(7)$ providing us with a projection $\tilde{p}: \mathrm{Spin}(7) \to S^7$. Dividing this with the center $\mathbb{Z}_2 \cong Z(\mathrm{SO}(8)) \subset \mathrm{SO}(8)$ we obtain another projection $p: \mathrm{SO}(7) \to \mathbb{R}P^7$. The geometric meaning of this map is straightforward: the preimage of a point of $\mathbb{R}P^7$ i.e. a line in $\mathbb{R}^8$ consists of those rotations which keep this line fixed therefore act only on a hyperplane perpendicular to this line: dimension counting shows that these transformations are exactly the automorphisms of the octonions hence their collection is isomorphic to G_2. Consequently the projection $p: \mathrm{SO}(7) \to \mathbb{R}P^7$ is the classical G_2-fibration of $\mathrm{SO}(7)$ over $\mathbb{R}P^7$. It has an associated homotopy exact sequence whose relevant segment for us is

$$\begin{tikzcd}[column sep=small]
 & & \pi_7(\mathrm{SO}(7),\mathrm{G}_2,e) \arrow[d, "p_*"', "\cong"] \arrow[dr, "\partial_*"] & & \\
\ldots \arrow[r] & \pi_7(\mathrm{SO}(7),e) \arrow[ur, "j_*"] & \pi_7(\mathbb{R}P^7, p(e)) & \pi_6(\mathrm{G}_2,e) \arrow[r] & \pi_6(\mathrm{SO}(7),e) \arrow[r] & \ldots
\end{tikzcd}$$

where $j: (\mathrm{SO}(7),e,e) \to (\mathrm{SO}(7),\mathrm{G}_2,e)$ is induced by the embedding $e \in \mathrm{G}_2 \subset \mathrm{SO}(7)$.

First let us compute $\pi_6(\mathrm{G}_2)$. Take $S^7 = \{2\cos t \cdot \mathbf{e}_0 + 2\sin t \cdot \mathbf{x} \,|\, 0 \leqq t \leqq \pi\,,\, \mathbf{x} \in S^6\}$. The conjugation (8) can be enhanced to an orthogonal transformation of the octonions which on a particular $\mathbf{u} \in \mathbb{O}$ has the form (again parantheses omitted)

$$\mathbf{u} \mapsto (2\cos t \cdot \mathbf{e}_0 + 2\sin t \cdot \mathbf{x})\mathbf{u}(2\cos t \cdot \mathbf{e}_0 + 2\sin t \cdot \mathbf{x})^{-1} = (\cos t \cdot \mathbf{e}_0 + \sin t \cdot \mathbf{x})\mathbf{u}(\cos t \cdot \mathbf{e}_0 - \sin t \cdot \mathbf{x}) \quad (11)$$

and provides us with a map from S^7 into $\mathrm{SO}(8)$. However this apparent $\mathrm{SO}(8)$ transformation of $\mathbb{O} \cong \mathbb{R}^8$ leaves $\mathrm{Re}\,\mathbb{O} \cong \mathbb{R}$ invariant i.e. acts only on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$ therefore it is actually an $\mathrm{SO}(7)$ transformation. This way we obtain a map $F: S^7 \to \mathrm{SO}(7)$ such that $[F] = 1 \in \pi_7(\mathrm{SO}(7)) \cong \mathbb{Z}$ i.e. its homotopy class is a generator [15]. Take now the 7-cell $e^7 := \{2\cos t \cdot \mathbf{e}_0 + 2\sin t \cdot \mathbf{x} \,|\, 0 \leqq t \leqq \frac{\pi}{3}\,,\, \mathbf{x} \in S^6\} \subset S^7$. Its boundary is $\partial e^7 = \{\mathbf{e}_0 + \sqrt{3}\,\mathbf{x} \,|\, \mathbf{x} \in S^6\}$ hence constitutes the inner automorphisms (8) of the octonions therefore via (11) it lies within $\mathrm{G}_2 \subset \mathrm{SO}(7)$. Consequently restriction to this 7-cell gives rise to a map $F|_{e^7}: (e^7, \partial e^7) \to (\mathrm{SO}(7), \mathrm{G}_2)$ satisfying $[F|_{e^7}] = 1 \in \pi_7(\mathrm{SO}(7),\mathrm{G}_2) \cong \pi_7(\mathbb{R}P^7) \cong \mathbb{Z}$ i.e. its homotopy class continues to be a generator. Taking into account that the third power of an inner automorphism is the identity it is clear that $j_*[F] = 3[F|_{e^7}]$. Consequently, since $\pi_6(\mathrm{SO}(7)) \cong 0$ we conclude from the homotopy exact sequence that $\pi_6(\mathrm{G}_2) \cong \mathbb{Z}/3\mathbb{Z}$, written as $\mathbb{Z}_3$, as desired.

Regarding the generator, it readily follows from (8) and (11) that $\partial(F|_{e^7}) = f$ where $f: S^6 \to \mathrm{G}_2$ is the map (9). Therefore $\partial_*[F|_{e^7}] = [f] \in \pi_6(\mathrm{G}_2)$. By exactness $\partial_* \neq 0$ and $F|_{e^7}$ represents the generator, hence its image f also represents a non-trivial element in $\pi_6(\mathrm{G}_2)$ which is the generator. $\square$

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