

Exotica and the status of the strong cosmic censor conjecture in four dimensions

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January 15, 2019

Abstract

An immense class of physical counterexamples to the four dimensional strong cosmic censor conjecture—in its usual broad formulation—is exhibited. More precisely, out of any closed and simply connected 4-manifold an open Ricci-flat Lorentzian 4-manifold is constructed which is not globally hyperbolic and no perturbation of it, in any sense, can be globally hyperbolic. This very stable non-global-hyperbolicity is the consequence of our open spaces having a “creased end” i.e., an end diffeomorphic to an exotic \mathbb{R}^4 . Open manifolds having an end like this is a typical phenomenon in four dimensions.

The construction is based on a collection of results of Gompf and Taubes on exotic and self-dual spaces, respectively, as well as applying Penrose’ non-linear graviton construction (i.e., twistor theory) to solve the Riemannian Einstein’s equation. These solutions then are converted into stably non-globally-hyperbolic Lorentzian vacuum solutions. It follows that the *plethora* of vacuum solutions we found cannot be obtained via the initial value formulation of the Einstein’s equation because they are “too long” in a certain sense (explained in the text). This different (i.e., not based on the initial value formulation but twistorial) technical background might partially explain why the existence of vacuum solutions of this kind has not been realized so far in spite of the fact that, apparently, their superabundance compared to the well-known globally hyperbolic vacuum solutions is overwhelming.

AMS Classification: Primary: 83C75, Secondary: 57N13, 53C28

Keywords: *Strong cosmic censor conjecture; Exotic \mathbb{R}^4 ; Twistors*

1 Introduction

Certainly one of the deepest open problems of contemporary classical general relativity is the validity or invalidity of the *strong cosmic censor conjecture* [24]. This is not only a technical conjecture of a particular branch of current theoretical physics: it deals with the very foundations of our rational description of Nature. Indeed, Penrose’ original aim in the 1960-70’s with formulating this conjecture

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was to protect causality in generic gravitational situations. We have the strong conviction that in the *classical* physical world at least, every physical event (possibly except the initial Big Bang) has a physical cause which is another and preceding physical event. Since mathematically speaking space-times having this property are called *globally hyperbolic*, our requirement can be formulated roughly as follows (cf. e.g. [30, p. 304]):

SCCC. *A generic (i.e., stable), physically relevant (i.e., obeying some energy condition) space-time is globally hyperbolic.*

We do not make an attempt here to survey the vast physical and mathematical literature triggered by the **SCCC** instead we refer to surveys [17, 23, 21]. Rather we may summarize the current situation as follows. During the course of time the originally single **SCCC** has fallen apart into several mathematical or physical versions, variants, formulations. For example there exists a generally working, mathematically meaningful but from a physical viewpoint rather weak version formulated in [30, p. 305] and proved in [5]. In another approach to the **SCCC** based on initial value formulation [30, Chapter 10], on the one hand, there are certain specific classes of space-times in which the **SCCC** allows a mathematically rigorous as well as physically contentful formulation whose validity can be established [23]; on the other hand counterexamples to the **SCCC** in this formulation also regularly appear in the literature however they are apparently too special, not “generic”. In spite of these sporadic counterexamples the overall confidence in the physicist community is that an appropriate form of the **SCCC** must hold true hence causality is saved.

However here we claim to exhibit an *abundance of generic counterexamples to the **SCCC*** whose first agent was announced in [6]. Informally speaking, the content of our main results here, namely Theorems 3.1 and 4.1 can be summarized as follows:

SCCC. *From every connected and simply connected closed (i.e., compact without boundary) smooth 4-manifold M one can construct an open (i.e., non-compact without boundary) smooth 4-manifold X_M and a smooth Ricci-flat Lorentzian metric g on it such that (X_M, g) is not globally hyperbolic. Moreover, any “sufficiently large” (in an appropriate topological sense) physical perturbation (X'_M, g') of this space cannot be globally hyperbolic, too.*

This very stable non-global-hyperbolicity follows because X_M as a smooth 4-manifold contains a “creased end” (see Figure 1), a typical four dimensional phenomenon.

What is then the resolution of the apparent contradiction between the well-known affirmative solutions and our negative result **SCCC** here? In this short introduction we just would like to draw attention to a historical aspect of the answer and try to offer more technical comments at the end of the paper. The Einstein equation as a non-linear partial differential equation on a 4-manifold is a quite transcendental object in the sense that there is yet no systematic way to solve it. So far the initial value formulation is the only known method which can provide sufficiently many solutions in various situations hence its investigation by Leray, Choquet-Bruhat, Lichnerowicz, Geroch in the 1950-60’s and by many others later cannot be overestimated. The initial value formulation starts off by considering an initial data set (S, h, k) with S being a smooth *three dimensional* manifold and h, k certain tensor fields on it satisfying (simpler) constraint equations; and out of these data it produces a solution (M, g) of the Lorentzian Einstein equation. An apparently innocuous technical by-product of the initial value formulation is that it fixes not only the metric but the smooth structure of the resulting space-time, too: the underlying *four dimensional* manifold M is always *diffeomorphic* to the product $S \times \mathbb{R}$ (with their unique smooth structures) by the celebrated Bernal–Sánchez theorem [2]. This technical nuance seemed to be not a problem at all for the physicist community by the time the initial value formulation came to existence.

Side-by-side with but quite isolated from these investigations mathematicians also made efforts to

understand the structure of smooth manifolds and they came up with unexpected issues. Since the early works of Whitney, Milnor in the 1950-60's followed by Casson, Kirby and others, it had been gradually realized that in higher dimensions topology and smoothness do not determine each other and their interaction gets particularly complicated in four dimensions. By the early 1980's it was recognized that essentially no known compact smoothable topological 4-manifold carries exactly one smooth structure; in fact in most of the well-understood cases they admit not only more than one but countably infinitely many different ones [10]. In the case of non-compact (relevant for physics) topological 4-manifolds there is even no obstruction against smooth structure and they typically accommodate an uncountably family of them [9]. A characteristic feature of these “exotic” smooth structures—unforeseeable in the 1950's—is that they are “creased” i.e., do not arise as smooth products of lower dimensional smooth structures. The striking discovery of exotic (or fake) \mathbb{R}^4 's (i.e., smooth 4-manifolds which are homeomorphic but not diffeomorphic to the usual \mathbb{R}^4) by Donaldson, Freedman, Taubes in the 1980's and investigated further by Akbulut, Bizaca, Gompf and others during the 1990's and 2000's is probably the most dramatic example of the general situation completely absent in other dimensions.

