A Global Uniqueness Theorem for Stationary Black Holes

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Abstract

A global uniqueness theorem for stationary black holes is proved as a direct consequence of the Topological Censorship Theorem and the topological classification of compact, simply connected four-manifolds.

1 Introduction

There is a remarkable interplay between differential geometry, the theory of differential equations and the physics of gravitation in the famous proof of the uniqueness of stationary black holes. The first proof was given in a series of papers by Carter, Hawking, Israel and Robinson (for a survey see [8], [12]). In the eighties a very elegant shorter proof was discovered by Mazur [10] who found a hidden symmetry of the electromagnetic and gravitational fields. These very deep and difficult investigations all were devoted to the uniqueness problem of the metric on a suitable four-manifold carrying a Lorentzian asymptotically flat structure in the spirit of Penrose's description of infinity of space-times.

More recently physicist's effort is addressed to the *topology* of the event horizons of general (i.e. non-stationary) black holes. The first theorems were proven by Hawking [8], [9] and later by Gannon, Galloway [6],[7] and others. Based on the celebrated Topological Censorship Theorem of Friedman, Schleich and Witt [5] and using energy conditions Chruściel and Wald gave a short proof that the event horizon of a stationary black hole in a "moment" is always a sphere.

The question naturally arises what can one say about the topology of the space-time itself in this case.

On the other hand the final step in the understanding of four-manifolds making use "classical" (i.e. non-physical) methods was done by Freedman in 1981 who gave a complete (topological) classification of compact simply connected four-manifolds.

In this paper, referring to the results of Chruściel–Wald and Freedman, we prove that a *global* uniqueness also holds for stationary black holes and more generally stationary space-times i.e. not only the metric but even the topology of the space-time in question is unique. Our method is based on a natural compactification of the space-time manifold and a careful study of a vector field extended to this compact manifold.

Truely speaking, this is not a very surprising result in light of the local uniqueness. However, it demonstrates the power of the theorems mentioned above.

2 Vector Fields

First we define the precise notion of a stationary, asymptotically flat space-time containing a black hole collecting the standard definitions. Let us summarize the properties of an asymptotically flat and empty space-time that we need; for the whole definitions see [12] and the notion of an asymptotically empty and flat space-time can be found in [8].

Let (M, g) be a space-time manifold and $x \in M$. then $J^{\pm}(x)$ is called the causal future and past of x respectively. If a space-time manifold (M, g) is *asymptotically flat and empty* then there exists a conformal inclusion $i : (M, g) \mapsto (\tilde{M}, \tilde{g})$ such that

- $\partial i(M) = \{i_0\} \cup \mathcal{I}^+ \cup \mathcal{I}^-$, where i_0 is the space-like infinity and the future and past null-infinities $\mathcal{I}^{\pm}$ satisfy $\mathcal{I}^{\pm} := \partial J^{\pm}(i_0) \setminus \{i_0\} = S^2 \times \mathbf{R}$;
- $\tilde{M} \setminus i(M) = \overline{J^+(i_0)} \cup \overline{J^-(i_0)}$;
- There exists a function $\Omega : \tilde{M} \mapsto \mathbf{R}^+$ which is smooth everywhere (except possibly at i_0) satisfying $\tilde{g}|_{i(M)} = \Omega^2 \left(i^{-1}\right)^* g$ and $\Omega|_{\partial i(M)} = 0$;
- Every null geodesic on $(\tilde{M}, \tilde{g})$ has future and past end-points on $\mathcal{I}^{\pm}$ respectively.

Definition. Let (M, g) be asymptotically flat. (M, g) is called *strongly asymptotically predictable* if there exists an open region $\tilde{V} \subset \tilde{M}$ such that $\tilde{V} \supset i(M) \cap J^-(\mathcal{I}^+)$ and $(\tilde{V}, \tilde{g}|_{\tilde{V}})$ is globally hyperbolic.

Remark. This definition provides that no singularities are visible for an observer in $\tilde{M} \cap \tilde{V}$. Moreover one can prove that $\tilde{M} \cap \tilde{V}$ can be foliated by Cauchy-surfaces S_t $(t \in \mathbf{R})$, see [8], [12].

