On Yang–Mills instantons over multi-centered gravitational instantons

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Abstract

In this paper we explicitly calculate the analogue of the 't Hooft $SU(2)$ Yang–Mills instantons on Gibbons–Hawking multi-centered gravitational instantons, which come in two parallel families: the multi-Eguchi–Hanson, or A_k ALE gravitational instantons and the multi-Taub–NUT spaces, or A_k ALF gravitational instantons. We calculate their energy and find the reducible ones. Following Kronheimer we also exploit the $U(1)$ invariance of our solutions and study the corresponding explicit singular $SU(2)$ magnetic monopole solutions of the Bogomolny equations on flat $\mathbb{R}^3$.

1 Introduction

Asymptotically Locally Euclidean or ALE and Asymptotically Locally Flat or ALF gravitational instantons are complete, non-compact hyper-Kähler four-manifolds which are intensively studied from both physical and mathematical sides recently. This paper is a continuation of the project of constructing $SU(2)$ Yang–Mills instantons on ALF gravitational instantons started in [10] and [11]. In [10] we identified all the reducible $SU(2)$ instantons on the Euclidean Schwarzschild manifold (which is ALF and Ricci flat, though not self-dual), and showed that these solutions, albeit not their reducibility, were already known to [6]. Then in [11] we went on and studied

$SU(2)$ instantons on the Taub–NUT space (which is an ALF gravitational instanton). Following [16] we exploited the self-duality of the metric, to obtain a family of $SU(2)$ instantons, which could be considered as the analogue of the 't Hooft solutions on $\mathbb{R}^4$. (For more historical remarks the reader is referred to [11].)

Here we carry out the generalization of [11] to the more general case of multi-Taub–NUT spaces (which are also ALF gravitational instantons). An interesting aspect of our paper is that what we do goes almost verbatim in the case of the other family of Gibbons–Hawking multi-centered gravitational instantons [13], namely the multi-Eguchi–Hanson spaces.

The only difference in the two cases will be the energy of some of the Yang–Mills instantons. Namely in the multi-Eguchi–Hanson case the infinity may contribute minus a fraction of one to the energy, while in the multi-Taub–NUT case the energy is always an integer.

We will make the calculation of the energy integral in the convenient framework of considering the $U(1)$-invariant instantons as singular monopoles on $\mathbb{R}^3$. We will follow [18] to write down this reduction and perform the integral. This way we will also exhibit explicit, but singular, solutions to the $SU(2)$ Bogomolny equation on $\mathbb{R}^3$. Remarkably we will get the same monopoles, regardless of which family of Gibbons–Hawking metrics we consider: the ALE or ALF.

Then we go on and identify the reducible $U(1)$ instantons among our solutions. They are interesting from the point of view of Hodge theory as their curvatures are L^2 harmonic 2-forms. Namely, only very recently has the dimension of the space of L^2 harmonic forms on ALF gravitational instantons been calculated in [14]. Remarkably we are able to identify a basis of generators for these L^2 harmonic 2-forms, arising in this paper as curvatures of reducible 't Hooft instantons.

In the last part of our paper we put everything together and look at the parameter spaces of solutions we uncovered in this paper and explain its properties in light of the results and point out some future directions to consider.

Our instantons are not new. Most of them were found during the early eighties by Aragone and Colaiacomo [1] and by Chakrabarti, Boutaleb-Joutei and Comtet in a series of papers [4]; for a review cf. [5]. The 't Hooft solutions and a few more general ADHM solutions in the Eguchi–Hanson case were written down explicitly in [3]. Finally, $SU(2)$ Yang–Mills instantons over A_k ALE gravitational instantons were classified by Kronheimer and Nakajima [19].

2 Instantons over the multi-centered spaces

In this section we generalize the method of [11], designed for finding Yang–Mills instantons on Taub–NUT space, to the multi-centered gravitational instantons (M_V, g_V) of Gibbons and Hawking [13][15].

In our previous paper we used the following result. Suppose (M, g) is a four-dimensional Riemannian spin-manifold which is self-dual and has vanishing scalar curvature. Consider the metric spin connection $\widetilde{\nabla}_S$ of the rescaled manifold $(M, \tilde{g})$ with $\tilde{g} = f^2 g$ where f is harmonic (i.e., $\triangle f = 0$ with respect to g). This $\mathfrak{so}(4)$-valued connection lives on the complex spinor bundle SM. Take the ∇^- component of $\widetilde{\nabla}_S$. This connection can be constructed as the projection onto the chiral spinor bundle S^-M, according to the splitting $SM = S^+M \oplus S^-M$ and can be regarded as an $\mathfrak{su}(2)^-$-valued connection. This is because the above splitting of the spinor bundle is compatible with the Lie algebra decomposition $\mathfrak{so}(4) \cong \mathfrak{su}(2)^+ \oplus \mathfrak{su}(2)^-$. In light of a result of Atiyah, Hitchin and Singer [2] (see also [11]) ∇^- is self-dual with respect to g. These ideas in the case of flat $\mathbb{R}^4$ were first used by Jackiw, Nohl and Rebbi [16], in the case of ALE gravitational instantons in [3] while for the Taub–NUT case by the authors [11] to construct plenty of new instantons. Our aim is to repeat this method in the present more general case.

If $\widetilde{\nabla}_S$ is represented locally by an $\mathfrak{so}(4)$-valued 1-form $\tilde{\omega}$ then we write A^- for the $\mathfrak{su}(2)^-$-valued connection 1-form of ∇^- in this gauge over a chart $U \subset M$.

Now we turn our attention to a brief description of the Gibbons–Hawking spaces denoted by M_V. This space topologically can be understood as follows. There is a circle action on M_V with k fixed points $p_1, \dots, p_k \in M_V$, called NUTs[1]. The quotient is $\mathbb{R}^3$ and we denote the images of the fixed points also by $p_1, \dots, p_k \in \mathbb{R}^3$. Then $U_V := M_V \setminus \{p_1, \dots, p_k\}$ is fibered over $Z_V := \mathbb{R}^3 \setminus \{p_1, \dots, p_k\}$ with S^1 fibers. The degree of this circle bundle around each point p_i is one.