From this perspective the cases for which the validity of the **SCCC** has been verified so far [17, 23] seem to be atypical hence essentially negligible ones; partly because affirmative answers have been obtained by the initial value formulation hence the underlying space-times in these affirmative solutions are never “creased”. On the contrary, our counterexample factory $\overline{\text{SCCC}}$ rests on typical features of smooth 4-manifolds. The only way to refute the general position adopted here when dealing with the **SCCC** is if one could somehow argue that general smooth 4-manifolds are too “exotic”, “fake” or “weird” from the aspect of physical general relativity. However from the physical viewpoint if the “summing over everything” approach to quantum gravity is correct then very general unconventional but still physical space-times should be considered, too [1, 4]; from the mathematical perspective non-linear partial differential equations like Einstein's equation are typically also solvable. Consequently both physically and mathematically speaking the true properties of general relativity cannot be revealed by understanding it only on simple atypical manifolds; the division of smooth 4-manifolds into “usual” and “unusual” ones can be justified only by conventionalism i.e., one has to evoke historical (and technical) arguments to pick up “usual” spaces from the bottomless sea of smooth 4-manifolds and abandon others. In fact our general position supporting $\overline{\text{SCCC}}$ fits well with the *four dimensional* (i.e., the original Einstein–Hilbert) *Lagrangian* formulation while the initial value formulation supporting **SCCC** rests on a *three dimensional Hamiltonian* reformulation of general relativity. Therefore, to summarize, in our opinion the choice between **SCCC** or $\overline{\text{SCCC}}$ reflects one's commitment toward one of these formulations of general relativity.

This paper, considered as a substantial generalization and technical improvement of [6], is organized as follows. Section 2 offers very general definitions of what one would expect to mean by a perturbation of a space-time and a counterexample to the **SCCC**. Section 3 contains the construction of complete Ricci-flat *Riemannian* manifolds based on twistor theory. Section 4 describes how to convert these solutions into stably non-globally-hyperbolic Ricci-flat *Lorentzian* manifolds i.e., counterexamples. Section 5 contains some concluding remarks and an outlook. Section 6 is an Appendix with a summary of the theory of Lebesgue integration in algebraic function fields, a tool has been used in Section 3.

Our notational convention throughout the text is that \mathbb{R}^4 will denote the four dimensional real vector space equipped with its standard differentiable manifold structure whilst R^4 or R_t^4 will denote various exotic (or fake) variants. The notation “ \cong ” will always mean “diffeomorphic to” whilst homeomorphism always will be spelled out as “homeomorphic to”. Finally we note that all set theoretical or topological operations (i.e., \subseteq , \cap , \cup , taking open or closed subsets, closures, complements, etc.) will be taken in a manifold M with its well-defined standard manifold topology throughout the text. In particular for \mathbb{R}^4 or the R_t^4 's this topology is the unique underlying manifold topology.

2 Definition of a counterexample

In agreement with the common belief in the physicist and mathematician community, formulating the strong cosmic censor conjecture in a mathematically rigorous way is obstructed by lacking an overall satisfactory concept of “genericity”. Consequently the main difficulty to find a “generic counterexample” to the **SCCC** lies not in its actual finding (indeed, most of the well-known basic solutions of Einstein’s equation provide violations of it) but rather in proving that the particular counterexample is “generic”. In this section we outnavigate this problem by mathematically formulating the concept of a certain counterexample which is logically stronger than a “generic counterexample” to the **SCCC**. Then we search for a counterexample of this kind making use of uncountably many large exotic \mathbb{R}^4 ’s.

A standard reference here is [30, Chapters 8,10]. By a *space-time* we mean a connected, four dimensional, smooth, time-oriented Lorentzian manifold without boundary. By a *(continuous) Lorentzian manifold* we mean the same thing except that the metric is allowed to be a continuous tensor field only.

Definition 2.1. *Let (S, h, k) be an initial data set for Einstein’s equation with (S, h) a connected complete Riemannian 3-manifold and with a fundamental matter represented by a stress-energy tensor T obeying the dominant energy condition. Let $(D(S), g|_{D(S)})$ be the unique maximal Cauchy development of this initial data set. Let (M, g) be a further maximal extension of $(D(S), g|_{D(S)})$ as a (continuous) Lorentzian manifold if exists. That is, $(D(S), g|_{D(S)}) \subseteq (M, g)$ is a (continuous) isometric embedding which is proper if $(D(S), g|_{D(S)})$ is still extendible and (M, g) does not admit any further proper isometric embedding. (If the maximal Cauchy development is inextendible then put simply $(M, g) := (D(S), g|_{D(S)})$ for definiteness.)*

The (continuous) Lorentzian manifold (M', g') is a perturbation of (M, g) relative to (S, h, k) if

(i) *M' has the structure*

$$M' := \text{the connected component of } M \setminus H \text{ containing } S$$

where, for a connected open subset U of M containing the initial surface i.e., $S \subset U \subseteq M$, the subset H is closed and satisfies $\emptyset \subseteq H \subseteq \partial U = \bar{U} \setminus U$ i.e., is a closed subset in the boundary of U (consequently $M' \subseteq M$ is open hence inherits a differentiable manifold structure);

(ii) *g' is a solution of Einstein’s equation at least in a neighbourhood of the initial surface $S \subset M'$ with a fundamental matter represented by a stress-energy tensor T' obeying the dominant energy condition at least in a neighbourhood of $S \subset M'$;*

(iii) *(M', g') does not admit further proper isometric embeddings and $(S, h') \subset (M', g')$ with $h' := g'|_S$ is a spacelike complete sub-3-manifold.*