Definition. Let (M, g) be a strongly asymptotically predictable space-time manifold. If $B := M \setminus J^-(\mathcal{I}^+)$ is not empty, then B is called a *black hole region* and $H := \partial B$ is its *event horizon.*

Remarks. Moreover we require (M, g) to be *stationary* i.e. there exists a future-directed time-like Killing field K on (M, g). In this case H is a three-dimensional null-surface in M, hence for each $t \in \mathbf{R}$, $H_t := H \cap S_t$ (S_t is a Cauchy-surface) is a two-dimensional surface in M. We shall assume that H_t (the event horizon in a "moment") is a two dimensional embedded, orientable, smooth, compact surface in M without boundary. This is the requirement the event horizon to be "regular". This condition is satisfied by physically relevant black hole solutions of Einstein's equations but there is no *a priori* reason to assume it.

Let (M, g) be a maximally extended space-time manifold as above. In the following considerations we shall focus on *one* outer, asymptotically flat region of it i.e. a part of (M, g) whose boundary at infinity in $\tilde{M}$ is *connected.* To get such an (incomplete) manifold we simply cut up (M, g) along one connected component of its event horizon. We shall continue to denote this separated part also by (M, g) (g is the original metric restricted to our domain). We can see that under the conformal inclusion i the manifold (M, g) has boundary $\partial i(M) = \tilde{H} \cup \mathcal{I}^+ \cup \{i_0\} \cup \mathcal{I}^-$ where $\tilde{H} = i(H)$ and $\tilde{H}$, $\mathcal{I}^\pm$ are *connected* now.

Proposition 1. *Let (M, g) be a space-time manifold. Then $K|_H \in \Gamma(TH)$.*

Proof. Assuming the existence of a point $p \in H$ such that $K_p \notin T_pH$, let $\gamma : (-1, 1) \mapsto M$ be a smooth integral curve of K satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = K_p$. Hence there is an $\varepsilon \in (-1, 1)$ such that $\gamma(-\varepsilon) \in B$ and $\gamma(\varepsilon) \notin B$. But this means that $\gamma(-\varepsilon) \in J^-(\mathcal{I}^+)$, since it can be connected by an integral curve of K to $\gamma(\varepsilon) \in J^-(\mathcal{I}^+)$. Hence this assumption led us to a contradiction. $\diamond$

Corollary. H is invariant under the flow generated by K on M and, being H a null-surface, $K|_H$ is a null vector field. $\diamond$

Remarks. Using a heuristic argument here we can identify H_t up to homeomorphism as follows. Since the boundary of (M, g) at infinity is homeomorphic to $S^2 \times \mathbf{R}$ we may assume due to the stationarity that H_t is homeomorphic to S^2. According to recent articles one can prove that this is indeed the case for a stationary black hole [6], [7], [8], [9]. For our purposes it is more important to refer to a stronger result of Chruściel and Wald [3].

They prove that an outer asymptotically flat region of a stationary space-time manifold satisfying the null energy condition is simply connected as a consequence of the Topological Censorship Theorem [5]. Under suitable additional hypothesis (e.g. the compactness of H_t) it follows that H_t is homeomorphic to a sphere.

Hence, with the aid of these results we get that H is homeomorphic to $S^2 \times \mathbf{R}$.

Now let us study the behaviour of the Killing field near the infinity! Let $\tilde{K} := i_* K$ induced by the inclusion i.

Proposition 2. *$\tilde{K}$ becomes a null vector field at infinity.*

Proof. Let $\tilde{\gamma}: \mathbf{R} \mapsto \tilde{M}$ be an inextendible integral curve of $\tilde{K}$! We may write:

$$\|\tilde{K}_{\tilde{\gamma}(t)}|_{i(M)}\|^2 = \tilde{g}|_{i(M)}(\tilde{K}_{\tilde{\gamma}(t)}, \tilde{K}_{\tilde{\gamma}(t)}) = \Omega^2(\tilde{\gamma}(t)) \left(i^{-1}\right)^* g(i_* K_{\gamma(t)}, i_* K_{\gamma(t)}) =$$
$$= \Omega^2(\tilde{\gamma}(t)) g(K_{\gamma(t)}, K_{\gamma(t)}).$$

