

Complex structure on the six dimensional sphere from a spontaneous symmetry breaking

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Dedicated to the memory of my father

Abstract

Existence of a complex structure on the 6 dimensional sphere is proved in this paper. The proof is based on re-interpreting a hypothetical complex structure as a classical ground state of a Yang–Mills–Higgs-like theory on S^6 . This classical vacuum solution is then constructed by Fourier expansion (dimensional reduction) from the obvious one of a similar theory on the 14 dimensional exceptional compact Lie group G_2 .

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1 Introduction

A classical result of Borel and Serre from 1951 states that among spheres the two and the six dimensional are the only ones which can carry almost complex structures [3, 4]. In case of the two-sphere the almost complex structure I is *unique* and stems from the embedding $S^2 \subset \text{Im}\mathbb{H}$ as the unit sphere hence I acts at $x \in S^2$ simply by multiplication with x itself. Equivalently I can be constructed through the identification $S^2 \cong \mathbb{C}P^1$ consequently it is *integrable* in the sense that it comes from a complex manifold structure. On the contrary the six-sphere admits a *plethora* of almost complex structures which are typically *non-integrable*. For example the analogous six dimensional almost complex structure I can be constructed [42, pp. 163-164] from the inclusion $S^6 \subset \text{Im}\mathbb{O}$ where \mathbb{O} refers to the octonions or Cayley numbers; however it is not integrable [17]. This can be directly proved by a lengthy calculation of its Nijenhuis tensor: since it is not zero this Cayley almost complex structure is not integrable by the Newlander–Nirenberg theorem [34]. Putting an orientation and a Riemannian metric g onto S^6 an abundance of other almost complex structures compatible with the orientation and the metric emerges as smooth sections of the projectivized positive chirality spinor bundle $P\Sigma^+$, or equivalently, of the positive twistor space $Z^+(S^6, [g])$ where $[g]$ denotes the conformal class of g , cf. [28, Chapter IV, §9]. Homotopy theory also can be used to construct almost complex structures [35]. In spite of

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the two dimensional case however, it has been unknown whether or not any of these or other almost complex structures are integrable [14, pp. 16-18]—or it has rather been believed that all of them are not-integrable hence the six-sphere cannot be a complex manifold at all [20, p. 424].

One source of this dismissive attitude might be an æsthetic aversion: if the most canonical and natural Cayley almost complex structure is not integrable then what else could be? Another one might be the feeling that the existence of a complex structure on S^6 would be somehow awkward: for example [24] it would allow one to perform non-Kählerian deformations of the standard complex structure on $\mathbb{C}P^3$; a “minor disaster” as LeBrun says in [29].

During the past six decades several works appeared which claimed to prove the non-existence however one-by-one they turned out to be erroneous. Examples are [1, 23] while the latest trial was Chern’s attack [15] in 2004 based on the link between S^6 and the exceptional group G_2 . Although it also seems to have a gap (cf. [7]), Chern’s approach has introduced interesting new techniques into gauge theories [31].

These failures are not surprising because this complex structure, if exists, cannot be grasped by conventional means. Indeed, let X denote this hypothetical compact complex 3-manifold i.e., an integrable almost complex structure J on S^6 . Regarding classical geometry, X cannot have anything to do with $S^6 \subset \mathbb{R}^7$ as the usual sphere with the standard round metric g_0 since J cannot be orthogonal with respect to it [29, 46] or even to a metric g in an open neighbourhood of g_0 [48]. That is, X cannot be constructed from a section of $Z^+(S^6, [g])$ in the sense above with a metric g “close” to g_0 (for some properties of the standard twistor space $Z^+(S^6, [g_0])$ cf. [33, 39, 46] and see also [47]). Since $\pi_k(X) \cong 0$ ($k = 1, \dots, 5$) X cannot be fibered over any lower dimensional manifold therefore algebraic topology does not help to cut down the problem to a simpler one. It also follows that $H^2(X; \mathbb{Z}) \cong 0$ consequently X cannot be projective algebraic or even Kähler hence cannot be approached by conventional algebraic geometry or analysis. In fact any meromorphic function on X must be constant [10] demonstrating that the algebraic dimension of X is zero consequently it is as far from being algebraic as possible. Lacking good functions, the inadequacy of powerful methods of complex analysis also follows. Last but not least we also know that X is a truly inhomogeneous complex manifold in the sense that $\text{Aut}_\theta X$ cannot act transitively and none of its orbits can be open [24] (actually this property would permit the existence of the aforementioned exotic complex projective spaces).

Nevertheless the Hodge numbers of X are known in some extent [19, 40]. We also mention that there is an extensive literature about various submanifolds of S^6 equipped with various structures. Far from being complete, for instance the almost complex submanifolds of the Cayley almost complex S^6 are studied in [18] and an excellent survey about the Lagrangian submanifolds of the Cayley nearly Kähler S^6 is [12].

The long resistance of the problem against proving non-existence may indicate that one should rather try to seek a complex structure on S^6 . However the irregular features of this hypothetical space X convince us that *asymmetry*, *transcendental* (i.e., non-algebraic) *methods* and *inhomogeneity* should play a key role in finding it.

Indeed, by a result of Wood [45] from a “physical” viewpoint the æsthetic Cayley almost complex structure is energetically remarkable unstable.¹ Therefore it is not surprising that if integrable almost complex structures on S^6 exist then they would appear “far” from the Cayley one. For instance Peng and Tang [35] recently have constructed a novel almost complex structure by twisting and extending the standard complex structure on $\mathbb{C}^3 \subset S^6$. It is a vast deformation of the Cayley almost complex structure and is integrable except a narrow equatorial belt in S^6 . However it is still orthogonal with respect to the standard round metric hence LeBrun’s theorem [29, 46] forbids its full integrability.

¹However for a contradictory result cf. [2].

One may try to seek X by minimizing another energy functional over a space somehow related to S^6 . Two questions arise: what this space should be and how the energy functional should look like?

As we have indicated above, the central difficulty with this hypothetical complex structure is that it should be compact and non-Kähler at the same time. However in fact plenty of such complex manifolds are known to exist for a long time: classical examples are the Hopf-manifolds [22] from 1948, the Calabi–Eckmann manifolds [8] from 1953; or the compact, even dimensional, simply connected Lie groups—as it was observed by Samelson [38] also in 1953. *A fortiori* our candidate is the compact simply connected 14 dimensional exceptional Lie group G_2 as a complex manifold which also arises as the total space of a non-trivial $SU(3)$ principal bundle over S^6 .

Regarding the second question a physicist is experienced that there is a close analogy between the existence of a *geometric structure* on a manifold on the mathematical side (cf. e.g. [25]) and the existence of a *spontaneous symmetry breaking* in classical Yang–Mills–Higgs-like systems on the physical side. This later process is familiar in both classical and quantum field theory [13, 26, 41]. Therefore we are tempted to construct an appropriate (non-linear) field theory on G_2 what we call a *Yang–Mills–Higgs–Nijenhuis theory*. Its dynamical variables are a Riemannian metric, a compatible gauge and a Higgs field with a usual Higgs potential but coupled to the gauge field through the Nijenhuis tensor; hence the Higgs field in the spontaneously broken classical vacuum state represents a complex structure on G_2 . Following the standard procedure used in dimensional reduction (cf. [13, 20, 26, 41] or for a recent application [43, 44]) we Fourier expand this classical vacuum state with respect to $SU(3)$ and expect that the ground Fourier mode—independent of the vertical directions—descends and gives rise to a classical vacuum solution in an effective Yang–Mills–Higgs–Nijenhuis field theory on S^6 . If this classical vacuum solution exists then it would represent a complex structure on S^6 . In checking that this is indeed the case we exploit the double role played by the group $SU(3)$: it can be used not only for Fourier expansion but at the same time to put an almost complex structure onto TS^6 by constructing the tangent bundle as a vector bundle associated to the $SU(3)$ principal bundle $\pi : G_2 \rightarrow S^6$ by the aid of the standard 3 dimensional complex representation of $SU(3)$. It also should become clear in the course of the construction why this approach breaks down in apparently similar situations like $SO(4n+1)/SO(4n) \cong S^{4n}$. It turns out that, in sharp contrast to the case of $G_2/SU(3) \cong S^6$, in the former cases the tangent bundles TS^{4n} cannot be constructed as *complex* associated vector bundles simply because the standard vector representations of $SO(4n)$'s are *real*. Guided by these heuristic ideas in what follows we hope to prove rigorously the existence of a complex structure on the six-sphere.

The paper is organized as follows. Sect. 2 summarizes the physical background motivation. It contains a formulation of a hierarchy of classical field theories with gauge symmetry describing various spontaneous symmetry breakings related to the existence of various geometric structures on the underlying manifold. Apparently these sort of spontaneous symmetry breakings are not distinguished by physicists. In particular we identify a so-called “Yang–Mills–Higgs–Nijenhuis field theory” on the tangent bundle of an even dimensional oriented Riemannian manifold describing a so-called “weak spontaneous symmetry breaking” which is the physical reformulation of the existence of a non-Kählerian integrable almost complex structure on the underlying manifold. Unfortunately this theory is highly non-linear hence solving its field equations directly or even the detailed analytical study of them is far beyond our technical limits.

Instead in Sect. 3 we rigorously develop a sort of global Fourier expansion which makes it possible to push down sections of vector bundles with lifted G -action over the total space P of a principal G -bundle to sections of quotient vector bundles over the base space $M = P/G$. However if we want to make a contact with the gauge symmetry inherently present in our formulation it turns out that this global Fourier expansion is not unique: Fourier expansion of a section is in general *not* gauge

equivalent to the Fourier expansion of the gauge transformed section. Or saying equivalently, the Fourier expansion of vector-valued sections is not unique in general and depends on the chosen lifted G -action on the vector bundle on P .

As a warming-up in Sect. 4 we construct the standard complex structure on S^2 with our tools developed here. Actually no Fourier expansion is required for this trivial example.

In Sect. 5 then our tools are used for S^6 as follows: after identifying G_2 with an $SU(3)$ bundle over S^6 we Fourier expand the horizontal part of the underlying almost complex tensor field of a complex structure on G_2 . The ground Fourier mode depends on the chosen Fourier expansion and—since simply being the average along the fibers—always descends to S^6 as a skew-symmetric tensor field but in general not as an almost complex tensor field. However we demonstrate that there exists an $SU(3)$ Fourier expansion which is truncated more precisely the corresponding ground Fourier mode coincides with the full horizontal component of the original integrable almost complex structure on G_2 ; hence it gives rise to an almost complex structure on S^6 (cf. Lemma 5.1 here). We also make it clear why the resulting almost complex structure on S^6 cannot be homogeneous in spite of the fact that the complex structure it stems from is homogeneous on G_2 . In short the reason is that the tangent spaces of S^6 are identified with the horizontal part of TG_2 not simply by the derivative of the projection $\pi : G_2 \rightarrow S^6$ as usual but instead in a fiberwise twisted manner (cf. Lemma 5.3 here). This section contains our main result, namely that this non-homogeneous almost complex structure is integrable hence there exists a complex manifold homeomorphic to the six-sphere (cf. Theorem 5.1 here).

2 Geometric structures and symmetry breaking

In this section we construct classical field theories with gauge symmetry whose spontaneously broken vacua give rise to important geometric structures on the underlying manifold. In particular a so-called Yang–Mills–Higgs–Nijenhuis field theory is exhibited with a spontaneously broken vacuum corresponding to a non-Kählerian complex structure.

Let M be an n dimensional real manifold and pick an abstract Lie group $G \subset GL(n, \mathbb{R})$. Recall that [25, Section 2.6] a G -structure on M is a principal sub-bundle of the frame bundle of M whose fibers are G and the structure group of this sub-bundle is also G . A G -structure is called *integrable* (or *torsion-free*) if there is a torsion-free connection ∇ on TM with $\text{Hol}\nabla$ being (a subgroup of) G . An integrable G -structure is interesting because the individual infinitesimal geometric structures on the tangent spaces stem from an underlying global non-linear structure on M .

As a starting setup suppose that a G -structure on M is given together with some connection on TM whose holonomy group satisfies $\text{Hol}\nabla \subseteq G$. Note that G plays a double role here. Pick a Lie subgroup $H \subset G$. We are going to describe two broad versions of a “spontaneous symmetry breaking” over M as follows. Our first concept is a *strong spontaneous symmetry breaking*, denoted by $G \rightrightarrows H$, in which G as the holonomy group of ∇ is reducible to (a subgroup of) H . That is, the connection satisfies that $\text{Hol}\nabla \subseteq H$.