The metric g_V on U_V looks like (cf. e.g. p. 363 of [8])

$$\mathrm{d}s^2 = V(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \frac{1}{V}(\mathrm{d}\tau + \alpha)^2, \tag{1}$$

where $\tau \in (0, 8\pi m]$ parametrizes the circles and $x = (x, y, z) \in \mathbb{R}^3$; the smooth function $V : Z_V \to \mathbb{R}$ and the 1-form $\alpha \in C^\infty(\Lambda^1 Z_V)$ are defined as follows:

$$V(x, \tau) = V(x) = c + \sum_{i=1}^{k} \frac{2m}{|x - p_i|}, \qquad \mathrm{d}\alpha = *_3 \mathrm{d}V. \tag{2}$$

Here c is a parameter with values 0 or 1 and $*_3$ refers to the Hodge-operation with respect to the flat metric on $\mathbb{R}^3$. We can see that the metric is independent of τ hence we have a Killing field on (M_V, g_V). This Killing field provides the above mentioned $U(1)$-action. Furthermore it is possible to show that, despite the apparent singularities in the NUTs, these metrics extend analytically over the whole M_V.

If $c = 0$ with k running over the positive integers we find the multi-Eguchi–Hanson spaces. If $c = 1$ we just recover the multi-Taub–NUT spaces. In particular if $c = 1$ and $k = 1$ then (1) is the Taub–NUT geometry on $\mathbb{R}^4$ (cf. Eq. (6) of [11]): this is easily seen by using the coordinate transformation $x^2 + y^2 + z^2 = (r - m)^2$. Note that under this transform V has the form (by putting the only NUT into the origin $r = m$)

$$V(r) = 1 + \frac{2m}{r - m}$$

i.e., coincides with the scaling function f found in the Taub–NUT case (cf. Eq. (8) in [11]) with $\lambda = 2m$. From here we guess that in the general case the right scaling functions will have the shape

$$f(x) = \lambda_0 + \sum_{i=1}^{k} \frac{\lambda_i}{|x - p_i|} \tag{3}$$

[1]The reason for writing the nut with block letters is the following. In 1951 Taub discovered an empty space solution of the Lorentzian Einstein equations [23] whose maximal analytical extensions were found by Newman, Unti and Tamburino in 1963 [20]. Hence this solution is referred to as Taub–NUT space-time.

On the other hand in 1978 Gibbons and Hawking presented a classification of known gravitational instantons taking into account the topology of the critical set of the Killing field appearing in theses spaces [8] [13]. Those whose critical set contains only isolated points were called “nuts” while another class having two dimensional spheres as singular sets were named as “bolts”. It is a funny coincidence that an example for the former class is provided by the generalization of the Riemannian version of Taub–NUT space-time, called multi-Taub–NUT space in the present work.

where by an inessential rescaling we can always assume that λ_0 is either 0 or 1.

We can prove that these are indeed harmonic functions. In order to put our formulas in the simplest form, we introduce the notation $(x, y, z) = (x^1, x^2, x^3)$ and will use Einstein summation convention.

First note that from the form of the metric, we have the following straightforward orthonormal tetrad of 1-forms on U_V:

$$\xi^0 = \frac{1}{\sqrt{V}}(\mathrm{d}\tau + \alpha), \quad \xi^1 = \sqrt{V}\mathrm{d}x^1, \quad \xi^2 = \sqrt{V}\mathrm{d}x^2, \quad \xi^3 = \sqrt{V}\mathrm{d}x^3. \tag{4}$$

The orientation is fixed such that $*(\xi^0 \wedge \xi^1 \wedge \xi^2 \wedge \xi^3) = 1$ which is the *opposite* to the orientation induced by any of the complex structures in the hyper-Kähler family. Let us thus take a general $U(1)$-invariant function $f : M_V \to \mathbb{R}$. It means that $f = f(x^1, x^2, x^3)$ does not depend on τ. Then we have on U_V:

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i = \frac{1}{\sqrt{V}}\frac{\partial f}{\partial x^i}\,\xi^i \quad (i = 1, 2, 3).$$

By using the above orthonormal tetrad (4), we see that

$$*\mathrm{d}f = \varepsilon^i{}_{jk}\frac{1}{\sqrt{V}}\frac{\partial f}{\partial x^i}\,\xi^j \wedge \xi^k \wedge \xi^0 = \varepsilon^i{}_{jk}\frac{\partial f}{\partial x^i}\,\mathrm{d}x^j \wedge \mathrm{d}x^k \wedge (\mathrm{d}\tau + \alpha).$$

(Note that $*$ is the Hodge star operator on (M_V, g_V).) Consequently

$$\triangle f = \delta \mathrm{d}f = - * \mathrm{d} * \mathrm{d}f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Thus we see that the $U(1)$-invariant f is harmonic on (M_V, g_V) if and only if it is harmonic on the flat $\mathbb{R}^3$. Positive harmonic functions on $\mathbb{R}^3$ which are bounded at infinity and have finitely many point-singularities, with at most inverse polynomial growth, have the shape

$$f(x) = \lambda_0 + \sum_i \frac{\lambda_i}{|x - q_i|},$$

where q_i's are finitely many points in $\mathbb{R}^3$. Note if we want our function $f : M_V \to \mathbb{R}$ to be a harmonic function with only point singularities, we need to place the q_i's at the NUTs p_i of the metric. Thus we have found all reasonable positive $U(1)$-invariant, harmonic functions on (M_V, g_V) with point singularities and bounded at infinity. They are of the form (3).