Remark. 1. It is crucial that in the original spirit of relativity theory we consider metric perturbations of the *four* dimensional space-time M (whilst keeping its underlying smooth structure fixed)—and not those of a *three* dimensional initial data set. This natural class of perturbations is therefore immense: it contains all connected manifolds M' satisfying

$$S \subset M' \subseteq M$$

i.e., contain the initial surface but perhaps being topologically different from the original manifold. The perturbed metric is a physically relevant solution of Einstein’s equation at least in the vicinity of $S \subset M'$ such that (M', g') is inextendible and $(S, h') \subset (M', g')$ is still spacelike and complete. In other words

these perturbations are physical solutions allowed to blow up along certain closed “boundary subsets” $\emptyset \subsetneq H \subset M$; the notation H for these subsets indicates that among them the (closure of the) Cauchy horizon $H(S)$ of (S, h, k) may also appear. Beyond the non-singular perturbations satisfying $H = \emptyset$ a prototypical example with $H \neq \emptyset$ is the physical perturbation (M', g') of the (maximally extended) undercharged Reissner–Nordström space-time (M, g) by taking into account the full backreaction of a pointlike particle or any classical field put onto the originally pure electro-vacuum space-time (“mass inflation”). In this case the singularity subset H is expected to coincide with the (closure of the) full inner event horizon of the Reissner–Nordström black hole which is the Cauchy horizon for the standard initial data set inside the maximally extended space-time [24]. A similar perturbation of the Kerr–Newman space-time is another example with $H \neq \emptyset$.

2. Accordingly, note that in the above definition of perturbation none of the terms “generic” or “small” have been used. This indicates that if such types of perturbations can be somehow specified then one should be able to recognize them among the very general but still physical perturbations of a space-time as formulated in Definition 2.1.

Now we are in a position to formulate in a mathematically precise way what we mean by a “robust counterexample” to the **SCCC** as formulated roughly in the Introduction.

Definition 2.2. *Let (S, h, k) be an initial data set for Einstein’s equation with (S, h) a connected complete Riemannian 3-manifold and with a fundamental matter represented by a stress-energy tensor T obeying the dominant energy condition. Assume that the maximal Cauchy development of this initial data set is extendible i.e., admits a (continuous) isometric embedding as a proper open submanifold into an inextendible (continuous) Lorentzian manifold (M, g) .*

*Then (M, g) is a robust counterexample to the **SCCC** if it is very stably non-globally hyperbolic i.e., all of its perturbations (M', g') relative to (S, h, k) are not globally hyperbolic.*

Remark. 1. Concerning its logical status it is reasonable to consider this as a *generic* counterexample because the perturbation class of Definition 2.1 is expected to contain all “generic perturbations” whatever they are. Consequently in Definition 2.2 we are dealing with a stronger statement than the logical negation of the affirmative sentence in **SCCC**.

2. The trivial perturbation i.e., the extension (M, g) itself in Definition 2.2 cannot be globally hyperbolic as observed already in [5, Remark after Theorem 2.1].

3 Riemannian considerations

Strongly influenced by [2, 3], in order to attack the **SCCC** we begin now our excursion into the weird world of the four dimensional exotic *ménagerie* (or rather *plethora*). A standard reference here is [10, Chapter 9]. Our aim in this section is to prove the following

Theorem 3.1. *Let M be a connected and simply connected, closed (i.e., compact without boundary) smooth 4-manifold. Then out of this manifold one can construct a Riemannian 4-manifold*

$$(X_M, g_0)$$

with the following properties.

The metric g_0 is a smooth Ricci-flat complete Riemannian metric on X_M . Furthermore, the space X_M is an open (i.e., non-compact without boundary) oriented smooth 4-manifold with a single so-called creased end. Here “creased” means that if $S \subset X_M$ is an arbitrary smoothly embedded sub-3-manifold then $X_M \not\cong S \times \mathbb{R}$ i.e., X_M does not split smoothly into the product of a 3-manifold and \mathbb{R} (with their unique smooth structures).

The proof of this theorem is based on a collection of strong and surprising results of Gompf [7, 8, 9] and Taubes [28, 29] and is rather involved. Therefore our plan to prove it is as follows: first we recall these results in precise forms in order to have a solid starting point and then through a sequence of technical lemmata we will arrive at the proof of Theorem 3.1 at the end of this section.

It is well-known that the Fubini–Study metric on the complex projective space $\mathbb{C}P^2$ with orientation inherited from its complex structure is self-dual (or half-conformally flat) i.e., the anti-self-dual part of its Weyl tensor vanishes; consequently the oppositely oriented complex projective plane $\overline{\mathbb{C}P^2}$ is anti-self-dual. A powerful generalization of this latter classical fact is Taubes’ construction of an abundance of anti-self-dual 4-manifolds; firstly we exhibit his result but now in an orientation-reversed form:

Theorem 3.2. (Taubes [29, Theorem 1.1]) *Let M be a connected, compact, oriented smooth 4-manifold. Let $\mathbb{C}P^2$ denote the complex projective plane with its usual orientation and let $\#$ denote the operation of taking the connected sum of manifolds. Then there exists a natural number $k_M \geq 0$ such that for all $k \geq k_M$ the modified compact manifold*

$$M \# \underbrace{\mathbb{C}P^2 \# \dots \# \mathbb{C}P^2}_k$$

admits a self-dual Riemannian metric. \diamond

Secondly we evoke a result which is a sort of summary of what is so special in four dimensions (i.e., absent in any other ones): we recall a special class of large exotic (or fake) \mathbb{R}^4 ’s whose properties we will need here are summarized as follows:

Theorem 3.3. (Gompf–Taubes, cf. [10, Lemma 9.4.2, Addendum 9.4.4 and Theorem 9.4.10]) *There exists a pair (R^4, K) consisting of a differentiable 4-manifold R^4 homeomorphic but not diffeomorphic to the standard \mathbb{R}^4 and a compact oriented smooth 4-manifold $K \subset R^4$ such that*

- (i) R^4 cannot be smoothly embedded into the standard \mathbb{R}^4 i.e., $R^4 \not\subset \mathbb{R}^4$ but it can be smoothly embedded as a proper open subset into the complex projective plane i.e., $R^4 \subsetneq \mathbb{C}P^2$;
- (ii) Take a homeomorphism $f : \mathbb{R}^4 \rightarrow R^4$, let $0 \in B_t^4 \subset \mathbb{R}^4$ be the standard open 4-ball of radius $t \in \mathbb{R}^+$ centered at the origin and put $R_t^4 := f(B_t^4)$ and $R_{+\infty}^4 := R^4$. Then

$$\{R_t^4 \mid r \leq t \leq +\infty \text{ such that } 0 < r < +\infty \text{ satisfies } K \subset R_r^4\}$$

is an uncountable family of nondiffeomorphic exotic \mathbb{R}^4 ’s none of them admitting a smooth embedding into \mathbb{R}^4 i.e., $R_t^4 \not\subset \mathbb{R}^4$ for all $r \leq t \leq +\infty$.