To clarify this concept, an example can be constructed as follows. Take a $2m$ real dimensional oriented Riemannian manifold (M, g) . Then TM is an $SO(2m)$ vector bundle associated with the standard representation of $SO(2m)$ and has a gauge group \mathcal{G}_{TM} consisting of fiber-preserving isomorphisms of TM which keep orientation and are orthogonal with respect to the metric g along each fiber. Let ∇ be an arbitrary $SO(2m)$ connection on TM and let R_∇ and T_∇ be its curvature and torsion respectively.

Consider moreover another associated bundle to the frame bundle making use of the adjoint representation of $SO(2m)$ on its Lie algebra. The fibers of this bundle $\text{Ad}M \subset \text{End}TM$ are $\mathfrak{so}(2m) \subset \text{End}\mathbb{R}^{2m}$. Let $\Phi \in C^\infty(M; \text{Ad}M)$ be its generic section. A geometric example of a section of this kind is

an orthogonal almost complex structure whose induced orientation agrees with the given one. Indeed, since for a real oriented vector space with a scalar product $(V, \langle \cdot, \cdot \rangle)$ the moduli space of orthogonal complex structures J compatible with the orientation is precisely $\text{SO}(V) \cap \mathfrak{so}(V) \subset \text{End } V$, it follows that we can look at any orthogonal almost complex structure with respect to g and compatible induced orientation as a section $J = \Phi \in C^\infty(M; \text{Ad}M)$. Let $\nabla : C^\infty(M; \text{Ad}M) \rightarrow C^\infty(M; \text{Ad}M \otimes \wedge^1 M)$ be the connection on $\text{Ad}M$, associated to the $\text{SO}(2m)$ connection on TM introduced above. Also fix a real number $e \neq 0$.

Then for instance an $\text{SO}(2m) \rightrightarrows \text{U}(m)$ strong spontaneous symmetry breaking arises by seeking the hypothetical vacuum solutions (see below) of the energy functional

$$\mathcal{E}(\nabla, \Phi) := \frac{1}{2} \int_M \left(\frac{1}{e^2} |R_\nabla|_g^2 + |\nabla\Phi|_g^2 + e^2 |\Phi\Phi^* - \text{Id}_{TM}|_g^2 \right) dV.$$

Various pointwise norms induced by the metric on various $\text{SO}(2m)$ vector bundles and the Killing form on $\mathfrak{so}(2m)$ are used here to calculate norms of tensor fields in the integral. We note that Φ^* is the pointwise adjoint (i.e., transpose) of the bundle map $\Phi : TM \rightarrow TM$ with respect to g .

$\mathcal{E}(\nabla, \Phi) \geq 0$ and it is invariant under $\text{SO}(2m)$ acting as gauge transformations $\alpha \in \mathcal{G}_{TM}$ given by $(\nabla, \Phi) \mapsto (\alpha\nabla\alpha^{-1}, \alpha\Phi\alpha^{-1})$. In fact $\mathcal{E}(\nabla, \Phi)$ is the usual Euclidean action of a Yang–Mills–Higgs theory from the physics literature formulated on TM . Hence, using physicists' terminology [13, 26, 41] the above functional can be regarded as the action functional of a *spontaneously broken $2m$ dimensional Euclidean $\text{SO}(2m)$ Yang–Mills–Higgs system* on TM . In this context ∇ is the “ $\text{SO}(2m)$ gauge field” and Φ is the “Higgs field in the adjoint representation”, e represents the “coupling constant” and the quartic polynomial $|\Phi\Phi^* - \text{Id}_{TM}|_g^2$ is the “Higgs potential”. If a smooth vacuum solution $\mathcal{E}(\nabla, \Phi, g) = 0$ exists then

$$\begin{cases} 0 = R_\nabla \\ 0 = \nabla\Phi \\ 0 = \Phi\Phi^* - \text{Id}_{TM}. \end{cases}$$

(Note that these are *not* the Euler–Lagrange equations of the system!) The first equation says that ∇ is flat. The third equation dictates $\Phi^2 = -\text{Id}_{TM}$ hence restricts the Higgs field to be an orthogonal almost complex structure $\Phi = J$ with respect to g . The second equation reduces the holonomy of ∇ in the usual way [25, Proposition 2.5.2] to (by flatness necessarily a discrete subgroup of) $\text{U}(m)$. Therefore this vacuum—if exists—describes a (not necessarily integrable) almost complex structure on M . The fact that for this vacuum solution $\text{Hol}\nabla \subset \text{U}(m) \subset \text{SO}(2m)$ justifies that this is indeed an example for a strong spontaneous symmetry breaking $\text{SO}(2m) \rightrightarrows \text{U}(m)$. The resulting structure is non-integrable and flat hence mathematically less interesting. We have included this example because of its relevance in physics.

This “illness” of the vacuum is cured if the curvature is replaced by the torsion. In passing consider a new theory with

$$\mathcal{E}(\nabla, \Phi, g) := \frac{1}{2} \int_M \left(\frac{1}{e^2} |\mathfrak{S}(R_\nabla)|_g^2 + |\nabla g|_g^2 + |\nabla\Phi|_g^2 + e^2 |\Phi\Phi^* - \text{Id}_{TM}|_g^2 \right) dV$$

where $\mathfrak{S}(R_\nabla)$ is the symmetrization of R_∇ given by

$$\mathfrak{S}(R_\nabla)(u, v)w := \mathfrak{S}(R_\nabla(u, v)w) = R_\nabla(u, v)w + R_\nabla(w, u)v + R_\nabla(v, w)u$$

for vector fields u, v, w on M . Bianchi's first identity says that

$$\mathfrak{S}(R_\nabla(u, v)w) = \mathfrak{S}(T_\nabla(T_\nabla(u, v), w) + \nabla_u T_\nabla(v, w))$$

demonstrating that $\mathfrak{S}(R_\nabla) = 0$ if and only if $T_\nabla = 0$. This functional is again non-negative and taking into account that $\alpha^*g = g$ it is acted upon by \mathcal{G}_{TM} as $(\nabla, \Phi, g) \mapsto (\alpha\nabla\alpha^{-1}, \alpha\Phi\alpha^{-1}, g)$ leaving the functional invariant. Although this theory looks physically less obvious this departure from physics yields a more familiar structure on the mathematical side. Namely, a hypothetical smooth minimizing configuration $\mathcal{E}(\nabla, \Phi, g) = 0$ gives rise to a Kähler structure on M . Note that $\text{Hol}\nabla \subseteq \text{U}(m) \subset \text{SO}(2m)$ hence this is again an (integrable) strong spontaneous symmetry breaking $\text{SO}(2m) \rightrightarrows \text{U}(m)$.

Returning to the original setup, our second concept is a *weak spontaneous symmetry breaking*, denoted as $G \rightrightarrows H$, in which G as the structure group of TM is reducible to (a subgroup of) H . That is, simply there is a H -structure on M but probably the connection is not compatible with this structure. A strong symmetry breaking always implies the corresponding weak one cf. e.g. [25, Theorem 2.3.6] however not the other way round.

An (integrable) weak symmetry breaking $\text{SO}(2m) \rightrightarrows \text{U}(m)$ is provided by the following example which goes along the lines of the previous cases. Consider the first order quadratic differential operator

$$N_\nabla : C^\infty(M; \text{Ad}M) \longrightarrow C^\infty(M; (\text{End}TM) \otimes \wedge^1 M) \cong C^\infty(M; TM \otimes \wedge^1 M \otimes \wedge^1 M)$$

whose shape on two vector fields u, v on M is defined to be

$$(N_\nabla\Phi)(u, v) := (\nabla_{\Phi u}\Phi)v - \Phi(\nabla_u\Phi)v - (\nabla_{\Phi v}\Phi)u + \Phi(\nabla_v\Phi)u$$

or equivalently in the particular gauge given by a local coordinate system (U, x^1, \dots, x^{2m}) it looks like [20, Eqn. 15.2.5]

$$((N_\nabla\Phi)|_U)^k_{ij} = \sum_{l=1}^{2m} \Phi^l_i (\nabla_l \Phi^k_j - \nabla_j \Phi^k_l) - \sum_{l=1}^{2m} \Phi^l_j (\nabla_l \Phi^k_i - \nabla_i \Phi^k_l), \quad i, j, k = 1, \dots, 2m. \quad (1)$$

Note that $N_\nabla\Phi$ transforms as a $(2, 1)$ -tensor under $\text{SO}(2m)$ acting on TM . This time set

$$\mathcal{E}(\nabla, \Phi, g) := \frac{1}{2} \int_M \left(\frac{1}{e^2} |\mathfrak{S}(R_\nabla)|_g^2 + |\nabla g|_g^2 + |N_\nabla\Phi|_g^2 + e^2 |\Phi\Phi^* - \text{Id}_{TM}|_g^2 \right) dV. \quad (2)$$

This functional is again invariant under \mathcal{G}_{TM} acting on (∇, Φ, g) as before and it is non-negative. Therefore it describes a sort of gauge theory on M what we call a *Yang–Mills–Higgs–Nijenhuis field theory*. The peculiarities here compared to the usual Yang–Mills–Higgs Lagrangian are that the curvature has again been replaced with the torsion, the connection is minimally coupled to the metric but its usual minimal coupling $\nabla\Phi$ to the Higgs field has been replaced with $N_\nabla\Phi$ regarded as a non-minimal quadratic coupling in (2). This further departure from the physical side gives rise again to a familiar structure on M on the mathematical side as follows.

If the smooth vacuum $\mathcal{E}(\Phi, \nabla, g) = 0$ is achieved in (2) over M then it satisfies

$$\begin{cases} 0 = \mathfrak{S}(R_\nabla) \\ 0 = \nabla g \\ 0 = N_\nabla\Phi \\ 0 = \Phi\Phi^* - \text{Id}_{TM}. \end{cases} \quad (3)$$

The first and second equations say that the $\text{SO}(2m)$ connection ∇ is the Levi–Civita connection of g . The fourth equation ensures us that $\Phi^2 = -\text{Id}_{TM}$ i.e., the Higgs field reduces to an orthogonal almost complex structure $\Phi = J$ with respect to the metric and is compatible with the orientation. Exploiting

this together with the fact that the Levi–Civita connection is torsion-free the operator $N_{\nabla}\Phi = N_{\nabla}J$ in the gauge (1) cuts down to

$$((N_{\nabla}J)|_U)^k_{ij} = \sum_{l=1}^{2m} J^l_i \left(\partial_l J^k_j - \partial_j J^k_l \right) - \sum_{l=1}^{2m} J^l_j \left(\partial_l J^k_i - \partial_i J^k_l \right), \quad i, j, k = 1, \dots, 2m$$

hence $N_{\nabla}J = N_J$ is the *Nijenhuis tensor* of the almost complex structure J given by

$$N_J(u, v) = -[u, v] + [Ju, Jv] - J[u, Jv] - J[Ju, v].$$

Consequently the third equation of (3) says that $N_J = 0$ that is, this almost complex structure is integrable on M and is orthogonal with respect to g . Therefore if a vacuum solution to (2) exists it makes M into a complex analytic m -space in light of the Newlander–Nirenberg theorem [34]. Note that this time the structure group of TM has been cut down to (a subgroup of) $U(m)$ but probably for the connection $\text{Hol}\nabla \not\subseteq U(m)$ hence we indeed obtain an example for an (integrable) weak spontaneous symmetry breaking $\text{SO}(2m) \Rightarrow U(m)$. Accordingly the resulting complex structure on M is more general than the Kähler one in the previous example.

We have described two different (i.e., strong and weak) symmetry breakings from $\text{SO}(2m)$ down to $U(m)$ over a Riemannian manifold. However for physicists these concepts coincide because for them a “spontaneous symmetry breaking” has a slightly different meaning. It simply means that although one starts with an $\text{SO}(2m)$ -invariant theory over M , the vacuum (∇, Φ, g) is stabilized by a subgroup of $\text{SO}(2m)$ isomorphic to $U(m)$ only. Of course this criteria holds in all of our cases.