Now we determine the Levi–Civitá connection of the re-scaled metric $\tilde{g} = f^2 g_V$ restricted to U_V. By using the trivialization (4) of the tangent bundle TU_V, the Levi–Civitá connection can be represented by an $\mathfrak{so}(4)$-valued 1-form $\tilde{\omega}$ on U_V. With the help of the Cartan equation we can write

$$\mathrm{d}\tilde{\xi}^i = -\tilde{\omega}^i_j \wedge \tilde{\xi}^j.$$

Taking into account that $\tilde{\xi}^i = f\xi^i$, this yields

$$\mathrm{d}\xi^i + \mathrm{d}(\log f) \wedge \xi^i = -\tilde{\omega}^i_j \wedge \xi^j.$$

As we have seen, f does not depend on τ, therefore we have an expansion like

$$\mathrm{d}(\log f) = \frac{1}{\sqrt{V}}\frac{\partial \log f}{\partial x^j}\,\xi^j \quad (j = 1, 2, 3).$$

Putting this and $\mathrm{d}\xi^i = -\omega^i_j \wedge \xi^j$ for the original connection into the previous Cartan equation, we get

$$\left(\omega^i_j + \frac{1}{\sqrt{V}}\frac{\partial \log f}{\partial x^j}\,\xi^i\right) \wedge \xi^j = \tilde{\omega}^i_j \wedge \xi^j,$$

consequently the local components of the new connection on U_V, after antisymmetrizing, have the shape

$$\tilde{\omega}^i_j = \omega^i_j + \frac{1}{\sqrt{V}}\left(\frac{\partial \log f}{\partial x^j}\,\xi^i - \frac{\partial \log f}{\partial x^i}\,\xi^j\right).$$

(Here it is understood that $\partial \log f/\partial x^0 = 0$.) By the aid of the Cartan equation in the original metric, the components of the original connection ω take the form on U_V

$$\omega^1_2 = -\frac{1}{2\sqrt{V}}\frac{\partial \log V}{\partial x^3}\xi^0 + \frac{1}{2\sqrt{V}}\frac{\partial \log V}{\partial x^2}\xi^1 - \frac{1}{2\sqrt{V}}\frac{\partial \log V}{\partial x^1}\xi^2$$

and

$$\omega^0_1 = -\frac{1}{2\sqrt{V}}\frac{\partial \log V}{\partial x^1}\xi^0 + \frac{1}{2\sqrt{V}}\frac{\partial \log V}{\partial x^3}\xi^2 - \frac{1}{2\sqrt{V}}\frac{\partial \log V}{\partial x^2}\xi^3,$$

and cyclically for the rest. Notice that from this explicit form we see that in this gauge the Levi–Civitá connection is itself self-dual i.e., $\omega^0_1 = \omega^2_3$ etc. (cf. p. 363 of [8]). Consequently it cancels out if we project $\tilde{\omega}$ onto the $\mathfrak{su}(2)^-$ subalgebra via

$$A^-_{\lambda_0,\dots,\lambda_k} = \frac{1}{4}\sum_{a=1}^{3}\sum_{i,j=0}^{3}\left(\overline{\eta}^i_{a,j}\tilde{\omega}^i_j\right)\,\overline{\eta}_a$$

where the 't Hooft matrices $\overline{\eta}^i$ are given by:

$$\overline{\eta}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \overline{\eta}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \overline{\eta}_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Using the identification $\mathfrak{su}(2)^- \cong \mathrm{Im}\,\mathbb{H}$ via $(\overline{\eta}_1, \overline{\eta}_2, \overline{\eta}_3) \mapsto (\mathbf{i}, -\mathbf{j}, -\mathbf{k})$ we get for $\nabla^-_{\lambda_0,\dots,\lambda_k}$ in the gauge (or bundle trivialization) given by (4) on U_V that

$$A^-_{\lambda_0,\dots,\lambda_k} = \frac{\mathbf{i}}{2\sqrt{V}}\left(\frac{\partial \log f}{\partial x^1}\xi^0 - \frac{\partial \log f}{\partial x^3}\xi^2 + \frac{\partial \log f}{\partial x^2}\xi^3\right) +$$
$$\frac{\mathbf{j}}{2\sqrt{V}}\left(\frac{\partial \log f}{\partial x^2}\xi^0 + \frac{\partial \log f}{\partial x^3}\xi^1 - \frac{\partial \log f}{\partial x^1}\xi^3\right) +$$
$$\frac{\mathbf{k}}{2\sqrt{V}}\left(\frac{\partial \log f}{\partial x^3}\xi^0 - \frac{\partial \log f}{\partial x^2}\xi^1 + \frac{\partial \log f}{\partial x^1}\xi^2\right).$$

This long but very symmetric expression can be written in a quite simple form as follows. Consider the following quaternion-valued 1-form ξ and the imaginary quaternion $\mathbf{d}(\log f)$ (we use the symbol "$\mathbf{d}$" to distinguish it from the real 1-form $\mathrm{d}(\log f)$)

$$\xi := \xi^0 + \xi^1\mathbf{i} + \xi^2\mathbf{j} + \xi^3\mathbf{k}, \quad \mathbf{d}(\log f) := \frac{\partial \log f}{\partial x^1}\mathbf{i} + \frac{\partial \log f}{\partial x^2}\mathbf{j} + \frac{\partial \log f}{\partial x^3}\mathbf{k}.$$

It is easily checked that with this notation the connection takes the simple shape

$$A^{-}_{\lambda_0,\dots,\lambda_k} = \mathrm{Im}\,\frac{\mathbf{d}(\log f)\,\xi}{2\sqrt{V}}. \tag{5}$$

This form emphasizes the analogy with the case of flat $\mathbb{R}^4$ (cf. p. 103 of [12]). By construction these connections are self-dual over (U_V, g_V); but we will prove in the next sections that they are furthermore gauge equivalent to smooth, self-dual connections over the whole (M_V, g_V) and have finite energy.

We remark that there is a familiar face in the crowd (5). This is the solution corresponding to the choice $f = V$. The Yang–Mills instanton (5) is then the same as the projection of the Levi–Civitá connection of M_V onto the other chiral bundle S^+M_V, which is easy to see from the form of the Levi–Civitá connection calculated earlier in this section. We denote this connection by ∇_{metric}. This is the solution which we called in [11] the Pope–Yuille solution of unit energy in the Taub–NUT case [21][4]. In the Eguchi–Hanson case this is the solution of energy 3/2 found in [4] and later again in [17].

