The universal von Neumann algebra of smooth four-manifolds with an application to gravity

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PLAN

* Rapid introduction to smooth 4-manifolds
* Construction of a von Neumann algebra (sketch)
* Construction of a new smooth 4-manifold invariant (sketch)
* Application to quantization of gravity in 4 dimensions (if time remains)

Smooth 4-manifolds

The basic problem: Assume that a topological space X admits a finite dimensional manifold structure. There are two essentially different possibilities: either X carries a C^0 or topological manifold structure X or a C^∞ or smooth manifold structure M. Given a toplogical manifold X, can it be refined in a compatible way (i.e., "smoothened") to an M, or equivalently: Does X admit any compatible smooth structure M (existence)? If yes, is this structure unique (uniqueness)?

Why is four dimensions so special?

(i) If $\dim_{\mathbb{R}} X \leqq 3$ then X always admits a smooth structure M which is unique (classical fact);

(ii) If $\dim_{\mathbb{R}} X \geqq 5$ and X is compact then it admits at most finitely (including zero) many different smooth structures (Sullivan);

(iii) If $\dim_{\mathbb{R}} X = 4$ then there exists a plethora of smooth structures and the situation is very complicated (Akbulut, Donaldson, Freedman, Gompf, Kirby, Taubes,...). It may happen that: X is compact and not smoothable at all (e.g. the simply connected space with intersection form E_8); it carries countably infinitely many smooth structures (e.g. the $K3$ surface); or we do not know yet how many (the case of S^4); X is not compact and the cardinality of different smooth structures reaches that of the continuum in ZFC set theory. Perhaps the most striking phenomenon:

Theorem

Let M be a smooth manifold which is homeomorphic to $\mathbb{R}^m$. Then M is diffeomorphic to $\mathbb{R}^m$ if $m \neq 4$. If $m = 4$ then there exist many non-countable families $\{R^4\}, \dots$ of pairwise non-diffeomorphic smooth 4-manifolds which are all homeomorphic but not diffeomorphic to $\mathbb{R}^4$ (such an R^4 is called a fake or exotic $\mathbb{R}^4$). $\diamond$

How to recognize or distinguish smoothness in four dimensions? (This question is important both mathematically and physically.) Try to construct computable but sensitive smooth 4-manifold invariants. Interestingly, four is the dimension of macroscopic physical space-time and the strongest invariants arrive from contemporary theoretical particle physics: The Donaldson invariant (from Yang–Mills or gauge theory) and the Seiberg–Witten invariant SW (from supersymmetric gauge theory). Why gravity theories (e.g. general relativity) cannot exhibit smooth invariants?

Remark

Two unsatisfactory properties of the Seiberg–Witten invariant (roughly): (i) If $\mathbb{C}P^2$ denotes the complex projective space then $SW(\mathbb{C}P^2) = 0$; (ii) If $\#$ denotes the connected sum operation on 4-manifolds then $SW(M\#N) = 0$. $\diamond$

Emergence of the II_1 hyperfinite factor

From the superabundance of smooth 4-manifolds one can destillate a single von Neumann algebra as follows:

Theorem (Etesi, 2017)

Let M be a connected oriented smooth 4-manifold. Making use of its smooth structure only, a von Neumann algebra $\mathfrak{R}(M)$ can be constructed which is geometric in the sense that it contains a norm-dense subalgebra of algebraic (i.e., formal) curvature tensors on M and $\mathfrak{R}(M)$ itself is a hyperfinite factor of type II_1 (hence is unique up to abstract isomorphism of von Neumann algebras).

The construction is based on three main steps and roughly (i.e. the technical details are suppressed) looks like this:

Let M be a connected oriented smooth 4-manifold and consider $T^{(p,q)}M$, the bundle of (p,q)-type tensors over M. Among these bundles $\wedge^2 T^*M \subset T^{(0,2)}M$, the bundle of 2-forms, is the only one which admits a natural (i.e. defined without any additional structure) pairing over M: Given $\alpha, \beta \in \Omega^2_c(M;\mathbb{C}) := C^\infty_c(M;\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C})$ a sesquilinear non-degenerate indefinite symmetric pairing

$$\langle \alpha, \beta \rangle_{L^2(M)} := \int_M \alpha \wedge \overline{\beta}$$

exists provided by the orientation and the smooth structure of M only.