We know that all integrable solutions to the previous strong problem—i.e., the Kähler structures—provide integrable solutions to the weak one—i.e., complex analytic structures—too. But we rather raise the question whether or not there are solutions strictly to the weaker problem which do not stem from strong solutions. The answer is yes, even in the compact case. In what follows let G be a connected compact even dimensional Lie group. Due to Samelson [38] homogeneous complex structures on G can be constructed as follows. Write $\mathfrak{g} := T_e G$ for the Lie algebra of G and let $\mathfrak{g}^{\mathbb{C}}$ be its complexification. A complex Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}^{\mathbb{C}}$ is called a *Samelson subalgebra* if $\dim_{\mathbb{C}} \mathfrak{s} = \frac{1}{2} \dim_{\mathbb{C}} (\mathfrak{g}^{\mathbb{C}})$ and $\mathfrak{s} \cap \mathfrak{g} = 0$ where now $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$. Infinitesimally \mathfrak{s} determines an almost complex structure $J_e : \mathfrak{g} \rightarrow \mathfrak{g}$ which satisfies that \mathfrak{s} is the $-\mathbf{i}$ -eigenspace of its complex linear extension $J_e^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$. Globally $\mathfrak{s} \subset \mathfrak{g}^{\mathbb{C}}$ gives rise to a complex Lie subgroup $S \subset G^{\mathbb{C}}$ of the complexification and there exists a diffeomorphism $G^{\mathbb{C}} \cong G \times S$; a complex structure on G induced by \mathfrak{s} arises by taking $G \cong G^{\mathbb{C}}/S$. Left translations act by biholomorphisms hence G is a compact homogeneous complex manifold. If $H^2(G; \mathbb{C}) \cong 0$ the space G cannot be projective algebraic (cf. e.g. [10, Lemma 4.1]) or even Kähler hence $\nabla J \neq 0$ with respect to any metric. An example is $\text{SU}(2) \times \text{SU}(2)$ as a complex 3-manifold.² Another example is the exceptional Lie group G_2 as a complex 7-manifold.

As a T -module with respect to the adjoint action of a maximal torus $T \subseteq G$ we have the usual decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\mu \in R^+} \mathfrak{g}_{\mu} \tag{4}$$

where $\mathfrak{t} = \mathfrak{t}^{\mathbb{C}} \cap \mathfrak{g}$ and $\mathfrak{g}_{\mu} = (\mathfrak{g}_{\mu}^{\mathbb{C}} \oplus \mathfrak{g}_{-\mu}^{\mathbb{C}}) \cap \mathfrak{g} = \mathfrak{g}_{-\mu}$ and R^+ is supposed to contain exactly one element from each pair $\{\mu, -\mu\}$ of real roots. The almost complex operator J_e has a corresponding decomposition. As a real vector space $\mathfrak{t} \cong \mathbb{R}^l$ where $l = \dim_{\mathbb{R}} \mathfrak{t}$ is the rank of G . Since $\dim_{\mathbb{R}} G$ is even, l is even

²Note that the smooth manifold $S^3 \times S^3$ can be given the structure of a complex manifold in the sense of Calabi–Eckmann [8], too. Although in principle this complex structure differs from the one stemming from the Samelson construction presented here they are in fact isomorphic [16, 30].

hence J_e restricted to \mathfrak{t} gives a complex structure J_0 on \mathfrak{t} providing an isomorphism $\mathfrak{t} \cong \mathbb{C}^{\frac{l}{2}}$. In addition, on any $\mathfrak{g}_\mu \cong \mathbb{R}^2$ we obtain a unique complex structure J_μ on \mathbb{R}^2 providing $\mathfrak{g}_\mu \cong \mathbb{C}$. Consequently at the unit $e \in G$ (hence everywhere) the integrable almost complex operator J representing the complex structure splits:

$$J_e = J_0 \oplus \bigoplus_{\mu \in R^+} J_\mu. \quad (5)$$

Suppose G is an even dimensional compact Lie group with $H^2(G; \mathbb{C}) \cong 0$ and take $M := G$ to be the base manifold in (2). Picking a representative J of the complex structure on M put an induced orientation as well as a metric g_J onto M for which J is orthogonal. Such metric exists by averaging any metric g with respect to J i.e., setting $g_J(u, v) := \frac{1}{2}g(u, v) + \frac{1}{2}g(Ju, Jv)$ for vector fields u, v on M . Consequently taking $\Phi := J$ and $g := g_J$ and ∇ to be its Levi–Civita connection we obtain that (∇, Φ, g) solves (3) on the compact space M i.e., it gives rise to a strictly weak spontaneous symmetry breaking $SO(2m) \Rightarrow U(m)$ where $\dim_{\mathbb{R}} M = 2m$. The spontaneously broken vacuum represents the complex structure on M which is not Kähler.

Finally we clarify a subtle conceptual ambiguity: how should one interpret the effect of a general gauge transformation on a given geometric structure? Consider $(\mathbb{R}^{2m}, J) \cong \mathbb{C}^m$ and pick an element $A \in GL(2m, \mathbb{R})$. On the one hand if A acts directly on \mathbb{R}^{2m} by $(x^1, \dots, x^{2m}) \mapsto A(x^1, \dots, x^{2m})$ then in this new frame J looks like AJA^{-1} . In this case A just represents a linear coordinate transformation on \mathbb{R}^{2m} hence “ AJA^{-1} is the same old complex structure in a new frame”. Globally, $f \in \text{Diff}(M)$ acts on a vector field by the chain rule as $u \mapsto f_*u$ hence on J by $J \mapsto f_*Jf_*^{-1}$ regarded as a “coordinate transformation”. Consequently the transformed structure is considered being equivalent to J . More generally an $\alpha \in \mathcal{G}_{TM}$ from the gauge group acts the same way: $J \mapsto \alpha J \alpha^{-1}$ therefore we obtain an embedding $\text{Diff}(M) \subset \text{Aut}(C^\infty(M; \text{End} TM)) \cong \mathcal{G}_{TM} / \{\pm \text{Id}_{TM}\}$. This natural embedding and the gauge theoretic formulation developed here dictates to extend this picture from diffeomorphisms to general gauge transformations as well. On the other hand if A does not act on \mathbb{R}^{2m} but acts on J itself by $J \mapsto AJA^{-1}$ then A describes a deformation of the complex structure hence “ AJA^{-1} is a new complex structure in the same old frame”. Correspondingly, in this picture \mathcal{G}_{TM} acts by global deformations $J \mapsto \alpha J \alpha^{-1}$ of the almost complex structure.

Principle. *Over an almost complex manifold there exists an abstract group operating in two essentially different ways. First, this group as the gauge group \mathcal{G}_{TM} describes the symmetries of a fixed almost complex manifold (M, J) by gauge transforming everything. Second, this group as a “deformation group” can also describe deformations of a given almost complex manifold (M, J) by deforming it into a new one (M, J') and leaving other objects intact.*

Therefore if this group acts as the gauge group describing the symmetries of a fixed almost complex manifold (M, J) by gauge transforming everything then in particular the Nijenhuis tensor (1) transforms as a $(2, 1)$ -tensor under such a symmetry transformation:

$$N_J = N_{\nabla J} \mapsto N_{\alpha \nabla \alpha^{-1}}(\alpha J \alpha^{-1}) = \alpha(N_{\nabla J}) = \alpha(N_J)$$

where $\alpha(T)$ denotes the natural action of $\alpha \in \mathcal{G}_{TM}$ on a (k, l) -type tensor T . But if this group acts as a “deformation group” by deforming J alone and not changing anything else then in particular the Nijenhuis tensor does not behave tensorially under such deformations:

$$N_J = N_{\nabla J} \mapsto N_{\nabla}(\alpha J \alpha^{-1}) = N_{\alpha J \alpha^{-1}} \neq \alpha(N_J).$$

These two roles played by the gauge group should not be mixed in the forthcoming paragraphs.

3 Fourier series expansion over principal bundles

In this technical section we collect some useful facts about expansion of functions into Fourier series over a connected compact Lie group G . A standard reference is for instance [21, 32]. Then we generalize this to Fourier expand sections of vector bundles admitting lifted G -actions over principal G -bundles. These results also exist in the literature although in a somewhat implicit form (cf. e.g. [5, Chapters II and III]). This generalized Fourier expansion is a rigorous mathematical tool for performing dimensional reduction in classical and quantum field theories (cf. e.g. [43, 44]).

Let G be a compact Lie group and let us denote by $\text{Irr}(G; \mathbb{C})$ the set of isomorphism classes of its finite dimensional complex irreducible representations. That is, if $\rho \in \text{Irr}(G; \mathbb{C})$ then there exists a finite dimensional complex vector space V_ρ and a homomorphism $\rho : G \rightarrow \text{Aut} V_\rho$. For a fixed (isomorphism class of) ρ pick a basis in V_ρ to write $(V_\rho, e_1, \dots, e_{\dim_{\mathbb{C}} V_\rho}) \cong \mathbb{C}^{\dim_{\mathbb{C}} V_\rho}$ and denote the corresponding matrix elements of $\gamma \in G$ as $\rho_{ij}(\gamma)$. Then $\rho_{ij} : G \rightarrow \mathbb{C}$ is a continuous function and the Peter–Weyl theorem [5, Theorem III.3.1] implies that

$$\left\{ \sqrt{\dim_{\mathbb{C}} V_\rho} \rho_{ij} : G \rightarrow \mathbb{C} \mid \rho \in \text{Irr}(G; \mathbb{C}); i, j = 1, \dots, \dim_{\mathbb{C}} V_\rho \right\}$$

forms a complete set of orthonormal functions in the Hilbert space $L^2(G; \mathbb{C})$ of square-integrable complex-valued functions on G with the usual scalar product $(f_1, f_2)_{L^2(G)} := (\text{Vol } G)^{-1} \int_G f_1 \bar{f}_2 d\gamma$. Here $d\gamma$ is a bi-invariant Haar measure on G and $\text{Vol } G = \int_G d\gamma$. A function $f \in L^2(G; \mathbb{C})$ then can be written as a formal sum

$$f(\gamma) \sim \sum_{\rho \in \text{Irr}(G; \mathbb{C})} \sqrt{\dim_{\mathbb{C}} V_\rho} \sum_{i,j=1}^{\dim_{\mathbb{C}} V_\rho} a_{\rho,ij} \rho_{ij}(\gamma)$$

where the constants $a_{\rho,ij} \in \mathbb{C}$ are given by

$$a_{\rho,ij} := \frac{1}{\text{Vol } G} \int_G f(\gamma) \bar{\rho}_{ij}(\gamma) d\gamma.$$

This expansion can be obtained in a more invariant (and abstract) way by the aid of the irreducible representations themselves without referring to the individual matrix elements. Right translation on G induces an infinite dimensional unitary representation of G from the left on $L^2(G; \mathbb{C})$ given by

$$(\gamma \cdot f)(\delta) := f(\delta\gamma)$$

for all $\delta \in G$. This representation—the *regular representation*—gives rise to an orthogonal decomposition

$$L^2(G; \mathbb{C}) \cong \bigoplus_{\rho \in \text{Irr}(G; \mathbb{C})} W(V_\rho)$$

where $W(V_\rho)$ is the isotypical summand for the finite dimensional irreducible representation V_ρ i.e., $W(V_\rho)$ is the L^2 -closure of the span of all subspaces in $L^2(G; \mathbb{C})$ isomorphic to V_ρ as G -modules (in fact in the case at hand $W(V_\rho)$ is the direct sum of $\dim_{\mathbb{C}} V_\rho$ copies of V_ρ). The continuous orthogonal projection $\Pi_\rho : L^2(G; \mathbb{C}) \rightarrow W(V_\rho)$ is given by the character

$$\chi_\rho := \text{Tr } \rho = \sum_{i=1}^{\dim_{\mathbb{C}} V_\rho} \rho_{ii}$$

of the corresponding representation as follows:

$$\Pi_\rho f := \frac{\dim_{\mathbb{C}} V_\rho}{\text{Vol } G} \int_G (\gamma \cdot f) \bar{\chi}_\rho(\gamma) d\gamma. \quad (6)$$

The associated formal sum

$$f \sim \sum_{\rho \in \text{Irr}(G; \mathbb{C})} \Pi_\rho f \quad (7)$$

is called the *Fourier expansion of f with respect to G* . If f is smooth then $\Pi_\rho f$'s are also smooth and (7) converges uniformly and pointwise equality holds.