To conclude this section, we write down the field strength or curvature of (5) over U_V. The field strength of a connection ∇ with connection 1-form A over a chart is $F = \mathrm{d}A + A\wedge A$. Therefore we can see by (5) that our field strength has the form over U_V

$$F^{-}_{\lambda_0,\dots,\lambda_k} = -\frac{\mathrm{d}V}{4V^{3/2}}\wedge \mathrm{Im}\left(\mathbf{d}(\log f)\,\xi\right) + \frac{1}{2\sqrt{V}}\mathrm{Im}\,\left(\mathbf{d}(\log f)\,\mathrm{d}\xi\right) +$$

$$\frac{1}{2\sqrt{V}}\mathrm{Im}\,\left(\mathrm{d}\mathbf{d}(\log f)\wedge\xi\right) + \frac{1}{4V}\left(\mathrm{Im}\,\left(\mathbf{d}(\log f)\,\xi\right)\wedge \mathrm{Im}\,\left(\mathbf{d}(\log f)\,\xi\right)\right).$$

The terms in the first line can be adjusted as follows. Using the identity

$$\mathrm{d}\xi = *_3\frac{\mathrm{d}V}{\sqrt{V}} - \frac{\mathrm{d}V}{2V}\wedge\overline{\xi}$$

we can write them in the form

$$\left(-\frac{\mathrm{d}V}{2V^{3/2}}\wedge\xi^0 + *_3\frac{\mathrm{d}V}{2V}\right)\mathbf{d}(\log f) = -\frac{\mathbf{d}(\log f)}{4V^2}\mathrm{Re}\,\left(\mathbf{d}V\xi\wedge\overline{\xi}\right)$$

with

$$\mathbf{d}V = \frac{\partial V}{\partial x^1}\mathbf{i} + \frac{\partial V}{\partial x^2}\mathbf{j} + \frac{\partial V}{\partial x^3}\mathbf{k}.$$

One immediately sees at this point that these two terms are self-dual with respect to g_V at least over U_V because $\xi\wedge\overline{\xi}$ is a basis for self-dual 2-forms. Self-duality of the remaining two terms is not so transparent; however a tedious but straightforward calculation assures us about it. So we can conclude that the connections $\nabla^{-}_{\lambda_0,\dots,\lambda_k}$ are self-dual with respect to g_V at least over U_V.

The action, or energy, or L^2-norm of the connection (if exists) is the integral

$$\|F^{-}_{\lambda_0,\dots,\lambda_k}\|^2 = \frac{1}{8\pi^2}\int\limits_{M_V}|F^{-}_{\lambda_0,\dots,\lambda_k}|^2_{g_V} = -\frac{1}{8\pi^2}\int\limits_{M_V}\mathrm{tr}\,\left(F^{-}_{\lambda_0,\dots,\lambda_k}\wedge *F^{-}_{\lambda_0,\dots,\lambda_k}\right). \tag{6}$$

Next we turn our attention to the extendibility of (5) over the NUTs and its asymptotical behaviour in order to calculate the above integral.

3 A gauge transformation around a NUT

Our next goal is to demonstrate that the self-dual connections just constructed are well-defined over the whole M_V up to gauge transformations. As we have seen, the gauge (5) contains only pointlike singularities hence if we could prove that the energy of $\nabla^-_{\lambda_0,\dots,\lambda_k}$ is finite in a small ball around a fixed NUT then, in light of the removable singularity theorem of Uhlenbeck [24] we could conclude that our self-dual connections extend through the NUTs after suitable gauge transformations. However the direct calculation of (6) is complicated because of its implicit character. Consequently it will be performed in the next section, here we write down a gauge transformation explicitly, such that the resulting connection will be easily extendible over the NUTs. To this end, we derive a useful decomposition of (5).

To keep our expressions as short as possible, we introduce further notations: let us write $r_j(x) := |x - p_j|$ and

$$V_i := c + \frac{2m}{r_i}, \qquad f_i := \lambda_0 + \frac{\lambda_i}{r_i},$$

and define the 1-form α_i on $\mathbb{R}^3$ by the equation $*_3 \mathrm{d}\alpha_i = \mathrm{d}V_i$. With these notations we introduce a new real valued function a_i on U_V as follows:

$$V =: a_i V_i. \tag{7}$$

One easily calculates

$$a_i = 1 + \frac{2m}{2m + cr_i}\sum_{j\neq i}\frac{r_i}{r_j}.$$

In the same fashion by putting the fixed NUT p_i into the origin of $\mathbb{R}^3$ (i.e., $p_i = 0$) we can write

$$\mathbf{d}(\log f) = -\frac{1}{f}(x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k})\sum_{j=1}^{k}\frac{\lambda_j}{r_j^3} + \frac{1}{f}\sum_{j\neq i}\frac{\lambda_j}{r_j^3}(p_j^1\mathbf{i} + p_j^2\mathbf{j} + p_j^3\mathbf{k}).$$

On the other hand,

$$\mathbf{d}(\log f_i) = -\frac{1}{f_i}\frac{\lambda_i}{r_i^3}(x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}),$$

therefore we can write for a real valued function b_i on U_V that

$$\mathbf{d}(\log f) =: b_i\mathbf{d}(\log f_i) + \mathbf{p}_i \tag{8}$$

where

$$b_i = \frac{1 + \sum\limits_{j\neq i}\frac{\lambda_j}{\lambda_i}\left(\frac{r_i}{r_j}\right)^3}{1 + \sum\limits_{j\neq i}\frac{\lambda_j}{\lambda_i + \lambda_0 r_i}\frac{r_i}{r_j}}$$

and we have introduced the function $\mathbf{p}_i : U_V \to \mathrm{Im}\,\mathbb{H}$ given by

$$\mathbf{p}_i := \frac{1}{f}\sum_{j\neq i}\frac{\lambda_j}{r_j^3}(p_j^1\mathbf{i} + p_j^2\mathbf{j} + p_j^3\mathbf{k}).$$