Step 1. Construction of a unital C^*-algebra over M. Let $\circledast$ be the adjoint operation on $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ formally defined by $\langle A^\circledast\alpha,\beta\rangle_{L^2(M)} := \langle\alpha, A\beta\rangle_{L^2(M)}$ for all $\alpha,\beta\in\Omega^2_c(M;\mathbb{C})$. Consider the $\circledast$-closed space

$$V(M) := \left\{A\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\,|\,A^\circledast\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\text{ exists}, r(A^\circledast A)<+\infty\right\}$$

defined by the $\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ spectral radius

$$r(B) := \sup_{\lambda\in\mathbb{C}}\left\{|\lambda|\,\middle|\,B-\lambda\mathrm{Id}_{\Omega^2_c(M;\mathbb{C})}\in\mathrm{End}(\Omega^2_c(M;\mathbb{C}))\text{ is not bijective}\right\}.$$

Then $\sqrt{r}$ turns out to be a norm and the corresponding completion of $V(M)$ renders $(V(M),\circledast)$ a C^*-algebra $\mathfrak{R}(M)$. This C^*-algebra is non-trivial in the sense that $\mathfrak{R}(M)$ contains the space of all bounded bundle morphisms i.e., $C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}))\cap V(M)$ as well as all orientation preserving diffeomorphisms of M i.e., $\mathrm{Diff}^+(M)$. Hence in particular it possesses a unit $1\in\mathfrak{R}(M)$.

Step 2. Construction of a finite trace von Neumann algebra over M. Given $A \in \mathfrak{R}(M)$ let $[[A]] := \sqrt{r(A^*A)}$ denote its C^*-algebra norm from Step 1. This norm on $\mathfrak{R}(M)$ can be improved to a Hermitian scalar product $(\,\cdot\,,\,\cdot\,) : \mathfrak{R}(M) \times \mathfrak{R}(M) \to \mathbb{C}$ which in the usual way looks like

$$\begin{aligned}(A, B) \;\; := \;\; & \frac{1}{2}\left([[A+B]]^2 - [[A]]^2 - [[B]]^2\right)\\ & + \frac{\sqrt{-1}}{2}\left([[A+\sqrt{-1}B]]^2 - [[A]]^2 - [[\sqrt{-1}B]]^2\right)\end{aligned}$$

rendering $\mathfrak{R}(M)$ a Hilbert space $\mathscr{H}(M)$ with underlying complete complex vector space isomorphic to $\mathfrak{R}(M)$. Moreover $\mathfrak{R}(M) \subset \mathfrak{B}(\mathscr{H}(M))$ turns out to be a von Neumann algebra with a functional $\tau : \mathfrak{R}(M) \to \mathbb{C}$ given by

$$\tau(A) := (A, 1)$$

such that $\tau(AB) = \tau(BA)$ and $\tau(1) = 1$.

Remark

A peculiarity of four dimensions. The $*$-subalgebra $C^\infty(M;\mathrm{End}(\wedge^2 T^*M\otimes_{\mathbb{R}}\mathbb{C}))\subset\mathrm{End}(\Omega^2_c(M;\mathbb{C}))$ of bundle morphisms contains the space of algebraic (i.e., formal) curvature tensors on M. E.g. if (M,g) is an oriented Riemannian 4-manifold then its Riemannian curvature tensor R_g is a member of this algebra: With respect to the decomposition of 2-forms into their (anti)self-dual parts it looks like

$$R_g=\begin{pmatrix}\frac{1}{12}\mathrm{Scal}+\mathrm{Weyl}^+ & \mathrm{Ric}_0\\ \mathrm{Ric}_0^* & \frac{1}{12}\mathrm{Scal}+\mathrm{Weyl}^-\end{pmatrix}$$

as a map

$$R_g:\begin{matrix}\Omega^+_c(M;\mathbb{C})\\ \bigoplus\\ \Omega^-_c(M;\mathbb{C})\end{matrix}\longrightarrow\begin{matrix}\Omega^+_c(M;\mathbb{C})\\ \bigoplus\\ \Omega^-_c(M;\mathbb{C})\end{matrix}$$

that is, $R_g\in C^\infty(M;\mathrm{End}(\wedge^2T^*M\otimes_{\mathbb{R}}\mathbb{C}))$ indeed. $\diamond$

Step 3. This von Neumann algebra is approximated by algebraic curvature tensors over M and is a II_1-type hyperfinite factor. The von Neumann algebra $\mathfrak{R}(M)$ is geometric in the sense that for every $A \in \mathfrak{R}(M)$ there exists a sequence $\{R_i(A) \in C^\infty(M; \mathrm{End}(\wedge^2 T^*M \otimes_{\mathbb{R}} \mathbb{C})) \cap V(M) \mid i \in \mathbb{N}\}$ with the property

$$\lim_{i \to +\infty} [[A - R_i(A)]] = 0$$

where $[[\cdot]]$ is the spectral radius norm for which $\mathfrak{R}(M)$ is complete. In particular $\mathfrak{R}(M)$ contains all bounded complexified algebraic (i.e., formal) curvature tensors on M. Moreover $\mathfrak{R}(M)$ is hyperfinite (essentially because M has a countable basis) and is a factor (since M is connected) and is of type II_1 (from Step 2). Consequently whatever M was, its $\mathfrak{R}(M)$ is unique up to abstract isomorphisms of von Neumann algebras.