We have used the third property of asymptotic flatness. However, using the fact that K is a Killing field on M, we can write

$$\|\tilde{K}_{\tilde{\gamma}(t)}|_{i(M)}\|^2 = a\Omega^2(\tilde{\gamma}(t)),$$

where $a := g(K_{\gamma(t_0)}, K_{\gamma(t_0)})$ is a constant for an arbitrary $t_0 \in \mathbf{R}$. But

$$\lim_{t \to \pm\infty} \Omega^2(\tilde{\gamma}(t)) = 0$$

because of the asymptotic flatness. $\diamond$

In the light of Proposition 1. and 2. we can see that K approaches a null vector field near the boundary of M. Hence it is straightforward to study the behaviour of null vector fields on M and $\tilde{M}$. Applying a smooth deformation to K on (M, g) we can produce a smooth, nowhere vanishing (but highly non-unique!) *null* vector field K_0 on (M, g) whose integral curves are inextendible geodesics. Denoting by $\tilde{K}_0$ the image of this field under i, i.e. $\tilde{K}_0 = i_* K_0$, $\tilde{K}_0$ has future and past end-points on $\mathcal{I}^{\pm}$ respectively by the fourth property of asymptotically flat space-times.

Remark. Of course, we could have started with this null vector field instead of the Killing field K. The reason for dealing with the naturally given Killing field was the attempt to exploit as much as possible the structure of a stationary, asymptotically flat space-time manifold.

Now let $\tilde{X}$ be an inextendible null vector field on $(\tilde{M}, \tilde{g})$ i.e. its integral curves are inextendible. We would like to study the extension of this field to the null infinities hence first we have to extend the domain of its integral curves, which is $\mathbf{R}$ in this moment. Let us suppose that this extended domain is the circle S^1.

Proposition 3. *Let $\tilde{X}$ be a null vector field on $(\tilde{M}, \tilde{g})$ with extended domain whose integral curves are geodesics. Then $\tilde{X}$ can be extended to $\mathcal{I}^{\pm}$ if and only if $\tilde{X}|_{\mathcal{I}^{\pm}} = 0$.*

Proof. Let $i(S) \cup \{i_0\} =: \tilde{S} \subset \tilde{M}$ be a space-like hypersurface and let $x \in \tilde{S} \setminus (\tilde{S} \cap \tilde{B})$ $(x \neq i_0)$. Then there exists an integral curve $\tilde{\gamma}: \mathbf{R} \mapsto \tilde{M}$ such that $\tilde{\gamma}(0) = x$. So we can find a point $q \in \mathcal{I}^+$ possessing the property

$$q = \lim_{t \to +\infty} \tilde{\gamma}(t).$$

Let us define a map $\phi : \tilde{S} \setminus (\tilde{S} \cap \tilde{B}) \mapsto \mathcal{I}^+$ by $\phi(x) = q$.

Let we assume that we have extended $\tilde{X}$ to $\mathcal{I}^+$ in a smooth manner and there is a $q \in \mathcal{I}^+$ such that $\tilde{X}_q \neq 0$! Hence there is a smooth curve $\tilde{\beta} : (-\varepsilon, \varepsilon) \mapsto \mathcal{I}^+$ for a suitable small positive number ε satisfying

$$\tilde{\beta}(0) = q, \quad \dot{\tilde{\beta}}(0) = \tilde{X}_q.$$

Obviously we can find a $\delta < \varepsilon$ such that for $q \neq q' \in U_q$

$$\tilde{\beta}(\delta) = q', \quad \dot{\tilde{\beta}}(\delta) = \tilde{X}_{q'} \neq 0.$$

But in this case there is an $x' \in U_x$ (U_x denotes a small neighbourhood of x) and an integral curve $\tilde{\gamma}'$ of $\tilde{X}$ such that $\phi(x') = q'$.

In other words q' satisfies

$$q' \in \mathbf{im}\tilde{\beta}, \quad q' \in \mathbf{im}\tilde{\gamma}',$$

and

$$q \in \mathbf{im}\tilde{\beta}, \quad q \notin \mathbf{im}\tilde{\gamma}'.$$

This means that q' is a branching point of an integral curve of $\tilde{X}$. But in this case even if $\tilde{X}_{q'}$ is well defined $-\tilde{X}_{q'}$ is not and this is a contradiction.