This class of manifolds is called the Gompf–Taubes large radial family. \diamond

Remark. 1. The fact that any member R_t^4 in this family is not diffeomorphic to \mathbb{R}^4 implies the counter-intuitive phenomenon that $R_t^4 \not\cong W \times \mathbb{R}$ i.e., R_t^4 does not admit any *smooth* splitting into a 3-manifold W and \mathbb{R} (with their unique smooth structures) in spite of the fact that such *continuous* splittings obviously exist. Indeed, from the contractibility of R_t^4 we can see that W must be a contractible open 3-manifold (a so-called *Whitehead continuum*) however, by an early result of McMillen [18], spaces of this kind always satisfy $W \times \mathbb{R} \cong \mathbb{R}^4$ i.e., their product with a line is always diffeomorphic to the standard \mathbb{R}^4 . We will call this property of (any) exotic \mathbb{R}^4 below as “creased”. The existence of many non-homeomorphic Whitehead continua have interesting consequences in the initial value formulation, too cf. e.g. [19].

2. From Theorem 3.3 we deduce that for all $r < t < +\infty$ there is a sequence of smooth proper embeddings

$$R_r^4 \subsetneq R_t^4 \subsetneq R_{+\infty}^4 = R^4 \subsetneq \mathbb{C}P^2$$

which are very wild in the following sense. The complement $\mathbb{C}P^2 \setminus R^4$ of the largest member R^4 of this family is homeomorphic to S^2 regarded as an only continuously embedded projective line in $\mathbb{C}P^2$; therefore, if not otherwise stated later, let us denote this complement as $S^2 := \mathbb{C}P^2 \setminus R^4 \subset \mathbb{C}P^2$ in order to distinguish it from the ordinary projective lines $\mathbb{C}P^1 \subset \mathbb{C}P^2$. If $\mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1$ is any holomorphic decomposition then $R^4 \cap \mathbb{C}P^1 \neq \emptyset$ (because otherwise $R^4 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$ would hold, a contradiction) as well as $S^2 \cap \mathbb{C}P^1 \neq \emptyset$ (because otherwise $H_2(R^4; \mathbb{Z}) \cong \mathbb{Z}$ would hold since $\mathbb{C}P^1 \subset \mathbb{C}P^2$ represents a generator of $H_2(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}$). Hence any ordinary projective line in $\mathbb{C}P^2$ is intersected by both R^4 and its complementum S^2 in $\mathbb{C}P^2$. This demonstrates that the members of the large radial family “live somewhere between” \mathbb{C}^2 and its projective closure $\mathbb{C}P^2$. However a more precise identification or location of them is a difficult task because these large exotic \mathbb{R}^4 ’s—although being honest differentiable 4-manifolds—are very transcendental objects [10, p. 366]: they require infinitely many 3-handles in any handle decomposition (like any other known large exotic \mathbb{R}^4) and there is presently¹ no clue as how one might draw explicit handle diagrams of them (even after removing their 3-handles). We note that the structure of small exotic \mathbb{R}^4 ’s i.e., which admit smooth embeddings into \mathbb{R}^4 , is better understood, cf. [10, Chapter 9].

Our last ingredient is the following *ménagerie* result of Gompf.

Theorem 3.4. (Gompf [9, Theorem 2.1]) *Let X be a connected (possibly non-compact, possibly with boundary) topological 4-manifold and let $X' := X \setminus \{ \text{one point of } X \}$ be the punctured manifold with a single point removed. Then the non-compact space X' admits noncountable many (with the cardinality of the continuum in ZFC set theory) pairwise non-diffeomorphic smooth structures. \diamond*

Remark. If for instance X is a connected compact smooth 4-manifold in Theorem 3.4 then Gompf’s construction goes as follows: take the maximal large R^4 from Theorem 3.3 and put $X' := X \# R^4$. This smooth 4-manifold is obviously homeomorphic to the punctured X' ; more generally, $X'_t := X \# R_t^4$ will produce uncountable many smooth structures on the unique topological 4-manifold underlying X'_t .

For our purposes and to begin with, we combine Theorems 3.2, 3.3 and 3.4 together as follows.

Lemma 3.1. *Out of any connected, closed (i.e., compact without boundary) oriented smooth 4-manifold M one can construct a connected, open (i.e., non-compact without boundary) oriented smooth Riemannian 4-manifold (X_M, g_1) which is self-dual but incomplete in general.*

Moreover X_M has a single creased end where “creased” means that if $S \subset X_M$ is any smoothly embedded sub-3-manifold then $X_M \not\cong S \times \mathbb{R}$ i.e., X_M does not split smoothly into the product of any smooth 3-manifold S and \mathbb{R} (with their unique smooth structures).