Now we proceed further and construct more general Fourier expansions. Let M be an arbitrary connected smooth manifold. As usual let G be a connected compact Lie group with a fixed bi-invariant Haar measure $d\gamma$ and consider a principal G bundle $\pi : P \rightarrow M$. This means that G has a free right action on P such that $P/G \cong M$. Suppose that there is a Riemannian metric on P . Also let E be a complex vector bundle $p : E \rightarrow P$ such that the right action of G on P lifts to E rendering it a vector bundle with some fixed smooth right G -action. Put some G -invariant Hermitian scalar product onto E ; this together with the metric on P gives a complex Hilbert space $L^2(P; E)$. A generic section is denoted by $s \in L^2(P; E)$. The right action of G on P and its lifted right action on E induces a continuous representation of G on $C^\infty(P; E)$ from the left given by

$$(\gamma \cdot s)(y) := s(y\gamma)\gamma^{-1}$$

whose unique continuous extension makes $L^2(P; E)$ into an infinite dimensional complex unitary G -module. We get again an orthogonal decomposition into isotypical summands

$$L^2(P; E) \cong \bigoplus_{\rho \in \text{Irr}(G; \mathbb{C})} W(V_\rho)$$

as before with continuous orthogonal projections $\Pi_\rho : L^2(P; E) \rightarrow W(V_\rho)$ given by fiberwise integration³

$$\Pi_\rho s := \frac{\dim_{\mathbb{C}} V_\rho}{\text{Vol } G} \int_G (\gamma \cdot s) \bar{\chi}_\rho(\gamma) d\gamma. \quad (8)$$

These considerations suggest to define the *global Fourier expansion with respect to G* of a section $s \in L^2(P; E)$ as the formal sum

$$s \sim \sum_{\rho \in \text{Irr}(G; \mathbb{C})} \Pi_\rho s. \quad (9)$$

If it happens that M is compact and $s \in C^\infty(P; E)$ then it follows from the general theory of Fourier expansions [32, Theorems 7.1 or 8.6] that also $\Pi_\rho s \in C^\infty(P; E)$ for all representations and (9) converges uniformly over P and pointwise equality holds.

The ground Fourier mode $s_1 := \Pi_1 s \in L^2(P; E)$ corresponding to the trivial representation $\rho \cong 1$ of G is nothing else than the average of s along the fibers. More precisely exploiting left-translation

³Although there are no canonical isomorphisms between the fibers of $\pi : P \rightarrow M$ and G the Haar measure on G can be pulled back to the fibers with any of these isomorphisms in an unambiguous way taking into account its translation invariance. Hence we obtain well-defined measures on the fibers.

invariance of the Haar measure it satisfies

$$\begin{aligned} s_1(y\delta) &= \frac{1}{\text{Vol } G} \int_G (\gamma \cdot s)(y\delta) d\gamma \\ &= \frac{1}{\text{Vol } G} \int_G s(y\delta\gamma) \gamma^{-1} d\gamma = \frac{1}{\text{Vol } G} \int_G s(y\gamma) (\delta^{-1}\gamma)^{-1} d\gamma = \left(\frac{1}{\text{Vol } G} \int_G s(y\gamma) \gamma^{-1} d\gamma \right) \delta \\ &= s_1(y) \delta \end{aligned}$$

for all $\delta \in G$. Making use of the right G -action on E we can form the natural collapsed bundle E/G over $P/G = M$. A vector in the fiber $(E/G)_x$ over $x \in M$ corresponds to a section of $E|_{\pi^{-1}(x)}$ consisting of the equivalence class of vectors with respect to the right G -action. Hence s_1 descends uniquely to a section $\tilde{s}_1 : M \rightarrow E/G$ of the collapsed bundle satisfying $\pi^* \tilde{s}_1 = s_1$.

By the aid of the Hermitian structure on E we have a gauge group \mathcal{G}_E . If $\alpha \in \mathcal{G}_E$ then it acts on sections by $s \mapsto \alpha s$ as usual. However we have seen that when performing Fourier expansions these sections are also acted upon by the group G as $s \mapsto \gamma \cdot s$ constructed above. This motivates to let \mathcal{G}_E act on G -actions as $\gamma \mapsto \gamma_\alpha$ given by

$$(\gamma_\alpha \cdot s)(y) := s(y\gamma) \gamma_\alpha^{-1}$$

where the α -twisted lifted right G -action $\gamma_\alpha : E_y \rightarrow E_{y\gamma}$ has the form

$$s(y) \gamma_\alpha := \alpha(y\gamma) ((\alpha^{-1}(y)s(y))\gamma).$$

This yields an identity

$$\gamma_\alpha \cdot (\alpha s) = \alpha(\gamma \cdot s).$$

With respect to the α -twisted lifted right G -action on E we can form again the collapsed bundle what we denote by $E/\alpha G$. Since by assumption the Hermitian structure on E is G -invariant we also obtain a gauge group $\mathcal{G}_{E/\alpha G}$ yielding a subgroup

$$\mathcal{G}_G := \pi^* \mathcal{G}_{E/\alpha G} \subset \mathcal{G}_E$$

consisting of gauge transformations which are constant along the fibers.⁴ Consequently if $\Pi_{\alpha,\rho}(\alpha s)$ denotes a Fourier mode of αs with respect to $s \mapsto \gamma_\alpha \cdot s$ and $\Pi_{\text{Id}_E,\rho} s := \Pi_\rho s$ is that of s with respect to $s \mapsto \gamma \cdot s$ then we can see from (8) that if in particular $\alpha \in \mathcal{G}_G$ then $\Pi_{\alpha,\rho}(\alpha s) = \alpha(\Pi_{\text{Id}_E,\rho} s)$. This means that the two Fourier expansions (9) are compatible with the gauge transformation. Since the twisted G -action $s \mapsto \gamma_\alpha \cdot s$ is a G -action on $L^2(P; E)$ on its own right we make the following

Definition 3.1. *Pick two gauge transformations $\alpha', \alpha'' \in \mathcal{G}_E$ and consider two G -actions on $L^2(P; E)$ of the form $s \mapsto \gamma_{\alpha'} \cdot s$ and $s \mapsto \gamma_{\alpha''} \cdot s$ respectively. These give rise to two global Fourier expansions (9) with respect to G on $L^2(P; E)$ given by the straightforwardly modified projections (8) respectively.*

These Fourier expansions are called gauge equivalent if there exists a $\beta \in \mathcal{G}_G \subset \mathcal{G}_E$ satisfying $\alpha'' = \beta \alpha'$.

Remark 3.1. 1. For different $\alpha \in \mathcal{G}_E$ the G -module structures on $L^2(P; E)$ are unitarily equivalent. However the induced Fourier expansions are not equivalent and their moduli space is the quotient $\mathcal{G}_G \backslash \mathcal{G}_E$ in some sense. Also notice that two gauge equivalent Fourier expansions on E over P give rise

⁴Note that $\mathcal{G}_G \subset \mathcal{G}_E$ is independent of $\alpha \in \mathcal{G}_E$.

to gauge equivalent ground modes on $E/\alpha'G \cong E/\alpha''G$ over $P/G = M$ with respect to $\mathcal{G}_{E/\alpha'G} \cong \mathcal{G}_{E/\alpha''G}$ which is the collapsed gauge group. Hence taking the ground mode is meaningful.

To simplify notation given $\alpha \in \mathcal{G}_E$ and its equivalence class $[\alpha] \in \mathcal{G}_G \setminus \mathcal{G}_E$ then $E/[\alpha]G$ will denote the isomorphism class of the corresponding collapsed bundle and $\mathcal{G}_{E/[\alpha]G}$ its gauge group.

2. The representation of G on $L^2(P; E)$ generalizes the single scalar-valued regular representation on $L^2(G; \mathbb{C})$ in two directions: it is a direct integral of a *family* of *vector-valued* regular representations. For instance assume that the vector bundle is globally trivial over P and pick a trivialization $\tau : E \rightarrow P \times \mathbb{C}^k$. It induces an action $\gamma_\tau := (\gamma, \text{Id}_{\mathbb{C}^{\text{rk}E}}) \in C^\infty(M; \text{Aut}P) \times \text{Aut}\mathbb{C}^{\text{rk}E}$ of G on E and all actions are of this form. One can see that with respect to this “trivial G -action” (8) just reduces to $\text{rk}_{\mathbb{C}}E$ copies of fiberwise integrals like (6) hence (9) gives back $\text{rk}_{\mathbb{C}}E$ -times the expansion (7) parameterized by the manifold M .

3. For clarity we remark that the Fourier modes (6) or (8) are of course independent of $\text{Vol}G$ (hence this volume will be taken from now on to be unit for instance).

4 The two dimensional sphere

After these preliminaries providing the general framework, let us focus our attention first to the case of the two-sphere. It will serve as a trivial warming-up exercise. We will demonstrate that S^2 admits both integrable strong and weak spontaneous symmetry breakings $\text{SO}(2) \rightrightarrows \text{U}(1)$ and $\text{SO}(2) \rightrightarrows \text{U}(1)$ respectively and of course these coincide by uniqueness hence both yield $S^2 \cong \mathbb{C}P^1$. This follows from the special isomorphism $\text{SO}(2) \cong \text{U}(1)$.

First consider an *integrable strong spontaneous symmetry breaking* $\text{SO}(2) \rightrightarrows \text{U}(1)$ over S^2 . Put the standard round metric g onto S^2 and let ∇ be the corresponding Levi–Civita connection. Fixing an orientation on S^2 as well we obtain a Hodge operator $*$: $\wedge^1 S^2 \rightarrow \wedge^1 S^2$ satisfying $*^2 = -\text{Id}_{\wedge^1 S^2}$. By the aid of the metric $\wedge^1 S^2 \cong TS^2$ therefore we come up with an almost complex tensor $I : TS^2 \rightarrow TS^2$. Since only the metric was used to construct it, we know that $\nabla I = 0$ moreover it is orthogonal for g . Putting these data (∇, I, g) into the corresponding Lagrangian we find

$$\mathcal{E}(\nabla, I, g) = \frac{1}{2} \int_{S^2} \left(\frac{1}{e^2} |\mathfrak{S}(R_\nabla)|_g^2 + |\nabla g|_g^2 + |\nabla I|_g^2 + |II^* - \text{Id}_{TS^2}|_g^2 \right) dV = 0$$

in other words (∇, I, g) gives rise to vacuum solution to the strong integrable spontaneous symmetry breaking $\text{SO}(2) \rightrightarrows \text{U}(1)$ yielding the usual Kähler structure on S^2 identifying it with $\mathbb{C}P^1$.

Secondly consider an *integrable weak spontaneous symmetry breaking* $\text{SO}(2) \rightrightarrows \text{U}(1)$ over S^2 . Because here we want to motivate the construction of Sect. 5 step-by-step we should begin with the quotient $\text{SO}(3)/\text{U}(1) \cong S^2$ given by the identifications $\text{SO}(3) \cong \text{Aut}\mathbb{H}$ and $S^2 \subset \text{Im}\mathbb{H}$. However $\text{SO}(3)$ is not an even dimensional compact Lie group hence it does not admit a complex structure *à la* Samelson therefore we even cannot make the very first step. To avoid this we will rather use another quotient $\text{SO}(4)/\text{U}(2) \cong S^2$. Since the following construction will just look like an overcomplicated version of the previous one we will only sketch its main steps and refer to Sect. 5 for the full details.