Similar argument holds for $\mathcal{I}^-$. $\diamond$

Hence we are naturally forced to find a null vector field tending to zero on the null infinities. However note that on $(\tilde{M}, \tilde{g})$ there is a natural cut-off function, namely Ω. Certainly there exists a $k \in \mathbf{N}$ such that the vector field defined by

$$\tilde{X}_0 := \Omega^k \tilde{K}_0$$

is a zero vector field restricted to $\mathcal{I}^\pm$ because of the third property of asymptotic flatness. Note that this new vector field can be extended as zero to i_0 as well: Surrounding i_0 by a small neighbourhood U one can see (since $U \cap (\mathcal{I}^+ \cup \mathcal{I}^-) \neq \emptyset$) $\tilde{X}_0$ is arbitrary small in U.

It is straightforward that $\tilde{X}_0$ is not transversal to $\mathcal{I}^\pm$ in the sense of [1] since it approaches the null-infinities as a tangential field. This would cause some difficulties later on in our construction. Fortunately one can overcome this non-transversality phenomenon by a general method. Due to standard transversality arguments [1] applying a *generic* small perturbation to $\tilde{X}_0$ in a suitable neighbourhood of $\mathcal{I}^\pm$ we can achieve that the perturbed field $\tilde{X}_\varepsilon$ will be transversal to the submanifolds of null-infinities (and even remains zero on them, of course).

3 The Compactification Procedure

Now let N_1 be a smooth four-manifold. We call a subset $C \subset N_1$ *a domain* of N_1 if it is diffeomorphic to a closed four-ball B^4. Let V, $U^\pm \subset N_1$ be domains and $j : \tilde{M} \mapsto N_1$ a smooth embedding satisfying the following conditions:

- $N_1 \setminus j(\tilde{M}) = \mathbf{int}V$;
- There exists a point $p_0 \in N_1$ and domains $U^\pm$ satisfying $U^+ \cap U^- = \{p_0\}$ such that $j(i_0) = p_0$ and $j(\mathcal{I}^\pm) \subset \partial U^\pm$ and $j(\overline{J^\pm(i_0)}) \subset U^\pm$;
- $\partial U^\pm \setminus j(\mathcal{I}^\pm) =: A^\pm \subset \partial V$;
- $j(\tilde{H}) \subset \partial V$;
- $j(\tilde{H}) \cap A^\pm = \emptyset$.

It is not difficult to see that such a j exists due to the topology of an outer region of an asymptotically flat stationary space-time containing a black hole and N_1 has no boundaries. Consider the vector field $Y_1 := j_*\tilde{X}_\varepsilon$. We wish to extend this field into the interior of V.

To be explicit take a coordinate map from N_1 to $\mathbf{R}^4$ under which V maps to the cylinder $V := B^3_1 \times [-1, 1]$ where B^3_r denotes a closed three-ball of radius r originated at the origin. Moreover let $W := D^3_2 \times (-1, 1)$ be another *open* neighbourhood (where D^3_r denotes the open ball respectively).

We assume that $j(\tilde{H})$ is given by the "belt" $\partial B^3_1 \times (-\frac{1}{2}, \frac{1}{2})$ and $A^\pm$ are represented by the top- and bottom-balls $B^3_1 \times \{\pm 1\}$. Note that this picture is consistent with the conditions for j (except that V is not a four-ball but a cylinder).

Note that by Proposition 1

$$Y_1|_{j(\tilde{H})} \in \Gamma(Tj(\tilde{H})).$$

In other words Y_1 has the form (0,0,0,1) on $j(\tilde{H})$. Hence we define a smooth vector field Z in W as follows: Let $q \in D^3_2$ and

- for $t \in [-1, -\frac{1}{2}]$
$$Z_{(q,t)} := (0, 0, 0, f_-(t));$$
where for the smooth cut-off function f_- the following holds:
$$f_-(t) = \begin{cases} 0 & \text{if } t \leq -1 \\ 1 & \text{if } t \geq -\frac{1}{2}; \end{cases}$$
- for $t \in (-\frac{1}{2}, \frac{1}{2})$
$$Z_{(q,t)} := (0, 0, 0, 1);$$
- for $t \in [\frac{1}{2}, 1]$
$$Z_{(q,t)} := (0, 0, 0, f_+(t));$$
where
$$f_+(t) = \begin{cases} 1 & \text{if } t \leq \frac{1}{2} \\ 0 & \text{if } t \geq 1. \end{cases}$$