Proof. Pick any connected, oriented, closed, smooth 4-manifold M . Referring to Theorem 3.2 let $k := \max(1, k_M) \in \mathbb{N}$ be a positive integer, put

$$\hat{X}_M := M \# \underbrace{\mathbb{C}P^2 \# \dots \# \mathbb{C}P^2}_k$$

¹More precisely in the year 1999, cf. [10].

and let \hat{g}_1 be a self-dual metric on it. Then (\hat{X}_M, \hat{g}_1) is a compact self-dual manifold. If $S^2 = \mathbb{C}P^2 \setminus R^4$ denotes the complement of $R^4 \subset \mathbb{C}P^2$ as in the *Remark* after Theorem 3.3 and $K \subset R^4$ is the compact subset as in part (ii) of Theorem 3.3 then put

$$X_M := M \# \underbrace{\mathbb{C}P^2 \# \dots \# \mathbb{C}P^2}_{k-1} \#_K (\mathbb{C}P^2 \setminus S^2) \cong M \# \underbrace{\mathbb{C}P^2 \# \dots \# \mathbb{C}P^2}_{k-1} \#_K R^4 \quad (1)$$

where the operation $\#_K$ means that the attaching point $y_0 \in R^4$ used to glue R^4 with $M \# \mathbb{C}P^2 \# \dots \# \mathbb{C}P^2$ satisfies $y_0 \in K \subset R^4$. The result is a connected, open 4-manifold (see Figure 1). From the proper smooth embedding $X_M \cong \hat{X}_M \setminus S^2 \subsetneq \hat{X}_M$ there exists a self-dual Riemannian metric $g_1 := \hat{g}_1|_{X_M}$ on X_M which is however in general non-complete.

Although being non-compact, if $S \subset X_M$ is any smoothly embedded sub-3-manifold then obviously $X_M \not\cong S \times \mathbb{R}$ i.e., X_M does not split *smoothly* into the product of a smooth 3-manifold S and \mathbb{R} (with their standard smooth structures) due to its exotic \mathbb{R}^4 -end i.e., the R^4 -factor present in its decomposition (1) above. \diamond

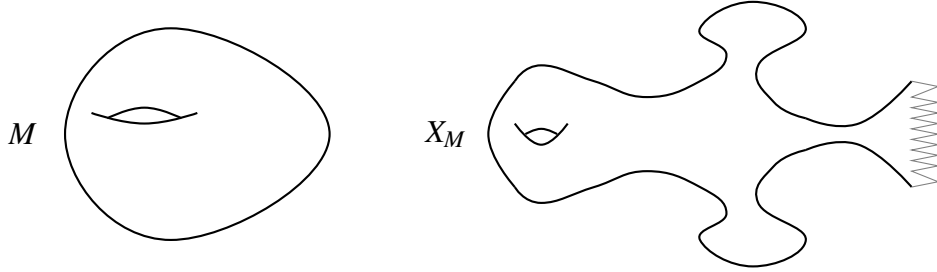


Figure 1. Construction of X_M out of M . The creased end of X_M is drawn by a gray zig-zag.

Next we improve the incomplete self-dual space (X_M, g_1) of Lemma 3.1 to a complete Ricci-flat space (X_M, g_0) by conformally rescaling g_1 with a suitable positive smooth function $\varphi : X_M \rightarrow \mathbb{R}^+$ which is a “multi-task” function in the sense that it kills both the scalar curvature and the traceless Ricci tensor of g_1 moreover blows up sufficiently fast along the exotic \mathbb{R}^4 -end of X_M to make the rescaled metric g_0 complete. A classical example serves as a motivation. Put $(\hat{X}_M, \hat{g}_1) := (S^4, \hat{g}_1)$ that is, the 4-sphere $S^4 \subset \mathbb{R}^5$ equipped with its standard orientation and round metric \hat{g}_1 inherited from the embedding. It is well-known that (S^4, \hat{g}_1) is self-dual and Einstein with non-zero cosmological constant i.e., not Ricci-flat. Put $X_M := S^4 \setminus \{\infty\} \cong \mathbb{R}^4$ to be the standard \mathbb{R}^4 ; then $g_1 := \hat{g}_1|_{\mathbb{R}^4}$ is an incomplete self-dual metric on \mathbb{R}^4 but picking $\varphi(r) := \frac{1}{1+r^2}$ where r is the radial coordinate on \mathbb{R}^4 from its origin i.e., φ vanishes exactly in $\{\infty\} \in S^4$, then $g_0 := \varphi^{-2} \cdot g_1$ is nothing but the standard flat metric on \mathbb{R}^4 which is of course complete and Ricci-flat. Hence $(X_M, g_0) := (\mathbb{R}^4, \varphi^{-2}g_1)$, the conformal rescaling of $(X_M, g_1) = (\mathbb{R}^4, g_1)$, is the desired complete Ricci-flat space in this simple case. In our much more general situation we shall use Penrose’ non-linear graviton construction (i.e., twistor theory) [20] to find conformal rescalings.

Remark. Let us first recall Penrose’ twistor method to solve the Riemannian vacuum Einstein equation (for a very clear introduction cf. [14, 15]). Consider the projectivized negative chiral spinor bundle $P(\hat{\Sigma}^-)$ over for instance the compact self-dual space (\hat{X}_M, \hat{g}_1) in Lemma 3.1; note that this bundle exists even if \hat{X}_M is not spin. Since in 4 dimensions $\hat{\Sigma}^-$, if exists, is a rank 2 complex vector bundle over \hat{X}_M , its projectivization $P(\hat{\Sigma}^-)$ is the total space of a smooth $\mathbb{C}P^1$ -fibration $\hat{p} : P(\hat{\Sigma}^-) \rightarrow \hat{X}_M$. The Levi-Civita connection of any metric on \hat{X}_M can be used to furnish the real 6-manifold $P(\hat{\Sigma}^-)$ with a canonical

almost complex structure; the fundamental observation of twistor theory is that this almost complex structure now is integrable because \hat{g}_1 is self-dual. The resulting complex 3-manifold $\hat{Z} \cong P(\hat{\Sigma}^-)$ is called the *twistor space* while the smooth fibration $\hat{p} : \hat{Z} \rightarrow \hat{X}_M$ the *twistor fibration* of (\hat{X}_M, \hat{g}_1) . The most important property of a twistor space of this kind is that its twistor fibers $\hat{p}^{-1}(x) \subset \hat{Z}$ for all $x \in \hat{X}_M$ fit into a locally complete complex 4-parameter family $\hat{X}_M^{\mathbb{C}}$ of projective lines $Y \subset \hat{Z}$ each with normal bundle $H \oplus H$, with H being the dual of the tautological line bundle over $Y \cong \mathbb{C}P^1$. Moreover, there exists a real structure $\hat{\tau} : \hat{Z} \rightarrow \hat{Z}$ defined by taking the antipodal maps along the twistor fibers $\mathbb{C}P^1 \cong \hat{p}^{-1}(x) \subset \hat{Z}$ for all $x \in \hat{X}_M \subset \hat{X}_M^{\mathbb{C}}$ which are therefore called “real lines” among all the lines in $\hat{X}_M^{\mathbb{C}}$. In other words, \hat{Z} is fibered exactly by the real lines $Y_x := \hat{p}^{-1}(x)$ for all $x \in \hat{X}_M$. Hence the real 4 dimensional self-dual geometry has been encoded into a 3 dimensional complex analytic structure in the sense that one can recover (\hat{X}_M, \hat{g}_1) just from \hat{Z} up to conformal equivalence.