Remark

$\mathfrak{R}(M)$ as a noncommutative enhancement of M. Take a closed (i.e. compact without boundary) oriented Riemannian 4-manifold (M, g) and let $\Delta : C^\infty(M;\mathbb{C}) \to C^\infty(M;\mathbb{C})$ be the associated Laplace operator acting on complex-valued functions together with $\{\mathrm{e}^{-t\Delta}\}_{t>0}$ the corresponding heat semigroup. The heat semigroup is a family of self-adjoint operators possesing a smooth kernel which means that on all $f \in L^2(M;\mathbb{C})$ (constructed by the aid of the metric g) the action of the heat semigroup can be written as

$$(\mathrm{e}^{-t\Delta} f)(x) = \int\limits_M k_M(t; x, y) f(y) \mathrm{d}y$$

where $k_M(t; x, y)$ is a smooth real function of $t > 0$ and $x, y \in M$.

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Therefore the assignment

$$x \longmapsto k_M\left(\frac{t}{2}; x, \cdot\right)\mathrm{Id}_{\wedge^2 T^*M\otimes_{\mathbb{R}}\mathbb{C}} \quad \text{for all } x \in M \text{ and fixed } t > 0$$

gives rise to a map $i_{M,t} : M \to \mathfrak{R}(M)$. By a result of Bérard–Besson–Gallot this map is in fact a (non-canonical) continuous embedding of M into a Cartan subalgebra of $\mathfrak{R}(M)$ such that

$$i^*_{M,t}(\cdot\,,\,\cdot) = g + \frac{t}{3}\left(\frac{1}{2}\mathrm{Scal} - \mathrm{Ric}\right) + O(t^2) \quad \text{as } t \downarrow 0$$

where $(\cdot\,,\,\cdot)$ is the scalar product on $\mathfrak{R}(M)$ (viewed as $\mathscr{H}(M)$). Moreover the image $i_{M,t}(M) \subset \mathfrak{R}(M)$ can be regarded as an orbit of $\mathrm{Diff}^+(M) \subset \mathrm{Inn}(\mathfrak{R}(M))$. $\diamond$

A new smooth 4-manifold invariant

The rich representation theory of the II_1 hyperfinite factor allows one to construct a smooth invariant as well:

Theorem (Etesi, 2017)

Let M be a connected oriented smooth 4-manifold and $\mathfrak{R}(M)$ its von Neumann algebra as before. Then $\mathfrak{R}(M)$ admits a representation on a certain separable Hilbert space $\mathscr{K}(M)$ over M such that the unitary equivalence class of this representation is invariant under orientation-preserving diffeomorphisms of M. Consequently the Murray–von Neumann coupling constant of this representation gives rise to a smooth invariant $\gamma(M) \in [0, 1)$. It behaves like $\gamma(M \setminus Y) = \gamma(M)$ under excision of homologically trivial submanifolds and $\gamma(M \# N) = (\gamma(M) + \gamma(N))/(1 + \gamma(M)\gamma(N))$ under connected sum.

Again very roughly the construction goes as follows:

First recall that if $\mathfrak{R}$ is a II_1 hyperfinite factor and $\mathscr{H}$ is a (left) $\mathfrak{R}$-module then there exists a map $\dim_{\mathfrak{R}} : \mathscr{H} \rightarrow [0, +\infty)$ called the $\mathfrak{R}$-dimension or the Murray–von Neumann coupling constant of the (left) $\mathfrak{R}$-module $\mathscr{H}$. It is a unitary invariant of the representation and gives rise to an isomorphism between equivalnce classes of (left, not necessarily irreducible) $\mathfrak{R}$-modules and $[0, +\infty)$.