Consider \mathbb{R}^4 equipped with an orientation and scalar product providing us with the group $\text{SO}(4)$. Taking any identification $\mathbb{R}^4 \cong \mathbb{C}^2$ we obtain a subgroup $\text{U}(2) \subset \text{SO}(4)$. Then $\text{SO}(4)$ acts on itself transitively from the left as well as $\text{U}(2) \subset \text{SO}(4)$ acts from the right. Dividing by this right action we obtain a principal $\text{U}(2)$ -bundle $\pi : \text{SO}(4) \rightarrow S^2$. A root decomposition of $\mathfrak{so}(4)$ with respect to one of its special maximal torus $T \subset \text{U}(2) \subset \text{SO}(4)$ gives rise to a well-defined left- $\text{SO}(4)$ -invariant splitting

$$T\text{SO}(4) = V \oplus H$$

such that $V|_{\pi^{-1}(x)} = T\pi^{-1}(x) \cong TU(2)$ for all $x \in S^2$ moreover H is also fixed by the root decomposition. The complexified adjoint representation of $U(2)$ on $\mathfrak{so}(4)^\mathbb{C}$ is reducible and decomposes as

$$\mathfrak{so}(4)^\mathbb{C} \cong \mathfrak{u}(2)^\mathbb{C} \oplus \mathbb{C} \oplus \bar{\mathbb{C}}$$

where $\mathfrak{u}(2)^\mathbb{C}$ is the complexified adjoint representation of $U(2)$ (still reducible!) and \mathbb{C} is the standard 1 dimensional complex representation of a certain $U(1) \subset U(2)$ and $\bar{\mathbb{C}}$ is its complex conjugate representation respectively. This gives rise to a refined left-invariant splitting

$$(TSO(4))^\mathbb{C} = V^\mathbb{C} \oplus L \oplus \bar{L}$$

such that the smooth complex line bundle L over $SO(4)$ satisfies $H^\mathbb{C} \cong L \oplus \bar{L}$ where $H^\mathbb{C}$ is the complexified horizontal bundle of the original splitting. We put two almost complex structures I_H and J_H onto $H \subset TSO(4)$ as follows. The first one, $I_H : H \rightarrow H$ is defined by requiring its complex linear extension $I_H^\mathbb{C} : H^\mathbb{C} \rightarrow H^\mathbb{C}$ to satisfy

$$(H, I_H) \cong H_I^{1,0} := L \subset H^\mathbb{C}$$

where $H_I^{1,0} \subset H^\mathbb{C}$ is the $+i$ -eigenbundle of $I_H^\mathbb{C}$. This almost complex structure is left- $SO(4)$ -invariant and right- $U(2)$ -invariant by its construction. Concerning the second one $J_H : H \rightarrow H$, let J be an integrable almost complex structure on $SO(4)$ found by Samelson as in Sect. 2. Consequently it has vanishing Nijenhuis tensor: $N_J = 0$. Moreover it turns out that it is blockdiagonal with respect to $TSO(4) = V \oplus H$ hence restricts to an almost complex structure $J_H : H \rightarrow H$. It is by construction left- $SO(4)$ -invariant.

Putting a left- $SO(4)$ -invariant and right- $U(2)$ -invariant metric g_0 onto $SO(4)$ and taking its restriction $g_H := g_0|_H$ as well restricting the orientation of $SO(4)$ induced by J to $H \subset TSO(4)$ we obtain a gauge group \mathcal{G}_H consisting of $SO(2)$ gauge transformations of H . Let

$$\nabla_H : C^\infty(SO(4); H) \times C^\infty(SO(4); H) \xrightarrow{\nabla_0|_H} C^\infty(SO(4); TSO(4)) \xrightarrow{P_H} C^\infty(SO(4); H)$$

be the restricted-projected connection where P_H is the g_0 -orthogonal bundle projection from $TSO(4)$ onto H . Since both I_H and J_H are left- $SO(4)$ -invariant there exists a gauge transformation $\alpha \in \mathcal{G}_H$ which rotates I_H into J_H i.e., $J_H = \alpha I_H \alpha^{-1}$. But if $U(1) \subseteq SO(2)$ denotes the group induced by I_H then taking into account that in fact $SO(2) \cong U(1)$ we find that $J_H = I_H$ as complex structures on H .

One can immediately see from our construction so far (especially from the fact that $J_H = I_H$)—or can Fourier expand J_H and g_H with respect to $[\text{Id}_H] \in \mathcal{G}_{U(2)} \setminus \mathcal{G}_H$ (cf. Definition 3.1)—that the constructed data (∇_H, J_H, g_H) descend to S^2 more precisely to

$$H/[\text{Id}_H]U(2) = TS^2.$$

The resulting triple (∇, J, g) on TS^2 is nothing else than the standard almost complex structure I and the round metric g with its Levi-Civita connection ∇ hence it satisfies

$$\mathcal{E}(\nabla, J, g) = \frac{1}{2} \int_{S^2} \left(\frac{1}{e^2} |\mathfrak{S}(R_\nabla)|_g^2 + |\nabla g|_g^2 + |N_\nabla J|_g^2 + e^2 |JJ^* - \text{Id}_{TS^2}|_g^2 \right) dV = 0$$

providing us with an integrable weak spontaneous symmetry breaking $SO(2) \Rightarrow U(1)$. Of course because $J = I$ on S^2 we conclude that the resulting complex structure just coincides with the standard one i.e., $S^2 \cong \mathbb{C}P^1$ again.

For the technical details we refer to a completely analogous (but technically more complicated) construction of Sect. 5.

5 The six dimensional sphere

The time has come to carefully perform an integrable weak spontaneous symmetry breaking procedure over the six-sphere.

To begin with, we ask if an *integrable strong spontaneous symmetry breaking* $\text{SO}(6) \Rightarrow \text{U}(3)$ exists over S^6 . First let us recall the topological origin why almost complex structures exist on the six-sphere. Take the standard representation of $\text{SO}(6)$ on \mathbb{R}^6 and let E be an $\text{SO}(6)$ vector bundle on S^6 associated to this representation. Up to isomorphism, these bundles are classified by the homotopy equivalence classes of maps from the equator of S^6 into $\text{SO}(6)$ i.e., by $\pi_5(\text{SO}(6)) \cong \mathbb{Z}$. In particular picking an isomorphism $E \cong TS^6$ is equivalent to the existence of an orientation and a Riemannian metric on S^6 . However a continuous map $f : S^5 \rightarrow \text{SO}(6)$ is homotopic to a continuous map $f' : S^5 \rightarrow \text{U}(3) \subset \text{SO}(6)$ because any particular embedding $i : \text{U}(3) \subset \text{SO}(6)$ induces a homomorphism $i_* : \pi_5(\text{U}(3)) \rightarrow \pi_5(\text{SO}(6))$ which is an isomorphism: since $\text{SO}(6)/\text{U}(3) \cong \mathbb{C}P^3$ and $\pi_k(\mathbb{C}P^3) \cong 0$ if $k = 5, 6$ the desired isomorphism is provided by the associated homotopy exact sequence of the $\text{U}(3)$ -fibration $\text{SO}(6) \rightarrow \mathbb{C}P^3$. Therefore in fact any $\text{SO}(6)$ vector bundle isomorphism $E \cong TS^6$ descends in a non-unique way to an $\text{U}(3)$ vector bundle isomorphism $E' \cong TS^6$ which is equivalent to saying that in addition to an orientation and a Riemannian metric g on S^6 there exists a further (non-unique) compatible almost complex structure $J : TS^6 \rightarrow TS^6$. That is $J^2 = -\text{Id}_{TS^6}$ in a manner such that it is orthogonal with respect to g which means that $g(Ju, Jv) = g(u, v)$ for all vector fields u, v on S^6 . One can assume that these structures are smooth.

A fixed g and J uniquely determine a smooth non-degenerate 2-form by $\omega(u, v) := g(Ju, v)$ which is not closed in general. In fact it cannot be closed at all: $\int_{S^6} \omega \wedge \omega \wedge \omega \sim \text{Vol}(S^6) > 0$ on the one hand but since $H^2(S^6; \mathbb{Z}) = 0$ if $d\omega = 0$ then $\omega = d\xi$ for a 1-form ξ hence one would find $\int_{S^6} \omega \wedge \omega \wedge \omega = \int_{S^6} d(\xi \wedge d\xi \wedge d\xi) = 0$ via Stokes' theorem on the other hand, a contradiction. Consequently with any choice of g and J we cannot expect the metric to be Kähler or even almost Kähler. The quadruple (S^6, g, J, ω) is called an *almost Hermitian structure* on S^6 .

The non-existence of a Kähler structure on S^6 can be reformulated as saying that for any (S^6, g, J, ω) the almost complex tensor is not parallel with respect to the connection ∇ on the endomorphism bundle, associated to the Levi-Civita connection i.e., $\nabla J \neq 0$. Or using our physical language: *no integrable strong spontaneous symmetry breaking* $\text{SO}(6) \Rightarrow \text{U}(3)$ exists over the six-sphere.

Therefore we proceed forward and try to construct an *integrable weak spontaneous symmetry breaking* $\text{SO}(6) \Rightarrow \text{U}(3)$ over S^6 . Consider a Yang–Mills–Higgs–Nijenhuis field theory (2) specialized to the six-sphere:

$$\mathcal{E}(\nabla, \Phi, g) := \frac{1}{2} \int_{S^6} \left(\frac{1}{e^2} |\mathfrak{G}(R_\nabla)|_g^2 + |\nabla g|_g^2 + |N_\nabla \Phi|_g^2 + e^2 |\Phi \Phi^* - \text{Id}_{TS^6}|_g^2 \right) dV. \quad (10)$$

Recall that the existence of a smooth solution to the corresponding vacuum equations (3) is equivalent to the existence of a non-Kählerian complex structure on S^6 . Our aim is therefore to prove the existence of a spontaneously broken vacuum in this theory. This task will be carried out by the aid of the Lie group G_2 and its complex structure in several steps below.

So let us consider the 14 dimensional connected, compact, simply connected exceptional Lie group G_2 . It acts on itself by *left-translations*. As it is well-known in addition letting $\text{SU}(3)$ act on G_2 from the *right* G_2 arises as the total space of an $\text{SU}(3)$ fibration $\pi : G_2 \rightarrow S^6$. More precisely (for a complete proof cf. e.g. [6],[37, pp. 306-311]) taking the identifications $G_2 \cong \text{Aut}\mathbb{O}$ and $S^6 \subset \text{Im}\mathbb{O}$ the group G_2 acts on S^6 transitively and the corresponding isotropy subgroup can be found as follows. Over a point $x \in S^6 \subset \text{Im}\mathbb{O}$ of unit length—with respect to the standard scalar product—the tangent space $T_x S^6$ can be

identified with a subspace $V_x \subset \mathbb{O}$ perpendicular to the \mathbb{R} -span of $\{1, x\}$ in \mathbb{O} . Orientation and the scalar product on \mathbb{O} restricts to V_x as well as a complex structure on V_x is induced by Cayley multiplication with x itself rendering V_x an oriented 3 dimensional complex vector space with a Hermitian scalar product. A fiber of our principal bundle over $x \in S^6$ then arises as the special unitary group of this V_x . This principal bundle is non-trivial: the group G_2 as a principal $SU(3)$ -bundle over S^6 corresponds to a generator of $\pi_5(SU(3)) \cong \mathbb{Z}$, cf. e.g. [11, Corollary 3].

First referring to this fibration and the Lie algebra structure on the tangent bundle of G_2 we construct a left-invariant splitting

$$TG_2 = V \oplus H$$

as follows. For the vertical bundle put $V|_{\pi^{-1}(x)} := T\pi^{-1}(x) \cong TSU(3)$ for all $x \in S^6$. Regarding the horizontal bundle $H \cong (TG_2)/V$ note that a positive root system P^+ of $SU(3)$ can be a subsystem of a positive root system R^+ of G_2 hence with the special Cartan subalgebra $\mathfrak{t} \subset \mathfrak{su}(3) \subset \mathfrak{g}_2$ the decomposition of \mathfrak{g}_2 into real root spaces (4) yields $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{m}$ where

$$\mathfrak{su}(3) = \mathfrak{t} \oplus \bigoplus_{\mu \in P^+} \mathfrak{g}_\mu, \quad \mathfrak{m} := \bigoplus_{\nu \in R^+ \setminus P^+} \mathfrak{g}_\nu$$

fixing the choice of $\mathfrak{m} \subset \mathfrak{g}_2$. Identifying both $\mathfrak{su}(3)$ and \mathfrak{m} with subspaces of left-invariant sections we fix $TG_2 = V \oplus H$ up to a Cartan subalgebra of $SU(3)$ as follows: we already know that $V_y = \text{ev}_y \mathfrak{su}(3)$ moreover put $H_y := \text{ev}_y \mathfrak{m}$ for all $y \in G_2$. In particular H is a trivial bundle over G_2 .