One can go further and raise the question how to recover precisely (\hat{X}_M, \hat{g}_1) itself from its conformal class, or more interestingly to us: how to get a Ricci-flat Riemannian 4-manifold (X_M, g_0) i.e., a solution of the (self-dual) Riemannian vacuum Einstein equation. Not surprisingly, to get the latter stronger structure, one has to specify further data on the twistor space. A fundamental result of twistor theory is that a solution of the 4 dimensional (self-dual) Riemannian vacuum Einstein equation is equivalent to the following set of data (cf. [14, 15]):

- * A complex 3-manifold Z , the total space of a holomorphic fibration $\pi : Z \rightarrow \mathbb{C}P^1$;
- * A complex 4-parameter family of holomorphically embedded complex projective lines $Y \subset Z$, each with normal bundle $H \oplus H$ (here H is the dual of the tautological bundle i.e., the unique holomorphic line bundle on $Y \cong \mathbb{C}P^1$ with $\langle c_1(H), [Y] \rangle = 1$);
- * A non-vanishing holomorphic section s of $K_Z \otimes \pi^*H^4$ (here K_Z is the canonical bundle of Z);
- * A real structure $\tau : Z \rightarrow Z$ such that Z is fibered by the τ -invariant elements $Y \subset Z$ of the family (these are called “real lines”) and τ coincides with the antipodal map $u \mapsto -\bar{u}^{-1}$ upon restricting to the real lines; moreover π and s are compatible with τ .

These data allow one to construct a Ricci-flat and self-dual (i.e., the Ricci tensor and the anti-self-dual part of the Weyl tensor vanishes) solution (X_M, g_0) of the *Riemannian Einstein’s vacuum equation* with vanishing cosmological constant as follows. The holomorphic lines $Y \subset Z$ form a locally complete family and fit together into a complex 4-manifold $X_M^{\mathbb{C}}$. This space carries a natural complex conformal structure by declaring two nearby points $y_1, y_2 \in X_M^{\mathbb{C}}$ to be null-separated if the corresponding lines intersect i.e., $Y_1 \cap Y_2 \neq \emptyset$ in Z . Infinitesimally this means that on every tangent space $T_y X_M^{\mathbb{C}} = \mathbb{C}^4$ a null cone is specified. Restricting this to the real lines singled out by τ and parameterized by an embedded real 4-manifold $X_M \subset X_M^{\mathbb{C}}$ we obtain the real conformal class $[g_0]$ of a Riemannian metric on X_M . The isomorphism $s : K_Z \cong \pi^*H^{-4}$ is essentially uniquely fixed by its compatibility with τ and π and gives rise to a volume form on X_M this way fixing the metric g_0 in the conformal class. Given the conformal class, it is already meaningful to talk about the projectivized negative chiral spinor bundle $P(\Sigma^-)$ over X_M with its induced orientation from the twistor space and Z can be identified with the total space of $P(\Sigma^-)$. This way we obtain a smooth twistor fibration $p : Z \rightarrow X_M$ whose fibers are $\mathbb{C}P^1$ ’s hence $\pi : Z \rightarrow \mathbb{C}P^1$ can be regarded as a parallel translation along this bundle over X_M with respect to a flat connection which is nothing but the induced negative spin connection of g_0 on Σ^- . Knowing the decomposition of the Riemannian curvature into irreducible components over an oriented Riemannian 4-manifold [25], this partial flatness of $P(\Sigma^-)$ implies that g_0 is Ricci-flat and self-dual. Finally note that, compared to the bare twistor space \hat{Z} of a self-dual manifold (\hat{X}_M, \hat{g}_1) above, the essential new

requirement for constructing a self-dual *Ricci-flat* space (X_M, g_0) is the existence of a holomorphic map π from the twistor space Z into $\mathbb{C}P^1$. We conclude our summary of the non-linear graviton construction by referring to [14, 15] for further details.

In the case of our situation set up in Lemma 3.1 twistor theory works as follows. Consider the compact self-dual space (\hat{X}_M, \hat{g}_1) from Lemma 3.1, take its twistor fibration $\hat{p} : \hat{Z} \rightarrow \hat{X}_M$ and let

$$p : Z \longrightarrow X_M$$

be its restriction induced by the smooth embedding $X_M \subsetneq \hat{X}_M$ i.e., $Z := \hat{Z}|_{X_M}$ and $p := \hat{p}|_{X_M}$. Then Z is a non-compact complex 3-manifold already obviously possessing all the required twistor data except the existence of a holomorphic mapping $\pi : Z \rightarrow \mathbb{C}P^1$.

Lemma 3.2. *Consider (X_M, g_1) as in Lemma 3.1 with its twistor fibration $p : Z \rightarrow X_M$ constructed above. If $\pi_1(X_M) = 1$ (i.e., the original compact manifold satisfies $\pi_1(M) = 1$) then there exists a holomorphic mapping $\pi : Z \rightarrow \mathbb{C}P^1$.*

Proof. Let $x_0 \in X_M$ be a fixed point belonging to the exotic \mathbb{R}^4 -factor R^4 of X_M in its decomposition (1). Our aim is to construct a holomorphic map

$$\pi : Z \longrightarrow p^{-1}(x_0) \cong \mathbb{C}P^1 \tag{2}$$

that we carry out in three steps.