As a next step, we have to examine these new objects around the NUT p_i. It is not difficult to see, via the explicit expressions for a_i, b_i and $\mathbf{p}_i$ that

$$\lim_{r_i(x)\to 0} a_i(x) = 1, \qquad \lim_{r_i(x)\to 0} b_i(x) = 1, \qquad \lim_{r_i(x)\to 0} |\mathbf{p}_i(x)|_{\mathbb{H}} = 0.$$

These observations show that in fact a_i, b_i and $\mathbf{p}_i$ are well defined on $U_V \cup \{p_i\}$. Now putting decompositions (7), (8) into (5) we arrive at the expression

$$A^-_{\lambda_0,\dots,\lambda_k} = \mathrm{Im}\,\frac{b_i \mathbf{d}(\log f_i)\xi}{2\sqrt{a_i V_i}} + \mathrm{Im}\,\frac{\mathbf{p}_i \xi}{2\sqrt{a_i V_i}}. \tag{9}$$

From here one immediately deduces that the first term (which formally coincides with the original 't Hooft solution on flat $\mathbb{R}^4$ when $c = 0$; and the solution on Taub–NUT space constructed in [11] when $c = 1$) is singular while the last term vanish in the NUT p_i. In order to analise the structure of the singular term more carefully, we introduce spherical coordinates around p_i i.e., the origin of $\mathbb{R}^3$:

$$x^1 := r_i \sin\Theta_i \cos\phi_i, \quad x^2 := r_i \sin\Theta_i \sin\phi_i, \quad x^3 := r_i \cos\Theta_i.$$

Here $\Theta_i \in (0, \pi]$ and $\phi_i \in (0, 2\pi]$ are the angles. In this way we rewrite the singular term as

$$\mathrm{Im}\,\frac{b_i \mathbf{d}(\log f_i)\xi}{2\sqrt{a_i V_i}} = \frac{b_i \lambda_i}{2(\lambda_0 r_i + \lambda_i)(r_i + 2m)} \left(-\frac{1}{a_i}(\mathrm{d}\tau + \alpha) + (r_i + 2m)\alpha_i \right) \mathbf{q}_i +$$

$$\frac{b_i \lambda_i}{2(\lambda_0 r_i + \lambda_i)} ((\sin\phi_i \mathbf{i} - \cos\phi_i \mathbf{j})\mathrm{d}\Theta_i - \mathbf{k}\mathrm{d}\phi_i)$$

where we have introduced the notation for the "radial" imaginary quaternion

$$\mathbf{q}_i := \sin\Theta_i \cos\phi_i \mathbf{i} + \sin\Theta_i \sin\phi_i \mathbf{j} + \cos\Theta_i \mathbf{k}$$

and have exploited the identity $\alpha_i = \cos\Theta_i \mathrm{d}\phi_i$ at several points. Now we can easily see that the above expression is singular because the quaternion $\mathbf{q}_i$ is ill-defined in the NUT p_i i.e., in the origin $r_i = 0$ (all the other terms involved are regular in p_i). Consequently we are seeking for a gauge transformation which rotates the quaternion $\mathbf{q}_i$ into $\mathbf{k}$, for example. This gauge transformation cannot be performed continuously over the whole $U_V \cong \mathbb{R}^3 \setminus \{p_1, \dots, p_k\} \times S^1$ therefore we introduce the two open subsets

$$U^+_V := \left\{ (x^1, x^2, x^3, \tau) \mid x^3 \geq 0, \text{ i.e., } \Theta_i \geq \frac{\pi}{2} \right\}, \quad U^-_V := \left\{ (x^1, x^2, x^3, \tau) \mid x^3 \leq 0, \text{ i.e., } \Theta_i \leq \frac{\pi}{2} \right\}.$$

Now it is not difficult to check that the gauge transformations $g^\pm_i : U^\pm_V \to SU(2)$ given by

$$g^\pm_i(\tau, r_i, \Theta_i, \phi_i) := \exp\left(\pm\mathbf{k}\frac{\phi_i}{2} \right) \exp\left(-\mathbf{j}\frac{\Theta_i}{2} \right) \exp\left(-\mathbf{k}\frac{\phi_i}{2} \right)$$

(here exp: $\mathfrak{su}(2) \to SU(2)$ is the exponential map) indeed rotate $\mathbf{q}_i$ into $\mathbf{k}$. (We remark that this gauge transformation is exactly the same which was used to identify the Charap–Duff instantons with Abelian ones over the Euclidean Schwarzschild manifold in [10].) Notice that $\exp(\mathbf{k}\phi_i) g^-_i = g^+_i$ showing that the two gauge transformations are related with an Abelian one along the equatorial plane $x^3 = 0$ i.e., $\Theta_i = \pi/2$.

We can calculate that (cf. [10])

$$g^\pm_i ((\sin\phi_i \mathbf{i} - \cos\phi_i \mathbf{j})\mathrm{d}\Theta_i - \mathbf{k}\mathrm{d}\phi_i)(g^\pm_i)^{-1} = -2 g^\pm_i \mathrm{d}\left(g^\pm_i\right)^{-1} \mp \mathbf{k}\mathrm{d}\phi_i$$

therefore we get for the gauge transformed connection $B^{\pm}_{\lambda_0,\dots,\lambda_k} := g_i^{\pm}A^{-}_{\lambda_0,\dots,\lambda_k}(g_i^{\pm})^{-1}+g_i^{\pm}\mathrm{d}\left(g_i^{\pm}\right)^{-1}$ by using decomposition (9) that

$$B^{\pm}_{\lambda_0,\dots,\lambda_k} = \frac{b_i\lambda_i}{2(\lambda_0 r_i+\lambda_i)(r_i+2m)}\left(-\frac{1}{a_i}(\mathrm{d}\tau+\alpha)+(r_i+2m)(\mp 1+\cos\Theta_i)\mathrm{d}\phi_i\right)\mathbf{k}+$$
$$\left(1-\frac{b_i\lambda_i}{\lambda_0 r_i+\lambda_i}\right)g_i^{\pm}\mathrm{d}\left(g_i^{\pm}\right)^{-1}+g_i^{\pm}\mathrm{Im}\,\frac{\mathbf{p}_i\xi}{2\sqrt{a_iV_i}}\left(g_i^{\pm}\right)^{-1}. \qquad (10)$$

Now we have reached the desired result: we can see that, approaching the NUT p_i from the north (i.e., along a curve whose points obey $x^3>0$) the terms written in the first line of $B^{+}_{\lambda_0,\dots,\lambda_k}$ remain regular while the (non-Abelian) terms of the second line vanish if $r_i=0$. The situation is exactly the same from the south if we use $B^{-}_{\lambda_0,\dots,\lambda_k}$. Moreover $B^{+}_{\lambda_0,\dots,\lambda_k}$ and $B^{-}_{\lambda_0,\dots,\lambda_k}$ are related via an Abelian gauge transformation along the equator $x^3=0$. Consequently in this new gauge our instantons are regular in the particular NUT p_i.