The complexification of this splitting can be refined. The complexified adjoint representation of $SU(3)$ on $\mathfrak{g}_2^{\mathbb{C}}$ is reducible and its decomposition into irreducible summands is given by

$$\mathfrak{g}_2^{\mathbb{C}} \cong \mathfrak{su}(3)^{\mathbb{C}} \oplus \mathbb{C}^3 \oplus \overline{\mathbb{C}^3}$$

where $\mathfrak{su}(3)^{\mathbb{C}}$ is the complexified adjoint representation while \mathbb{C}^3 and $\overline{\mathbb{C}^3}$ are the standard 3 dimensional complex representation of $SU(3)$ and its complex conjugate respectively. This induces a refined left-invariant decomposition

$$(TG_2)^{\mathbb{C}} = V^{\mathbb{C}} \oplus Z \oplus \overline{Z}$$

of $(TG_2)^{\mathbb{C}} = V^{\mathbb{C}} \oplus H^{\mathbb{C}}$ such that the smooth complex vector bundle Z over G_2 satisfies $H^{\mathbb{C}} = Z \oplus \overline{Z}$ where $H^{\mathbb{C}} := H \otimes \mathbb{C}$ is the complexified horizontal bundle.

Secondly we consider two canonical almost complex structures I_H and J_H on $H \subset TG_2$. The first one, $I_H : H \rightarrow H$ is defined by requiring that its complex linear extension $I_H^{\mathbb{C}} : H^{\mathbb{C}} \rightarrow H^{\mathbb{C}}$ satisfies

$$(H, I_H) \cong H_I^{1,0} := Z \subset H^{\mathbb{C}}$$

where $H_I^{1,0} \subset H^{\mathbb{C}}$ is the $+\mathbf{i}$ -eigenbundle of $I_H^{\mathbb{C}}$. Note that by construction $I_H : H \rightarrow H$ is a left- G_2 -invariant and right- $SU(3)$ -invariant almost complex structure which is moreover not integrable.⁵

The other one, $J_H : H \rightarrow H$ is constructed as follows. We already know from Sect. 2 that there exists a left- G_2 -invariant integrable almost complex tensor J on G_2 à la Samelson. It has of course vanishing Nijenhuis tensor

$$N_J = 0 \tag{11}$$

moreover via (5) it admits a left-invariant global blockdiagonal decomposition

$$J = \begin{pmatrix} J_V & 0 \\ 0 & J_H \end{pmatrix} \tag{12}$$

⁵We note that by right $SU(3)$ -invariance it descends and gives the Cayley almost complex structure on S^6 .

where $J_V \in C^\infty(G_2; \text{End}V)$ and $J_H \in C^\infty(G_2; \text{End}H)$. In particular $J_H^2 = -\text{Id}_H$ and in this way we obtain a smooth trivial rank 3 complex vector bundle (H, J_H) over G_2 . That is, as a complex vector bundle

$$(H, J_H) \cong H_J^{1,0} \subset H^\mathbb{C}$$

where $H_J^{1,0}$ is the $+\mathbf{i}$ -eigenbundle of $J_H^\mathbb{C} : H^\mathbb{C} \rightarrow H^\mathbb{C}$.

Thirdly we construct an invariant metric on G_2 . Fix an orientation on G_2 induced by J . Also fix a Riemannian metric g_0 on G_2 which is invariant under both left- G_2 -translations induced by the left-action of G_2 and fiberwise right- $\text{SU}(3)$ -translations induced by the right-action of $\text{SU}(3) \subset G_2$. Such metric g_0 easily arises by averaging an arbitrary metric g on G_2 with respect to $G_2 \times \text{SU}(3)$ equipped with the measure $dy \otimes d\gamma$ where dy and $d\gamma$ are the unit-volume bi-invariant Haar measures on G_2 and $\text{SU}(3)$ respectively:

$$g_0(u, v) := \int_{G_2 \times \text{SU}(3)} g((L_y)_*(R_\gamma)_*u, (L_y)_*(R_\gamma)_*v) dy \otimes d\gamma.$$

If the metric is normalized such that the induced volume form satisfies $\int_{G_2} dV_0 = 1$ then as left-invariant measures $dV_0 = dy$. The associated Levi-Civita connection will be denoted by ∇_0 . Taking the restricted orientation and metric $g_0|_H =: g_H$ we also have an associated gauge group \mathcal{G}_H consisting of $\text{SO}(6)$ gauge transformations of the horizontal bundle. Take the restricted-projected connection

$$\nabla_H : C^\infty(G_2; H) \times C^\infty(G_2; H) \xrightarrow{\nabla_0|_H} C^\infty(G_2; TG_2) \xrightarrow{P_H} C^\infty(G_2; H) \quad (13)$$

where P_H is the left- G_2 -equivariant g_0 -orthogonal bundle projection from TG_2 onto H . The metric furthermore admits a unique Hermitian extension $g_H^\mathbb{C}$ to $H^\mathbb{C} \cong H \oplus \mathbf{i}H$ given by

$$g_H^\mathbb{C}(u_1 + \mathbf{i}u_2, v_1 + \mathbf{i}v_2) := g_H(u_1, v_1) + g_H(u_2, v_2) + \mathbf{i}(g_H(u_2, v_1) - g_H(u_1, v_2))$$

(complex conjugate linear in its second variable) yielding the complexified gauge group $\mathcal{G}_H^\mathbb{C}$ consisting of $\text{U}(6)$ gauge transformations with respect to $g_H^\mathbb{C}$.

By the aid of this gauge group the almost complex structures J_H and I_H on the horizontal bundle can be compared. By construction I_H commutes with the fiberwise right-action of $\text{SU}(3) \subset G_2$ hence for all $y \in G_2$ and $\gamma \in \text{SU}(3)$ we know that

$$I_{H_{y\gamma}}(R_\gamma)_* = (R_\gamma)_*I_{H_y}. \quad (14)$$

In addition I_H is invariant under the left- G_2 -action hence $H_I^{1,0} \subset H^\mathbb{C}$ is a left-invariant bundle moreover I_H is orthogonal for g_H . But J_H is also invariant under the left- G_2 -action hence $H_J^{1,0} \subset H^\mathbb{C}$ is also left-invariant by construction. Therefore these two sub-bundles can be rotated into each other within $H^\mathbb{C}$ i.e., there exists a gauge transformation $\alpha \in \mathcal{G}_H^\mathbb{C}$ such that $H_J^{1,0} = \alpha H_I^{1,0}$ i.e., there exists an induced real gauge transformation $\alpha \in \mathcal{G}_H$ such that for all $y \in G_2$

$$J_{H_y} = \alpha(y)I_{H_y}\alpha^{-1}(y)$$

holds. Note that in any point $\alpha(y) \notin \text{U}(3)$ with $\text{U}(3) \subset \text{SO}(6)$ being induced by I_{H_y} because I_{H_y} and J_{H_y} are not equivalent; but at least $\alpha(y) \in \text{SO}(6)$ where $\text{SO}(6)$ is by definition induced by g_H and the orientation.⁶ Hence in particular it follows that J_H is also orthogonal with respect to g_H .

⁶In our terminology developed at the end of Sect. 2 the element $\alpha \in \mathcal{G}_H$ acts as a *deformation* of J_H into I_H .

It follows from (12) that $J_H J_H^* = \text{Id}_H$ and (1) together with (12) also imply that $N_{\nabla_H} J_H = N_{\nabla_0|_H} J_H$. Moreover we deduce from (13) that $|\nabla_H g_H|_{g_H} \leq |(\nabla_0|_H) g_H|_{g_H}$ and taking into account that $\nabla_0|_H$ is torsion-free we also find $\mathfrak{S}(R_{\nabla_H}) = \mathfrak{S}(P_H R_{\nabla_0|_H})$ hence $|\mathfrak{S}(R_{\nabla_H})|_{g_H} \leq |\mathfrak{S}(R_{\nabla_0|_H})|_{g_H}$. Therefore

$$\begin{aligned} \mathcal{E}(\nabla_H, J_H, g_H) &= \frac{1}{2} \int_{G_2} \left(\frac{1}{e^2} |\mathfrak{S}(R_{\nabla_H})|_{g_H}^2 + |\nabla_H g_H|_{g_H}^2 + |N_{\nabla_H} J_H|_{g_H}^2 + e^2 |J_H J_H^* - \text{Id}_H|_{g_H}^2 \right) dV_0 \\ &\leq \text{horizontal part of } \mathcal{E}(\nabla_0, J, g_0) \leq \mathcal{E}(\nabla_0, J, g_0) \end{aligned}$$

and via (11) the functional (2) specialized to G_2 satisfies $\mathcal{E}(\nabla_0, J, g_0) = 0$. Our constructions so far can be then summarized as follows: the triple (∇_H, J_H, g_H) represents a smooth vacuum solution (3) to the Yang–Mills–Higgs–Nijenhuis field theory (2) restricted to the horizontal bundle $H \subset TG_2$ i.e.,

$$\mathcal{E}(\nabla_H, J_H, g_H) = 0. \quad (15)$$

We want to use this vacuum solution to obtain a vacuum solution to (10). Our technical tool to achieve this will be a carefully constructed global Fourier expansion with respect to $\text{SU}(3)$ of the vacuum Higgs field J_H .

As we have seen in Sect. 3 for this purpose we need an action of $\text{SU}(3)$ on the Hilbert space of sections of the horizontal bundle $H \subset TG_2$. Given any $\gamma \in \text{SU}(3)$ and $u_y \in H_y$ consider the real linear isomorphism $\gamma_\alpha : H_y \rightarrow H_{y\gamma}$ whose shape is

$$u_y \gamma_\alpha := (\pi_*|_{H_{y\gamma}} \alpha^{-1}(y\gamma))^{-1} (\pi_*|_{H_y} \alpha^{-1}(y)) u_y$$

or in other words, making use of the fiberwise right $\text{SU}(3)$ -translation $R_\gamma : G_2 \rightarrow G_2$ we put

$$u_y \gamma_\alpha := (\alpha(y\gamma)(R_\gamma)_* \alpha^{-1}(y)) u_y. \quad (16)$$

Dividing by this α -dependent action we get an abstract isomorphism $H/_{[\alpha]} \text{SU}(3) \cong TS^6$ for the quotient as a plain real vector bundle.

The space $C^\infty(G_2; H)$ carries an associated representation of $\text{SU}(3)$ from the left defined by

$$(\gamma_\alpha \cdot u)(y) := u(y\gamma) \gamma_\alpha^{-1} \quad (17)$$

for all $u \in C^\infty(G_2; H)$. Consider the real Hilbert space $L^2(G_2; H)$ as the completion of $C^\infty(G_2; H)$ with respect to the L^2 scalar product $(u, v)_{L^2(G_2)} := \int_{G_2} g_H(u(y), v(y)) dV_0$. Extending (17) from $C^\infty(G_2; H)$ to this space we obtain continuous real representations of $\text{SU}(3)$ parameterized by \mathcal{G}_H . These representations are orthogonal because the metric g_H is right-invariant with respect to $\text{SU}(3)$. The construction can be complexified and we come up with continuous complex unitary representations of $\text{SU}(3)$ on the fixed complex Hilbert space $L^2(G_2; H^{\mathbb{C}}) \cong L^2(G_2; H) \otimes \mathbb{C}$ parameterized by the complexified gauge group $\mathcal{G}_H^{\mathbb{C}}$. All of these representations are unitarily equivalent.

Now we apply the general construction of Sect. 3 to our situation by taking the compact group G to be $\text{SU}(3)$, the total space P to be G_2 and the auxiliary vector bundle E on it to be $\text{End}H$. Note that $J_H \in C^\infty(G_2; \text{End}H)$ because $\text{Ad}G_2 \subset \text{End}TG_2$. Picking a gauge transformation $\alpha \in \mathcal{G}_H$ there is an action of $\text{SU}(3)$ on $L^2(G_2; \text{End}H)$ induced by (17). Consequently the smooth Higgs field J_H admits an associated $\mathcal{G}_{\text{SU}(3)}$ -equivariant global Fourier expansion (9) of the shape

$$J_H = \sum_{\rho \in \text{Irr}(\text{SU}(3); \mathbb{C})} \Pi_{\alpha, \rho} J_H \quad (18)$$

and here pointwise equality holds.