Firstly over an exotic $R^4 \subset \mathbb{C}P^2$ we construct by classical means holomorphic maps parameterized by ideal points $x \in \mathbb{C}P^2 \setminus R^4$. It is known that $\hat{Z}(\mathbb{C}P^2) \cong P(T^*\mathbb{C}P^2)$ i.e., the twistor space of the complex projective space can be identified with its projective cotangent bundle. Consequently $\hat{Z}(\mathbb{C}P^2)$ can be described as the flag manifold $F_{12}(\mathbb{C}^3)$ consisting of pairs $(\mathfrak{l}, \mathfrak{p})$ where $0 \in \mathfrak{l} \subset \mathbb{C}^3$ is a line and $\mathfrak{l} \subset \mathfrak{p} \subset \mathbb{C}^3$ is a plane containing the line. Then in the twistor fibration $\hat{p} : \hat{Z}(\mathbb{C}P^2) \rightarrow \mathbb{C}P^2$ of the complex projective space \hat{p} sends $(\mathfrak{l}, \mathfrak{p}) \in F_{12}(\mathbb{C}^3)$ into the point $[\mathfrak{l}] \in \mathbb{C}P^2$ corresponding to $\mathfrak{l} \subset \mathbb{C}^3$. This is a smooth $\mathbb{C}P^1$ -fibration over $\mathbb{C}P^2$. Part (i) of Theorem 3.3 tells us that $R^4 \subset \mathbb{C}P^2$. Writing $Z(R^4) := \hat{Z}(\mathbb{C}P^2)|_{R^4}$ and $p := \hat{p}|_{R^4}$ the restricted twistor fibration $p : Z(R^4) \rightarrow R^4$ is topologically trivial i.e., $Z(R^4)$ is homeomorphic to $R^4 \times S^2 \cong \mathbb{R}^4 \times S^2$ because R^4 is contractible.² Take a starting pair $(\mathfrak{l}, \mathfrak{p}) \in Z(R^4)$ with a running point $[\mathfrak{l}] \in R^4$. Fix a target point $[\mathfrak{l}_0] \in R^4$ with $p^{-1}([\mathfrak{l}_0]) \subset Z(R^4)$ consisting of terminating pairs $(\mathfrak{l}_0, \mathfrak{p}_0)$. Fix an ideal point $x \in \mathbb{C}P^2 \setminus R^4$ hence surely $x \neq [\mathfrak{l}]$ as well as $x \neq [\mathfrak{l}_0]$.

Now we construct a map $\pi_x : Z(R^4) \rightarrow p^{-1}([\mathfrak{l}_0]) \cong \mathbb{C}P^1$ as follows. By the aid of the Fubini–Study metric one can talk about distances and angles on $\mathbb{C}P^2$. Then surely $d([\mathfrak{l}], x) > 0$ therefore there exists a unique projective line in $\mathbb{C}P^2$ passing through $[\mathfrak{l}]$ and x and precisely *two* perpendicular bisectors of the corresponding two segments along this projective line.³ Let $\ell \subset \mathbb{C}P^2$ be one continuous choice of these perpendicular bisectors as $[\mathfrak{l}] \in R^4$ varies. Now, take $(\mathfrak{l}, \mathfrak{p}) \in p^{-1}([\mathfrak{l}] \in R^4) \subset Z(R^4)$; there is a unique intersection point $[\mathfrak{p}] \cap \ell \in \mathbb{C}P^2$ and consider the unique line $m \subset \mathbb{C}P^2$ connecting $[\mathfrak{p}] \cap \ell$ and the ideal point x . We denote this operation so far as $P_x([\mathfrak{p}]) = m$. Next, since $d(x, [\mathfrak{l}_0]) > 0$, we can repeat the whole procedure replacing the running $[\mathfrak{l}] \in R^4$ with the fixed target point $[\mathfrak{l}_0] \in R^4$. That is, let $\ell_0 \subset \mathbb{C}P^2$ be a fixed perpendicular bisector of the line through x and $[\mathfrak{l}_0]$; then there is a unique point $m \cap \ell_0$ and

²This is a necessary topological condition for the existence of the map (2). The full twistor fibration $\hat{p} : \hat{Z}(\mathbb{C}P^2) \rightarrow \mathbb{C}P^2$ is non-trivial, neither its restriction to the punctured space $\mathbb{C}P^2 \setminus \{x\}$.

³Recall how to construct them. First let $y \in \mathbb{C}P^2$ be a midpoint along one segment connecting $[\mathfrak{l}]$ and x i.e., satisfying $d([\mathfrak{l}], y) = d(y, x) = \frac{1}{2}d([\mathfrak{l}], x)$; second viewing the distinct points $[\mathfrak{l}]$ and x in $\mathbb{C}P^2$ as two linearly independent vectors in \mathbb{C}^3 let $y_\infty \in \mathbb{C}P^2$ be the point corresponding to their complex vectorial product in \mathbb{C}^3 . Then y_∞ is not on the segment hence $y_\infty \neq y$ and a perpendicular bisector arises by taking the unique projective line in $\mathbb{C}P^2$ through y and y_∞ .

finally, define the pair $(\ell_0, \mathfrak{p}_0) \in p^{-1}([\ell_0])$ such that $[\mathfrak{p}_0] \subset \mathbb{C}P^2$ is the unique line connecting $m \cap \ell_0$ with $[\ell_0] \in R^4$. Again, denote this operation by $R_x(m) = [\mathfrak{p}_0]$. In short,

$$\pi_x((\ell, \mathfrak{p})) := (\ell_0, \mathfrak{p}_0) \text{ where } \mathfrak{p}_0 \subset \mathbb{C}^3 \text{ is the line } [\mathfrak{p}_0] \subset \mathbb{C}P^2 \text{ satisfying } R_x(P_x([\mathfrak{p}])) = [\mathfrak{p}_0] \quad (3)$$

(see Figure 2 for a construction of this map in projective geometry). It is a classical observation that this map is well-defined and holomorphic; in particular it is the identity on $p^{-1}([\ell_0]) \subset Z(R^4)$ i.e., $\pi_x((\ell_0, \mathfrak{p}_0)) = (\ell_0, \mathfrak{p}_0)$. For clarity we note that $\pi_x : Z(R^4) \rightarrow p^{-1}([\ell_0])$ is single-valued along R^4 in spite of the fact that π_x itself extends over the larger punctured space $\mathbb{C}P^2 \setminus \{x\} \supset R^4$ and is double-valued there due to the ambiguity in the choice of the perpendicular bisector; however fortunately one cannot pass continuously to another branch of π_x without crossing somewhere the infinitely distant 2-sphere $\mathbb{C}P^2 \setminus R^4 \subset \mathbb{C}P^2$.