Then, essentially using the standard GNS technique only, out of M and $\mathfrak{R}(M)$ and $\mathscr{H}(M)$ as before, one constructs a Hilbert space $\{0\} \subseteqq \mathscr{K}(M) \subsetneqq \mathscr{H}(M)$ and a representation ρ_M of $\mathfrak{R}(M)$ on this Hilbert space. If $P_M : \mathscr{H}(M) \rightarrow \mathscr{K}(M)$ is the orthogonal projection then $P_M \in \mathfrak{R}(M)$ and $\dim_{\mathfrak{R}(M)} \mathscr{K}(M) = \tau(P_M) \in [0, 1)$ hence is an invariant of the representation. Finally putting

$$\gamma(M) := \tau(P_M)$$

we obtain a smooth invariant of M itself.

The Hilbert space $\mathscr{K}(M)$ arises as follows. Consider M and its $\mathfrak{R}(M)$ as before. The previous Hilbert space completion $\mathscr{H}(M)$ of $\mathfrak{R}(M)$ carries a left action π_M of $\mathfrak{R}(M)$ by multiplication hence $\mathscr{H}(M)$ is in fact the unique standard left $\mathfrak{R}(M)$-module (therefore $\dim_{\mathfrak{R}(M)}\mathscr{H}(M) = 1$). Pick a pair (Σ, ω) consisting of an (immersed) closed oriented surface $\Sigma \looparrowright M$ and a (not necessarily compactly supported!) 2-form $\omega \in \Omega^2(M; \mathbb{C})$ which is also closed i.e., $\mathrm{d}\omega = 0$. Consider the continuous $\mathbb{C}$-linear functional $F_{\Sigma,\omega} : \mathfrak{R}(M) \to \mathbb{C}$ by continuously extending the geometric map

$$A \longmapsto \frac{1}{2\pi\sqrt{-1}} \int\limits_{\Sigma} A\omega$$

from $V(M)$ to $\mathfrak{R}(M)$. Let $\{0\} \subseteqq I_{\Sigma,\omega} \subseteqq \mathfrak{R}(M)$ be the closure in the norm $[[\,\cdot\,]]$ on $\mathfrak{R}(M)$ of the subset of elements $A \in \mathfrak{R}(M)$ satisfying $F_{\Sigma,\omega}(A^*A) = 0$. In fact obviously $\{0\} \subsetneqq I_{\Sigma,\omega}$ and it is a left-multiplicative ideal for all pairs (Σ, ω).

One can show furthermore that either $I_{\Sigma,\omega} \subsetneqq \mathfrak{R}(M)$ (i.e. is not trivial) and is essentially independent of (Σ, ω) if $F_{\Sigma,\omega}(1) \neq 0$ or $I_{\Sigma,\omega} = \mathfrak{R}(M)$ (i.e. trivial) hence independent of (Σ, ω) if $F_{\Sigma,\omega}(1) = 0$. Exploiting $\mathfrak{R}(M) \cong \mathscr{H}(M)$ as complete complex vector spaces put

$$\mathscr{K}(M) := (I^{\perp}_{\Sigma,\omega}, (\,\cdot\,,\,\cdot\,)|_{I^{\perp}_{\Sigma,\omega}})$$

and define $\rho_M : \mathfrak{R}(M) \to \mathfrak{B}(\mathscr{K}(M))$ to be

$$\rho_M := \begin{cases} \pi_M|_{\mathscr{K}(M)} \text{ on } \mathscr{K}(M) \neq \{0\} \text{ if possible (then } \tau(P_M) \neq 0), \\ \pi_M|_{\mathscr{K}(M)} \text{ on } \mathscr{K}(M) = \{0\} \text{ otherwise (then } \tau(P_M) = 0). \end{cases}$$

The choice is unambigously determined by the topology of M and in the first case $\gamma(M) = \tau(P_M) \neq 0$ while $\gamma(M) = \tau(P_M) = 0$ in the second case.

Some properties of the invariant:

Let M, N be connected, oriented smooth 4-manifolds.

(i) (Reversing orientation.) $\gamma(M) = \gamma(\overline{M})$;

(ii) (Excision.) Let $\emptyset \subsetneqq Y \subset M$ be a submanifold so that $M \setminus Y \subsetneqq M$ is connected and the embedding $i : M \setminus Y \to M$ induces an isomorphism $i_* : H_2(M \setminus Y; \mathbb{Z}) \to H_2(M; \mathbb{Z})$ on the 2nd homology. Then $\gamma(M \setminus Y) = \gamma(M)$;

(iii) (Gluing.) The smooth invariant of the connected sum $M\#N$ satisfies

$$\gamma(M\#N) = \frac{\gamma(M) + \gamma(N)}{1 + \gamma(M)\gamma(N)} \quad .$$

Some calculations with the invariant:

(i) $\gamma(S^4) = 0$, $\gamma(\mathbb{R}^4) = 0$ and $\gamma(R^4) = 0$ for all fake R^4's;

(ii) Take $x \in [0,1)$ and put $R_0(x) := 0, R_1(x) := x, \dots, R_k(x) := \frac{x+R_{k-1}(x)}{1+xR_{k-1}(x)}, \dots$ and put $y := \gamma(\mathbb{C}P^2) = \gamma(\overline{\mathbb{C}P^2}) \neq 0$. Then for every connected, simply connected, closed 4-manifold M there exists a number $n \in \{0\} \cup \mathbb{N}$ such that $\gamma(M) = R_n(y)$. (*Proof*: For every pair (M, N) of connected, simply connected closed 4-manifolds there exist integers k_1, l_1 and k_2, l_2 such that $M \# k_1 \mathbb{C}P^2 \# l_1 \overline{\mathbb{C}P^2} \cong N \# k_2 \mathbb{C}P^2 \# l_2 \overline{\mathbb{C}P^2}$. Then put M arbitrary and $N := S^4$ and apply the gluing principle.) For instance $\gamma(\mathbb{C}P^1 \times \mathbb{C}P^1) = \gamma(\mathbb{C}P^1 \tilde{\times} \mathbb{C}P^1) = R_2(y) = \frac{2y}{1+y^2}$ and $\gamma(K3_{\mathrm{standard}}) = R_{22}(y)$.

Interesting question: What is the numerical value of $y \in (0,1)$?

Application to gravity

So far the mathematical construction was:

$$M \not\cong N \not\cong \ldots \Longrightarrow \mathfrak{R}(M) \cong \mathfrak{R}(N) \cong \ldots$$

i.e. out of any possible connected smooth 4-manifold a unique (a II_1 hyperfinite factor) von Neumann algebra has been constructed; all smooth 4-manifolds are embedded into it and this algebra is moreover generated by algebraic curvature tensors.

Let us symbolically, formally, etc. reverse this: Consider an abstractly given II_1 type hyperfinite factor and regard it as the collection of "all possible 4-spaces, curvature tensors, etc.":

$$\mathfrak{R} \Longrightarrow M \not\cong N \not\cong \ldots$$

We can declare the existence of a quantum theory with properties:

(i) $\mathfrak{R}$ is its algebra of observables, interpreted as curvature tensors in a 4 dimensional gravity theory;

(ii) The unique standard left $\mathfrak{R}$-module $\mathscr{H}$ is its state space and the standard representation $\pi : \mathfrak{R} \to \mathfrak{B}(\mathscr{H})$ is its unique quantum representation (in the sense of Haag) corresponding to a unique infinite temperature phase (in the sense of KMS theory) of the theory (because $\mathscr{H}$ comes from the unique tracial state τ on $\mathfrak{R}$);

(iii) The various $\mathfrak{R}$-modules $\rho_M : \mathfrak{R} \to \mathfrak{B}(\mathscr{K}(M))$, $\rho_N : \mathfrak{R} \to \mathfrak{B}(\mathscr{K}(N))$,... are its various classical representations corresponding to non-unique spontaneously broken finite temperature phases (because the $\mathscr{K}(M)$'s come from various non-tracial states $F_{\Sigma,\omega}$ on $\mathfrak{R}$);

(iv) The unique finite trace τ on $\mathfrak{R}$ can be used to calculate the expectation value $\tau(AB)$ of an observable $A \in \mathfrak{R}$ in a state $B \in \mathscr{H} \cong \mathfrak{R}$ (e.g. syntactically i.e. formally-mathematically if (M, g) is a space-time then its expectation value in another space-time state (N, h) is $\tau(R_g R_h) \in \mathbb{C}$ and is correctly defined; we expect that in a quantum theory of gravity such formal expectation values acquire even meanining i.e. appear at the semantical i.e. experimental-physical level of the theory as well);

(v) Dynamics is provided by the Tomita–Takesaki modular Hamiltonians Δ in the various phases i.e. representations. In the unique quantum i.e. tracial or infinite temperature phase $\Delta = 1$ hence the dynamics is trivial; in the various classical i.e. non-tracial or finite temperature phases $\Delta \neq 1$ hence the dynamics is non-trivial. Thermodynamical origin of time? (von Weizsäcker 1939, Connes–Rovelli 1994)

Further details to be worked out...

References:

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See also:

http://www.math.bme.hu/~etesi/publ.html

Thank you!