Lemma 5.1. *Let J_H be the horizontal part of a left- G_2 -invariant integrable and let I_H be the horizontal part of a left- G_2 -invariant and right $SU(3)$ -invariant non-integrable almost complex structure on G_2 respectively. Consider a gauge transformation $\alpha \in \mathcal{G}_H$ with respect to the metric g_H satisfying $J_H = \alpha I_H \alpha^{-1}$ as above.*

Then the global Fourier expansion (18) corresponding to $\alpha \in \mathcal{G}_H$ satisfies $J_H = \Pi_{\alpha,1} J_H$ i.e., all the higher Fourier modes vanish: $\Pi_{\alpha,\rho} J_H = 0$ with $\rho \neq 1$. Consequently J_H uniquely descends to an endomorphism

$$J_{H/[\alpha]SU(3)} : H/[\alpha]SU(3) \longrightarrow H/[\alpha]SU(3)$$

over $G_2/SU(3) \cong S^6$.

Moreover this Fourier expansion is well-defined up to an element of $\mathcal{G}_{SU(3)} \subset \mathcal{G}_H$ (cf. Definition 3.1) i.e., depends only on the class $[\alpha] \in \mathcal{G}_{SU(3)} \setminus \mathcal{G}_H$.

Proof. Exploiting the identities (14) and (16) we calculate the ground mode $\Pi_{\alpha,1} J_H$ in (18) over a point $y \in G_2$ as follows:

$$\begin{aligned} \Pi_{\alpha,1} J_{H_y} &= \int_{SU(3)} (\gamma \alpha \cdot J_{H_y}) d\gamma \\ &= \int_{SU(3)} (\gamma \alpha^{-1} J_{H_{y\gamma}} \gamma \alpha) d\gamma \\ &= \int_{SU(3)} (\gamma \alpha^{-1} (\alpha(y\gamma) I_{H_{y\gamma}} \alpha^{-1}(y\gamma)) \gamma \alpha) d\gamma \\ &= \int_{SU(3)} \left(\alpha(y) (R_{\gamma^{-1}})_* \alpha^{-1}(y\gamma) \alpha(y\gamma) I_{H_{y\gamma}} \alpha^{-1}(y\gamma) \alpha(y\gamma) (R_{\gamma})_* \alpha^{-1}(y) \right) d\gamma \\ &= \int_{SU(3)} \left(\alpha(y) (R_{\gamma^{-1}})_* I_{H_{y\gamma}} (R_{\gamma})_* \alpha^{-1}(y) \right) d\gamma \\ &= \int_{SU(3)} (\alpha(y) I_{H_y} \alpha^{-1}(y)) d\gamma \\ &= \int_{SU(3)} J_{H_y} d\gamma \\ &= J_{H_y}. \end{aligned}$$

Therefore completeness gives $\Pi_{\alpha,\rho} J_H = 0$ if $\rho \neq 1$ and we obtain the result.

Moreover this result obviously remains unchanged if $\alpha \in \mathcal{G}_H$ is multiplied from the left with a vertically constant gauge transformation $\beta \in \mathcal{G}_{SU(3)} \subset \mathcal{G}_H$ as claimed. \diamond

Lemma 5.2. *Making use of the gauge transformation $\alpha \in \mathcal{G}_H$ of Lemma 5.1 and the associated Fourier expansion the horizontal part g_H of the metric on $H \subset TG_2$ satisfies $g_H = \Pi_{\alpha,1} g_H$. Consequently g_H uniquely descends to a metric $g_{H/[\alpha]SU(3)}$ on the bundle $H/[\alpha]SU(3)$ over $G_2/SU(3) \cong S^6$.*

Additionally, the associated connection ∇_H in (13) uniquely descends to a connection

$$\nabla_{H/[\alpha]SU(3)} : C^\infty(S^6; H/[\alpha]SU(3)) \longrightarrow C^\infty(S^6; (H/[\alpha]SU(3)) \otimes \wedge^1 S^6)$$

compatible with $g_{H/[α]SU(3)}$ and gives rise to the corresponding curvature tensor

$$R_{\nabla_{H/[α]SU(3)}} : C^\infty(S^6; H/[α]SU(3)) \longrightarrow C^\infty(S^6; (H/[α]SU(3)) \otimes \wedge^2 S^6).$$

Proof. The metric g_H is right-SU(3)-translation invariant: for any $u, v \in C^\infty(G_2; H)$ one finds $g_H(u, v) = g_H((R_\gamma)_*u, (R_\gamma)_*v)$ with $\gamma \in SU(3)$; as well as it is of course invariant under its own gauge group: $g_H(u, v) = g_H(\alpha u, \alpha v)$ with $\alpha \in \mathcal{G}_H$. Therefore by (16) we find that

$$g_H(u, v) = g_H(u\gamma_\alpha, v\gamma_\alpha) \quad (19)$$

i.e., it is invariant under any twisted right-action. In particular we can Fourier expand the metric with respect to the action of SU(3) on $L^2(G_2; H^* \otimes H^*)$ induced by (17) with $\alpha \in \mathcal{G}_H$ used in Lemma 5.1. Taking account (19) a similar calculation as in Lemma 5.1 demonstrates that

$$g_H = \Pi_{\alpha, 1} g_H$$

hence g_H descends to a metric $g_{H/[α]SU(3)}$ on the bundle $H/[α]SU(3)$ over S^6 .

Moreover the compatibility equation $\nabla_H g_H = 0$ says for arbitrary horizontal vector fields $u, v, w \in C^\infty(G_2; H)$ that

$$d(g_H(u, v))w = g_H((\nabla_H)_w u, v) + g_H(u, (\nabla_H)_w v). \quad (20)$$

Therefore

$$\begin{aligned} d(g_H(u\gamma_\alpha, v\gamma_\alpha))w &= g_H((\nabla_H)_w(u\gamma_\alpha), v\gamma_\alpha) + g_H(u\gamma_\alpha, (\nabla_H)_w(v\gamma_\alpha)) \\ &= g_H((\nabla_H)_w(u\gamma_\alpha)\gamma_\alpha^{-1}\gamma_\alpha, v\gamma_\alpha) + g_H(u\gamma_\alpha, (\nabla_H)_w(v\gamma_\alpha)\gamma_\alpha^{-1}\gamma_\alpha) \\ &= g_H((\nabla_H)_w(u\gamma_\alpha)\gamma_\alpha^{-1}, v) + g_H(u, (\nabla_H)_w(v\gamma_\alpha)\gamma_\alpha^{-1}). \end{aligned} \quad (21)$$

A comparison of (19), (20) and (21) shows that $\nabla_H(\cdot) = \nabla_H((\cdot)\gamma_\alpha)\gamma_\alpha^{-1}$ i.e., ∇_H commutes with the twisted right-SU(3)-translations, too. Consequently we find that the restricted Levi-Civita connection ∇_H also descends uniquely to a connection with an induced curvature as claimed. \diamond

We proceed further and choose a particular real vector bundle isomorphism

$$\varphi : TS^6 \longrightarrow H/[α]SU(3).$$

The endomorphism $J_{H/[α]SU(3)} : H/[α]SU(3) \rightarrow H/[α]SU(3)$ of Lemma 5.1 can be transported to an almost complex structure $J : TS^6 \rightarrow TS^6$ by the formula

$$J := \varphi^{-1}(J_{H/[α]SU(3)})\varphi.$$

Pulling back the metric and the connection in Lemma 5.2 we also get a Riemannian metric

$$g := (g_{H/[α]SU(3)})(\varphi \times \varphi)$$

(i.e., $g(\tilde{u}, \tilde{v}) := g_{H/[α]SU(3)}(\varphi(\tilde{u}), \varphi(\tilde{v}))$ for all $\tilde{u}, \tilde{v} \in C^\infty(S^6; TS^6)$) and a compatible connection

$$\nabla := \varphi^{-1}(\nabla_{H/[α]SU(3)})\varphi$$

on TS^6 . The triple (∇, J, g) depends only on $[\alpha] \in \mathcal{G}_{\text{SU}(3)} \setminus \mathcal{G}_H$. Indeed, if $\psi : TS^6 \rightarrow H/[\alpha]\text{SU}(3)$ is another isomorphism then $\psi^{-1} \circ \varphi : TS^6 \rightarrow TS^6$ is a general gauge transformation of the tangent bundle hence by our **Principle** from Sect. 2 we can suppose that the particular choice of the vector bundle isomorphism between the bundles TS^6 and $H/[\alpha]\text{SU}(3)$ over S^6 does not effect the geometric structure induced by (∇, J, g) on S^6 . In the particular case of an almost complex structure we checked at the end of Sect. 2 that $N_\nabla J$ transforms as a tensor under the induced action of $\psi^{-1} \circ \varphi \in \mathcal{G}_{TS^6}$.

We make a digression here and prove the counter-intuitive fact that

Lemma 5.3. *The almost complex structure $J : TS^6 \rightarrow TS^6$ is not homogeneous.*

Proof. Our strategy to prove this will be as follows. It is well-known (cf., e.g. [9]) that homogeneous almost complex structures on S^6 are parameterized by $\text{SO}(7)/\text{G}_2 \cong \mathbb{R}P^7$ and are orthogonally equivalent to the standard Cayley one I with respect to the standard metric on S^6 . Therefore if we can prove that J just constructed above is not orthogonally equivalent to I then we are done.

There exists a special isomorphism

$$\varphi_\alpha : TS^6 \cong H/[\text{Id}_H]\text{SU}(3) \longrightarrow H/[\alpha]\text{SU}(3)$$

induced by the gauge transformation $\alpha \in \mathcal{G}_H$ from Lemma 5.1. For an arbitrary $\beta \in \mathcal{G}_H$ let us denote sections of the corresponding quotient bundle as $[u]_\beta \in C^\infty(S^6; H/[\beta]\text{SU}(3))$ consisting of fiberwise equivalence classes of γ_β -invariant sections $u \in C^\infty(\text{G}_2; H)$, cf. Sect. 3. Then a tangent vector $\tilde{u} \in C^\infty(S^6; TS^6)$ can be written in the form $[u]_{\text{Id}_H} \in C^\infty(S^6; H/[\text{Id}_H]\text{SU}(3))$. Since (16) gives an identity $\alpha(u\gamma_{\text{Id}_H}) = (\alpha u)\gamma_\alpha$ we simply put

$$\varphi_\alpha(\tilde{u}) = \varphi_\alpha([u]_{\text{Id}_H}) := [\alpha u]_\alpha$$

with inverse $\varphi_\alpha^{-1}([u]_\alpha) = [\alpha^{-1}u]_{\text{Id}_H}$. Because elements of the form $\alpha u \in C^\infty(\text{G}_2; H) \subset C^\infty(\text{G}_2; T\text{G}_2)$ are gauge transformed objects the almost complex structure $J_H : H \rightarrow H$ also acts on them in its *gauge transformed* form $\alpha(J_H) = \alpha J_H \alpha^{-1}$ by our **Principle**. Hence recalling the construction of J and writing $J_H = \alpha I_H \alpha^{-1}$ as a usual *deformation* of I_H (cf. the remarks at the end of Sect. 2) we obtain

$$J\tilde{u} = \varphi_\alpha^{-1}(J_{H/[\alpha]\text{SU}(3)}(\varphi_\alpha([u]_{\text{Id}_H}))) = [\alpha^{-1}(\alpha J_H \alpha^{-1})\alpha u]_{\text{Id}_H} = [J_H u]_{\text{Id}_H} = [\alpha I_H \alpha^{-1} u]_{\text{Id}_H}.$$

This means that the operator $\alpha I_H \alpha^{-1}$, when restricted to $\text{SU}(3)$ -invariant sections, descends to TS^6 hence on this subspace it coincides with its ground mode in its Fourier expansion with respect to $\text{Id}_H \in \mathcal{G}_H$. Consequently $[\alpha I_H \alpha^{-1} u]_{\text{Id}_H}$ can be calculated analytically as $\Pi_{\text{Id}_H, 1}(\alpha I_H \alpha^{-1})$. Repeating the steps of the proof of Lemma 5.1 we find for an $y \in \text{G}_2$ that

$$\Pi_{\text{Id}_H, 1}(\alpha(y)I_{H_y}\alpha^{-1}(y)) = \int_{\text{SU}(3)} \left((R_{\gamma^{-1}})_* \alpha(y\gamma)I_{H_y\gamma}\alpha^{-1}(y\gamma)(R_\gamma)_* \right) d\gamma.$$