By performing the same transformations around all NUTs $p_1,\dots,p_k$ we can see that in fact the instantons (5) extend smoothly across all the NUTs.

4 Kronheimer's singular monopoles and the energy

In this section we identify our $U(1)$-invariant instantons over the Gibbons–Hawking spaces with monopoles over flat $\mathbb{R}^3$ carrying singularities in the (images of the) NUTs $p_1,\dots,p_k$. This identification enables us to calculate the energy of our solutions as well. In this section we will follow Kronheimer's work [18].

Remember S^-M_V is an $SU(2)$ vector bundle over M_V and the $U(1)$ action can be lifted from M_V to S^-M_V. Our instantons $\nabla^{-}_{\lambda_0,\dots,\lambda_k}$ are self-dual, $U(1)$-invariant $SU(2)$-connections on this bundle. If we choose an $U(1)$-invariant gauge in S^-M_V (for example (5) or (10) is suitable) then $A^{-}_{\lambda_0,\dots,\lambda_k}$ becomes an $U(1)$-invariant $\mathfrak{su}(2)^-$-valued 1-form which we can uniquely express as

$$A^{-}_{\lambda_0,\dots,\lambda_k} = \pi^*A-\pi^*\Psi(\mathrm{d}\tau+\alpha)$$

where A and Ψ are a 1-form and a 0-form on Z_V and $\pi: U_V\to Z_V$ is the projection. Dividing S^-M_V by the $U(1)$-action we obtain a $SU(2)$ vector bundle E over Z_V together with a pair (A,Ψ) on it. Omitting π^* we calculate

$$F^{-}_{\lambda_0,\dots,\lambda_k} = \nabla A^{-}_{\lambda_0,\dots,\lambda_k} = (F-\Psi\mathrm{d}\alpha)-\nabla\Psi\wedge(\mathrm{d}\tau+\alpha). \qquad (11)$$

Here we have used the notation $F=\nabla A$. One finds that the self-duality $F^{-}_{\lambda_0,\dots,\lambda_k} = *F^{-}_{\lambda_0,\dots,\lambda_k}$ of the original connection is equivalent to $*_3(F-\Psi\mathrm{d}\alpha)=V\nabla\Psi$ or $*_3F=\nabla(V\Psi)$ since $\mathrm{d}\alpha=*_3\mathrm{d}V$. Putting $\Phi:=V\Psi$ we can write

$$F=*_3\nabla\Phi$$

which is the Bogomolny equation for the pair (A,Φ) on Z_V. This shows that the pair (A,Φ) can be naturally interpreted as a magnetic vectorpotential and a Higgs field while F as a magnetic field on Z_V. Notice however that in this case the Higgs field Φ is singular at the images of the NUTs hence the reason we have to use the punctured $\mathbb{R}^3$ denoted by Z_V.

In our case, by using (5) we can write down the pair (A,Φ) explicitly. We easily find that

$$\Phi=\frac{\mathbf{d}(\log f)}{2},\qquad A=\mathrm{Im}\,\frac{\mathbf{d}(\log f)\mathbf{i}}{2}\mathrm{d}x^1+\mathrm{Im}\,\frac{\mathbf{d}(\log f)\mathbf{j}}{2}\mathrm{d}x^2+\mathrm{Im}\,\frac{\mathbf{d}(\log f)\mathbf{k}}{2}\mathrm{d}x^3. \qquad (12)$$

In this framework one can find a simple formula for the energy we are seeking for. In what follows define Z^R_ε to be the intersection of a large ball of radius R in $\mathbb{R}^3$ containing all the NUTs and the complements of small balls of radius ε_i surrounding the NUTs p_i. Putting (11) into (6) we can write

$$8\pi^2\|F^-_{\lambda_0,\dots,\lambda_k}\|^2 = \int\limits_{M_V} |(F-\Psi{\rm d}\alpha)-\nabla\Psi\wedge({\rm d}\tau+\alpha)|^2_{g_V} =$$

$$-\int\limits_{M_V} V\left(\frac{1}{V^2}|F-\Psi{\rm d}\alpha|^2+|\nabla\Psi|^2\right)\wedge({\rm d}\tau+\alpha) = 16\pi m\lim_{\substack{R\to\infty\\ \varepsilon_i\to 0}}\int\limits_{Z^R_\varepsilon} V|\nabla\Psi|^2$$

taking into account the Bogomolny equation in the form $F-\Psi{\rm d}\alpha = *_3V\nabla\Psi$. By writing $2V|\nabla\Psi|^2 = {\rm d}*_3{\rm d}(V|\Psi|^2)$ and exploiting Stokes' theorem we arrive at the useful formula

$$8\pi^2\|F^-_{\lambda_0,\dots,\lambda_k}\|^2 = 8\pi m\lim_{\substack{R\to\infty\\ \varepsilon_i\to 0}}\int\limits_{\partial Z^R_\varepsilon} *_3{\rm d}\left(\frac{|\Phi|^2}{V}\right).$$

In the above expressions $|\cdot\,,\,\cdot|$ stands for the Killing norm of $\mathfrak{su}(2)$ induced by the Euclidean metric on $\mathbb{R}^3$ i.e., it is the standard Killing norm hence it is equal to *twice* the usual norm square of a quaternion under the identification $\mathfrak{su}(2)\cong{\rm Im}\,\mathbb{H}$; e.g. $|\Phi|^2 = 2|\Phi|^2_{\mathbb{H}}$.