Clearly Z is a smooth vector field in W and $Z|_{A^\pm} = 0$. Now take a smooth cut-off function $\rho : \mathbf{R}^4 \mapsto \mathbf{R}^+$ satisfying $\rho|_V = 1$ and being zero on the complement of W. Define the extension of Y_1 by

$$\rho Z + (1 - \rho)Y_1.$$

It is obvious that the extended field (also denoted by Y_1) is a smooth vector field on N_1 due to the transversality of the original field to $\mathcal{I}^\pm$ (more precisely it has not been defined in $U^\pm$ yet) and satisfies $Y_1|_{\partial U^\pm} = 0$.

As a final step let us apply a smooth homotopy for N_1 contracting the four-balls $U^\pm$ to the point p_0. In this way we get a smooth compact manifold N_0 without boundaries and, due to the transversality conditions on Y_1, a well-defined smooth vector field Y_0 on it. It is clear that Y_0 has only one (degenerated) isolated singular point, namely p_0. Its index is $+2$ as easy to see due to the fourth property of asymptotic flatness.

Theorem. *N_0 is homeomorphic to the four-sphere S^4.*

Proof. First it is not difficult to see that N_0 is simply connected. Note that N_0 is a union of an outer region of the original stationary, asymptotically flat space-time M and a solid torus-like space T homeomorphic to $B^3 \times S^1$ with one B^3 pinched into a point (namely which corresponds to p_0). Choosing p_0 as a base point consider the loop l in N_0 representing the generator of the fundamental group $\pi_1(T) = \mathbf{Z}$. Clearly this loop can be deformed continuously into $M \cup \{p_0\}$. But referring again to the theorem of Chruściel and Wald [3] the deformed loop l is homotopically trivial since M is simply connected. It is obvious that every other loops in N_0 are contractible proving $\pi_1(N_0) = 0$.

Secondly, we have constructed a smooth vector field Y_0 on N_0 having one isolated singular point p_0 with index $+2$. Taking into account that N_0 is a smooth, compact manifold without boundaries this means that

$$\chi(N_0) = 2$$

according to the classical Poincaré–Hopf Theorem [2] [10].

The Euler characteristic of a simply connected compact four-manifold S always has the form $\chi(S) = 2 + b_2$ where b_2 denotes the rank of its intersection form (or second Betti number). In our case this rank is zero, hence our form is of even type. By the uniqueness of the trivial zero rank matrix representing this even form and referring to the deep theorem of Freedman [4] which gives a full classification of simply connected compact topological four-manifolds in terms of their intersection matrices we deduce that N_0 is homeomorphic to the four-sphere since its even intersection form is given by the same zero-matrix. $\diamond$

Corollary. The uniqueness of N_0 implies the uniqueness of the original space-time manifold (more precisely one connected piece of its outer region) M since we simply have to remove a singular solid torus T from N_0 and this can be done in a unique way being N_0 simply connected. Hence, taking into account the explicit example of the Kerr-solution, this outer region is always homeomorphic to $S^2 \times \mathbf{R}^2$. $\diamond$

4 Conclusion

We have proved that topological uniqueness holds for space-times carrying a stationary black hole. Note that we could prove only a topological equivalence although our constructed manifold N_0 carries a smooth structure, too. It would be interesting to know if this smooth structure was identical to the standard one in light of the unsolved problem of the four dimensional Poincaré-conjecture in the smooth category.

From the physical point of view this uniqueness is important if we are interested in problems concerning the whole structure of such space-time manifolds.

For example one may deal with the description of the vacuum structure of Yang-Mills fields on the background of a gravitational configuration containing a black hole or some other singularity (taking the singularity theorems into consideration this question is very general and natural). One would expect that a black hole may have a strong influence on these structures and using our theorem presented here we can study this problem effectively in our following paper.

Moreover we hope that our construction works for non-stationary i.e. higher genus black holes as well giving an insight into the structure of such more general space-time manifolds.

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6 References

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