Using right- $\text{SU}(3)$ -translation to identify $\text{End}(H|_{\pi^{-1}(x)})$ with $\text{SU}(3) \times \text{End}H_y$ for some fixed $y \in \pi^{-1}(x)$ we can suppose that $\alpha(y(\cdot)) : \text{SU}(3) \rightarrow \text{End}H_y$ and $I_{H_y(\cdot)} : \text{SU}(3) \rightarrow \text{End}H_y$ are functions and the latter

being constant by its construction. Therefore we come up with

$$\begin{aligned} \Pi_{\text{Id}_H,1}(\alpha(y)I_{H_y}\alpha^{-1}(y)) &= \int_{\text{SU}(3)} (\alpha(y\gamma)I_{H_{y\gamma}}\alpha^{-1}(y\gamma)) d\gamma \\ &= \int_{\text{SU}(3)} (\text{Ad}_{\alpha^{-1}(y\gamma)}(I_{H_{y\gamma}})) d\gamma \\ &= \left(\int_{\text{SU}(3)} \text{Ad}_{\alpha^{-1}(y\gamma)} d\gamma \right) I_{H_y}. \end{aligned}$$

Introducing the real orthogonal representation $\sigma_y : \text{SU}(3) \rightarrow \text{Aut}(\text{End}H_y)$ where $\sigma_y(\gamma) := \text{Ad}_{\alpha^{-1}(y\gamma)}$ we conclude that

$$\Pi_{\text{Id}_H,1}(\alpha(y)I_{H_y}\alpha^{-1}(y)) = \mathbf{P}_y(I_{H_y})$$

where $\mathbf{P}_y := \int_{\text{SU}(3)} \sigma_y(\gamma) d\gamma$ is a finite dimensional projection onto the invariant subspace of σ_y :

$$\mathbf{P}_y : \text{End}H_y \longrightarrow (\text{End}H_y)_{\text{SU}(3)}.$$

It is easy to check that σ_y is a reducible representation and its invariant subspace is spanned by two invariant operators of H_y namely the identity operator of H_y and the complex multiplication on H_y induced by the embedding $\alpha(y\text{SU}(3)) \subset \text{SO}(6)$. This complex structure on H_y is obviously different from (H_y, I_{H_y}) —because $\alpha(y\gamma) \notin \text{U}(3) \subset \text{SO}(6)$ induced by I_{H_y} —consequently \mathbf{P}_y indeed projects I_{H_y} non-trivially. It is also clear from the construction that $\mathbf{P}_y(I_{H_y})$ is $\text{SU}(3)$ -invariant consequently if $\pi(y) = x$ then $\mathbf{P}_y(I_{H_y})$ descends unambiguously from H_y to $(H/[\text{Id}_H]\text{SU}(3))_x \cong T_x S^6$ and it coincides with $J_x : T_x S^6 \rightarrow T_x S^6$. In other words there exists an element $a \in C^\infty(S^6; \text{Aut}(TS^6))$ defined by

$$J =: aIa^{-1}$$

where I is the Cayley almost complex structure on S^6 . Taking into account that g_H projects onto the standard metric g_0 on $H/[\text{Id}_H]\text{SU}(3) \cong TS^6$ we conclude that the automorphism $a \in C^\infty(S^6; \text{Aut}(TS^6))$ is a non-orthogonal element with respect to the standard metric g_0 on S^6 . This is because $a(x) \in \text{Aut}(T_x S^6)$ with $x \in S^6$ arises as the average of g_H -orthogonal transformations $\alpha(y\gamma) \in \text{Aut}H_y$ when γ runs over $\text{SU}(3) \cong \pi^{-1}(x) \subset G_2$ and orthogonality is lost during averaging. Consequently J is not orthogonally equivalent to the Cayley structure as claimed.

Finally we record that the metric for which J is orthogonal has the form $g = g_0(a \times a)$ on S^6 . \diamond

The time has come to return to our field theory (10). So consider S^6 with its induced orientation from the one used on G_2 and also take the triple (∇, J, g) constructed on S^6 . The connection ∇ on TS^6 has a corresponding curvature tensor R_∇ . It is already meaningful to consider the symmetrized part $\mathfrak{S}(R_\nabla)$ of this induced curvature operator on TS^6 . Define the smooth function $f : S^6 \rightarrow \mathbb{R}^+$ by $dV = fd(y\text{SU}(3))$ where dV is the volume form to g and $d(y\text{SU}(3))$ denotes the standard coset measure on $G_2/\text{SU}(3) \cong S^6$ induced by $dy = dV_0$ on G_2 . Inserting the equality $1 = \int_{\text{SU}(3)} d\gamma$ into (10) and making use of the Fubini

formula [5, Proposition I.5.16]) for the coset $G_2/SU(3) \cong S^6$ we write

$$\begin{aligned} \mathcal{E}(\nabla, J, g) &= \frac{1}{2} \int_{S^6} \left(\frac{1}{e^2} |\mathfrak{S}(R_\nabla)|_g^2 + |\nabla g|_g^2 + |N_\nabla J|_g^2 + e^2 |JJ^* - \text{Id}_{TS^6}|_g^2 \right) dV \\ &= \frac{1}{2} \int_{S^6} \left(\int_{SU(3)} \left(\frac{1}{e^2} |\mathfrak{S}(R_\nabla)|_g^2 + |\nabla g|_g^2 + |N_\nabla J|_g^2 + e^2 |JJ^* - \text{Id}_{TS^6}|_g^2 \right) d\gamma \right) fd(ySU(3)) \\ &= \frac{1}{2} \int_{G_2} \left(\frac{1}{e^2} |\mathfrak{S}(R_{\nabla_H})|_{g_H}^2 + |\nabla_H g_H|_{g_H}^2 + |N_{\nabla_H} J_H|_{g_H}^2 + e^2 |J_H J_H^* - \text{Id}_H|_{g_H}^2 \right) (\pi^* f) dV_0 \end{aligned}$$

consequently by the aid of (15) we find that

$$0 \leq \mathcal{E}(\nabla, J, g) \leq \|\pi^* f\|_{L^\infty(G_2)} \mathcal{E}(\nabla_H, J_H, g_H) = 0.$$

Therefore (∇, J, g) is a smooth vacuum solution to the Yang–Mills–Higgs–Nijenhuis theory (10).

Putting all of our findings so far together we obtain that J is a smooth everywhere integrable almost complex structure on S^6 consequently we have arrived at the following result:

Theorem 5.1. *Assume that the Principle in Sect. 2 holds. Then there exists a unique smooth integrable almost complex structure J on S^6 given by the Higgs field in the weakly spontaneously broken vacuum (∇, J, g) of the Yang–Mills–Higgs–Nijenhuis theory (10) on S^6 .*

Consequently up to isomorphism there exists at least one compact complex manifold X whose underlying real manifold is homeomorphic to the six dimensional sphere. \diamond

Remark 5.1. 1. The new integrable almost complex structure on S^6 is nothing but the horizontal component J_H of an integrable almost complex structure J on G_2 projected onto S^6 by a non-trivial twisted projection given by (16). By [36, p. 123] the moduli space of complex structures on G_2 is $\mathbb{C}^+ \sqcup \mathbb{C}^-$. However one can demonstrate⁷ that any of these complex structures J projects onto the same J_H on the horizontal sub-bundle $H \subset TG_2$ consequently the constructed structure on S^6 is unique.

2. We have seen in Lemma 5.3 that although $Y \cong G_2$ as a complex manifold is homogeneous since left translations act by biholomorphisms i.e., $[(L_y)_*, J] = 0$ for all $y \in G_2$ this property does *not* descend to $X \cong S^6$. A standard way to get a homogeneous structure on S^6 is to try to push down $J_H : H \rightarrow H$ onto S^6 with respect to the canonical mapping $\pi_*|_H : H \rightarrow TS^6$. For this to happen J_H should commute with the adjoint action of G_2 , see [27, Volume II, Chapter X.6]. However one can demonstrate by an explicit calculation⁸ that for J_H this fails. In our language (cf. Definition 3.1) this procedure would look like this: J_H on H should be pushed down by the canonical Fourier expansion corresponding to $[\text{Id}_H] \in \mathcal{G}_{SU(3)} \setminus \mathcal{G}_H$ to an endomorphism $J_{H/[\text{Id}_H]SU(3)}$ of the quotient bundle $H/[\text{Id}_H]SU(3)$ over $G_2/SU(3) \cong S^6$. Then the induced canonical isomorphism $H/[\text{Id}_H]SU(3) \cong TS^6$ would yield a homogeneous integrable almost complex structure on S^6 . We note that although this is impossible for J_H , it is possible to perform the same Fourier expansion on the other non-integrable almost complex structure I_H on G_2 with $[\text{Id}_H] \in \mathcal{G}_{SU(3)} \setminus \mathcal{G}_H$. This way we just recover the Cayley almost complex structure I on S^6 .

Instead we are forced to push down J_H from H to an endomorphism $J_{H/[\alpha]SU(3)}$ on the bundle $H/[\alpha]SU(3)$ by a non-canonical Fourier expansion making use of a twisted action with a non-trivial element $[\alpha] \in \mathcal{G}_{SU(3)} \setminus \mathcal{G}_H$ as in Lemma 5.1.

⁷Thanks go to N.A. Dautseva for calculating explicitly the whole $\mathbb{C}^+ \sqcup \mathbb{C}^-$ family of integrable almost complex structures on G_2 in the origin $e \in G_2$ i.e., on $g_2 = T_e G_2$.

⁸See Footnote 7.

3. It is worth mentioning that a comparison of certain results offers an independent check of our Theorem 5.1 here. Let Y be G_2 equipped with a complex structure as before. Inserting the decomposition $TY = V \oplus H$ into $\wedge^{p,q}(TY)$ we obtain

$$\wedge^{p,q}(TY) \cong \wedge^{p,q}(V) \oplus \wedge^{p,q}(H) \oplus \dots$$

and let $\psi^{p,q} \in [\psi^{p,q}] \in H^{p,q}(Y; \mathbb{C})$ be a representative of a non-zero Dolbeault cohomology class. Its restriction $\psi^{p,q}|_H$ can be cut down to S^6 via Fourier expansion as before. Let X be a compact complex manifold homomorphic to S^6 with the complex structure coming from Y . This way one obtains an element $\tilde{\psi}^{p,q} \in C^\infty(\wedge^{p,q}X; \mathbb{C})$. It may happen that $[\tilde{\psi}^{p,q}] \in H^{p,q}(X; \mathbb{C})$ i.e., it represents a Dolbeault cohomology class.

We know the following things. On the one hand it follows from [36, Proposition 4.5] that $h^{0,1}(Y) = 1$ and $h^{0,2}(Y) = 0$. On the other hand it is proved in [19, 40] that $h^{0,1}(X) = h^{0,2}(X) + 1$. Therefore it is suggestive to expect that $h^{0,1}(X) = 1$ and $h^{0,2}(X) = 0$. Similarly, we know from [36, Proposition 4.5] that $h^{2,0}(Y) = 0$, $h^{1,1}(Y) = 1$, $h^{1,0}(Y) = 0$ and $h^{1,2}(Y) = 1$ meanwhile [40, Proposition 3.1] provides us that $h^{2,0}(X) + h^{1,1}(X) = h^{1,0}(X) + h^{1,2}(X) + 1$. This suggests that $h^{2,0}(X) = 0$, $h^{1,1}(X) = 1$, $h^{1,0}(X) = 0$ but $h^{1,2}(X) = 0 \neq 1$. However in spite of these naive considerations we notice that the general relationship between the Hodge numbers of Y and X is certainly not straightforward.

4. It is also worth pointing out again in the retrospective why our whole construction breaks down in the very similar situation of $SO(4n+1)/SO(4n) \cong S^{4n}$. In our understanding the crucial difference between these quotients and $G_2/SU(3) \cong S^6$ is as follows. Although in the former case $SO(4n)$ can be used to Fourier expand a complex structure on $SO(4n+1)$, it fails to furnish TS^{4n} with an almost complex structure because the standard representation of $SO(4n)$ is *real*. Meanwhile in the later case $SU(3)$ can be used not only for Fourier expansion but also to construct an almost complex structure on S^6 through its 3 dimensional *complex* representation. This is an exceptional phenomenon occurring only in six dimensions and was exploited in Lemma 5.1.

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