In order to determine the exact value of the action of our solutions, we simply have to calculate the contributions of each components of the boundary ∂Z^R_ε, in other words, the NUTs p_i and the infinity of $\mathbb{R}^3$.

First we can see that, using (7) and (8), for small ε_i there is an expansion

$$\frac{|\Phi|^2}{V} = \frac{1}{2}\left|\left(-\frac{b_i\lambda_i}{\varepsilon_i(\lambda_0\varepsilon_i+\lambda_i)}\mathbf{q}_i+\mathbf{p}_i\right)\sqrt{\frac{\varepsilon_i}{a_i(c\varepsilon_i+2m)}}\right|^2_{\mathbb{H}} = \begin{cases} 1/(4m\varepsilon_i)+O(1) & \text{if } \lambda_i\neq 0,\\ 0 & \text{if } \lambda_i = 0.\end{cases}$$

This implies that

$$\left|{\rm d}\left(\frac{|\Phi|^2}{V}\right)\right| = \begin{cases} |-1/(4m\varepsilon_i^2)+O(1/\varepsilon_i)| & \text{if } \lambda_i\neq 0,\\ 0 & \text{if } \lambda_i = 0.\end{cases}$$

(For clarity we remark that the outermost $|\cdot|$ in the last expression is the Euclidean norm on $\mathbb{R}^3$.) However in the above integral there is a contribution $-4\pi\varepsilon_i^2$ by the spheres (the minus sign comes from the orientation) consequently each NUT p_i together with the condition $\lambda_i\neq 0$ contributes a factor $8\pi^2$ to the integral i.e., 1 to the energy. Hence the total contribution is n where n stands for the number of non-zero λ_i's. Clearly $0\leq n\leq k$.

To see the contribution of infinity, we have to understand the fall-off properties of the function $|\Phi|^2/V$. Clearly this is not modified if we put all the NUTs into the origin of $\mathbb{R}^3$. Thus asymptotically our functions take the shape

$$V = c+\frac{2mk}{R},\qquad f = \lambda_0+\frac{1}{R}\sum_{i=1}^k\lambda_i =: \lambda_0+\frac{\lambda}{R}.$$

Putting these expressions into $|\Phi|^2/V$ and expanding it into $1/R$ terms one finds the following for large R:

$$\frac{|\Phi|^2}{V}=\frac{1}{2}\left|-\frac{\lambda}{R(\lambda_0R+\lambda)}\mathbf{q}\sqrt{\frac{R}{cR+2mk}}\right|^2_{\mathbb{H}}=\begin{cases}1/(4mkR)+O(1/R^2) & \text{if } \lambda_0=0 \text{ and } c=0,\\ O(1/R^2) & \text{otherwise.}\end{cases}$$

Consequently

$$\left|\mathrm{d}\left(\frac{|\Phi|^2}{V}\right)\right|=\begin{cases}|-1/(4mkR^2)+O(1/R^3)| & \text{if } \lambda_0=0 \text{ and } c=0,\\ |O(1/R^3)| & \text{otherwise.}\end{cases}$$

Since now again $4\pi R^2$ is the volume of the large sphere (notice that there is no minus sign because of the orientation) we get that the contribution of infinity is $-8\pi^2/k$ to the above integral i.e., $-1/k$ to the energy in the case of the multi-Eguchi–Hanson space $c=0$ with the special limit $\lambda_0=0$. Otherwise the contribution of infinity is zero. Also notice that if $n=0$ then $f=\lambda_0$ is a constant hence the energy is certainly zero.

We summarize our findings in the following

Theorem 4.1 *The connection $\nabla^-_{\lambda_0,\dots,\lambda_k}$ as given in (5) is a smooth self-dual $SU(2)$ Yang–Mills connection and has energy* 0 *if all $\lambda_i=0$ ($i>0$) i.e., $n=0$ (in which case $\nabla^-_{\lambda_0,0,\dots,0}$ is the trivial connection), otherwise we have*

$$\|F^-_{\lambda_0,\dots,\lambda_k}\|^2=\begin{cases}n-(1/k) & \textit{if } \lambda_0=0,\ c=0,\\ n & \textit{otherwise,}\end{cases}$$

where k refers to the number of NUTs while n is the number of non-zero λ_i's ($i>0$). ◇

5 The Abelian solutions

In this section we show that (5) is reducible to $U(1)$ if and only if for an $i=0,\dots,k$ we have $\lambda_i>0$ (for simplicity we take $\lambda_i=1$), while $\lambda_j=0$ if $j\neq i$ and $j=0,\dots,k$. First take the case when $\lambda_0\neq 0$ but the other λ's vanish, then our solution (5), which we denote by ∇_0, is trivial. Now suppose that $\lambda_i\neq 0$ for $i=1,\dots,k$ but the others vanish. Take the NUT p_i and consider the new gauge (10). In the case at hand the second line vanishes and (10) reduces to the manifestly Abelian instanton ∇_i with $B^\pm_i:=B^\pm_{0,\dots,0,1,0,\dots,0}$ (with 1 is in the ith place):

$$B^\pm_i=\left(-\frac{\mathrm{d}\tau+\alpha}{Vr_i}+(\mp 1+\cos\Theta_i)\mathrm{d}\phi_i\right)\frac{\mathbf{k}}{2}.$$

But in fact these connections are gauge equivalent because $B^+_i+\frac{1}{2}\mathbf{k}\mathrm{d}\phi_i=B^-_i-\frac{1}{2}\mathbf{k}\mathrm{d}\phi_i$, and ∇_i locally can be written as

$$B_i=\left(-\frac{\mathrm{d}\tau+\alpha}{Vr_i}+\alpha_i\right)\frac{\mathbf{k}}{2}, \tag{13}$$

where $*_3\mathrm{d}\alpha_i=\mathrm{d}V_i$. The curvature $F_i=\nabla B_i$ of this Abelian connection are the L^2 harmonic 2-forms found by [22] in the multi-Taub–NUT case.

Now take any reducible $SU(2)$ instanton of the form (5). Then in a suitable gauge it can be brought to the form

$$\sum_i \mu_i B_i,$$

where μ_i are real numbers. This follows by applying a result from [14] which claims that the 2-forms F_i generate the space of L^2 harmonic 2-forms on our space. Now note that the corresponding Higgs field of this instanton is $\left(\sum\limits_i \frac{\mu_i}{r_i}\right)\frac{\mathbf{k}}{2}$. Since $|\Phi|$ is gauge invariant we have the following identity around a particular NUT p_i via (8) and (12):

$$|\Phi|^2 = \frac{1}{2}\left(\frac{\mu_i}{\varepsilon_i} + \sum_{j\neq i}\frac{\mu_j}{r_j}\right)^2 = \frac{1}{2}\left|-\frac{b_i\lambda_i}{\varepsilon_i(\lambda_0\varepsilon_i + \lambda_i)}\mathbf{q}_i + \mathbf{p}_i\right|^2_{\mathbb{H}}.$$

Provided it is not identically zero the right hand side nowhere vanishes. We can deduce that the μ_i's have to be all non-positive or non-negative (otherwise $|\Phi|^2$ would be zero at some point). Without loss of generality we suppose they are all non-negative. Moreover the right hand side times ε_i^2 tends to either 0 or 1/2 when ε_i tends to 0. Therefore we have that μ_i is either 0 or 1. Moreover for large R we have $1/(2R^2)\sum\limits_i \mu_i$ for the left hand side while the right hand side asymptotically looks like

$$|\Phi|^2 = \begin{cases} 1/(2R^2) + O(1/R^3) & \text{if } \lambda_0 = 0, \\ O(1/R^3) & \text{if } \lambda_0 \neq 0. \end{cases}$$

Thus this last expansion implies that $\lambda_0 = 0$ and only one $\mu_i = 1$, the rest vanishes. This in turn shows that only one λ_i is not zero which proves the following

Theorem 5.1 *An instanton in the form (5) is reducible if and only if for an $i = 0, \ldots, k$ we have $\lambda_i \neq 0$ and $\lambda_j = 0$ for $j = 0, 1, \ldots, i-1, i+1, \ldots, k$; in this case it can be put into the form (13).* $\diamond$

6 Conclusion

In this paper we have explicitly calculated the analogue of the 't Hooft $SU(2)$ instantons for multi-centered metrics of k NUTs. We found a k parameter family of such solutions (parametrized by $\lambda_0, \ldots, \lambda_k$ all non-negative numbers modulo an overall scaling) one for each NUTs. The structure of the space of solutions could be best visualized as an intersection of the positive quadrant and the unit ball in $\mathbb{R}^k$. The k flat sides of this body correspond to the solutions $\lambda_i = 0$ for an $i = 1, \ldots, k$, while the spherical boundary component corresponds to $\lambda_0 = 0$. The vertices of this body correspond to the reducible solutions, the origin corresponding to the trivial solution; the rest to the non-Abelian solutions (5). The energy of the solutions are k at the interior points of the body, while reduces by 1 for every $\lambda_i = 0$ for $i = 1, \ldots, k$ and by $1/k$ if $\lambda_0 = c = 0$. In order to see the boundary solutions as ideal solutions of energy k we can imagine an ideal Dirac delta connection of energy 1 at each NUT p_i for which $\lambda_i = 0$ in the Taub–NUT case; the situation is the same for the multi-Eguchi–Hanson case except that add a further Dirac delta connection of energy $1/k$ at infinity if $\lambda_0 = 0$.

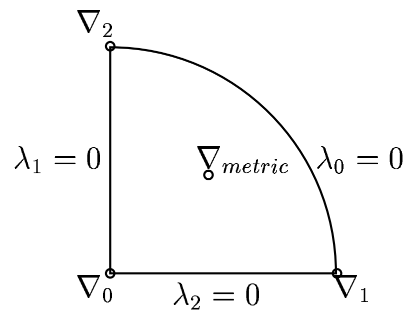

Figure 1: Case of 2-Taub–NUT

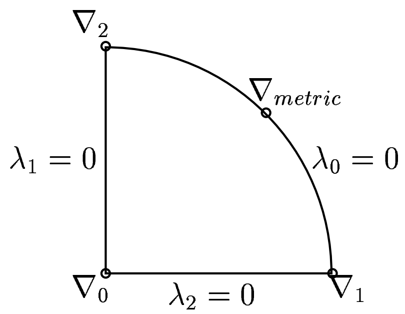

Figure 2: Case of Eguchi–Hanson

A puzzling feature of this description in the multi-Taub–NUT case is at the interior of the spherical boundary component of our solution space, where $\lambda_0 = 0$ but no other λ_i is zero. These solutions are not reducible by Theorem 5.1 but have energy k by Theorem 4.1. However they are singular points in our solution space. Thus either there are other solutions on the other side of the spherical boundary, or the moduli space of energy k instantons will have singularities at non-reducible points, a phenomenon which does not occur when the underlying 4-manifold is compact or ALE. The relation to the multi-Eguchi–Hanson case, explained in the next paragraph, however might point to the second possibility.

Following [18] we have also studied our $U(1)$ invariant instantons as singular monopoles on $\mathbb{R}^3$. Interestingly we got the same monopoles (12) from the two cases ($c = 0, 1$). This raises the possibility to take a $U(1)$ invariant instanton on a multi-Eguchi–Hanson space (where all the solutions were classified in [19]) consider it as a singular monopole on $\mathbb{R}^3$ and then pull it back to a $U(1)$ invariant instanton on the corresponding multi-Taub–NUT space. This method might lead to the construction of (all) $U(1)$ invariant instantons on a multi-Taub–NUT space and if so it would indeed exhibit singular points in the non-reducible part of the moduli space as we explained in the paragraph above. We postpone a more thorough discussion of these ideas for a future work as well as the construction of the spectral data [7] [18] for the singular monopoles.

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