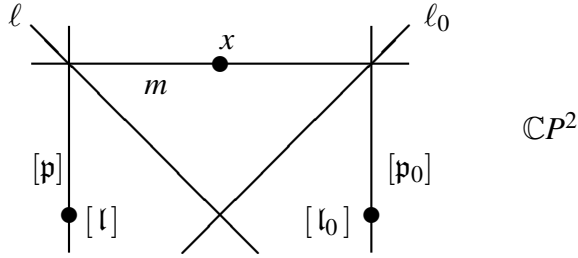


Figure 2. Two-step construction of the map π_x satisfying $\pi_x((\ell, \mathfrak{p})) = (\ell_0, \mathfrak{p}_0)$.

Secondly we fuse all the maps $\pi_x : Z(R^4) \rightarrow p^{-1}([\ell_0])$ in (3), when $x \in \mathbb{C}P^2 \setminus R^4$ runs through the complement of the exotic \mathbb{R}^4 , into a single-valued holomorphic map $\pi : Z(R^4) \rightarrow p^{-1}([\ell_0])$ by applying the concept of Lebesgue integration of algebraic-function-field-valued functions summarized in the Appendix. Assume that with a fixed ideal point $x \in \mathbb{C}P^2 \setminus R^4$ the holomorphic map (3) is given; take now a *different* ideal point $y \in \mathbb{C}P^2 \setminus R^4$ with its corresponding holomorphic map $\pi_y : Z(R^4) \rightarrow p^{-1}([\ell_0])$ into the *same* target space. Then there exists a commutative diagram

$$\begin{array}{ccc} Z(R^4) & \xrightarrow{\pi_x} & p^{-1}([\ell_0]) \\ & \searrow \pi_y & \downarrow f_{yx} \\ & & p^{-1}([\ell_0]) \end{array}$$

with f_{yx} being a holomorphic map satisfying $f_{yx} = \text{Id}_{p^{-1}([\ell_0])}$. Pick an affine coordinate system (u, v) on a coordinate ball $U \subset \mathbb{C}P^2$ centered about $[\ell_0] \in U$ i.e., $(u([\ell_0]), v([\ell_0])) = (0, 0)$. In this coordinate system any affine line $[\mathfrak{p}_0] \cap U$ passing through $[\ell_0]$ looks like $(u([\mathfrak{p}_0]), v([\mathfrak{p}_0])) = (u, zu)$ with $z \in \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$ hence $(\ell_0, \mathfrak{p}_0) = z$ provides us with an identification $p^{-1}([\ell_0]) \cong \mathbb{C}P^1$. However it is known for a long time that a holomorphic map from $\mathbb{C}P^1$ into itself is a rational function in a single variable; consequently under this identification $f_{yx} : p^{-1}([\ell_0]) \rightarrow p^{-1}([\ell_0])$ can be described by a unique element R_{yx} of the algebraic function field $\mathbb{C}(z)$, the complex rational functions in one variable z , satisfying $R_{yx}(z) = z$. That is, there exist complex-coefficient polynomials $P_{yx}(z) = a_m(y)z^m + \dots + a_1(y)z + a_0(y)$ and $Q_{yx}(z) = b_n(y)z^n + \dots + b_1(y)z + b_0(y)$ such that $R_{yx}(z) = \frac{P_{yx}(z)}{Q_{yx}(z)}$ and $R_{yx}(z) = z$ implies $a_1(x) = 1$ and $b_0(x) = 1$ and all the rest being zero at x . In this context for a fixed $(\ell, \mathfrak{p}) \in Z(R^4)$ it is worth

regarding $\pi_x((l, p))$ as a particular choice for z in the abstractly given algebraic function field $\mathbb{C}(z)$ and denoting this coordinatized $(\mathbb{C}(z), \pi_x)$ simply as $\mathbb{C}(\pi_x)$. We eventually come up with

$$\pi_y((l, p)) = R_{yx}(\pi_x((l, p))) = \frac{P_{yx}(\pi_x((l, p)))}{Q_{yx}(\pi_x((l, p)))}$$

and the coefficients $a_i, b_j : \mathbb{C}P^2 \setminus R^4 \rightarrow \mathbb{C}$ of P_{yx} and Q_{yx} respectively, are at least continuous functions assuming perhaps zero values, therefore the degrees of P_{yx} and Q_{yx} can jump as y runs through the ideal points. Nevertheless, exploiting the compactness of $\mathbb{C}P^2 \setminus R^4$ (homeomorphic to S^2) and the continuity of the coefficients, one can see that there exist overall constants $N_x \in \mathbb{N}$ and $K_x \in \mathbb{R}^+$ such that

$$\max \left(\sup_{y \in \mathbb{C}P^2 \setminus R^4} \deg P_{yx}, \sup_{y \in \mathbb{C}P^2 \setminus R^4} \deg Q_{yx} \right) \leq N_x, \quad \max_{0 \leq i, j \leq N_x} \left(\sup_{y \in \mathbb{C}P^2 \setminus R^4} |a_i(y)|, \sup_{y \in \mathbb{C}P^2 \setminus R^4} |b_j(y)| \right) \leq K_x.$$

Let $S^2 \subset \mathbb{R}^3$ denote the standard 2-sphere with its inherited orientation, smooth structure and round metric and let $i : S^2 \rightarrow \mathbb{C}P^2$ be a continuous embedding such that $i : S^2 \rightarrow \mathbb{C}P^2 \setminus R^4$ is a homeomorphism onto the complement. In this way the coefficients of P_{yx} and Q_{yx} give rise to continuous functions on the standard 2-sphere via pullback and we obtain a continuous function $i^*y \mapsto R_{i^*y, x}$ from S^2 into $\mathbb{C}(\pi_x)$. Writing $d(i^*y)$ for the usual volume-form on S^2 with respect to its orientation and round metric we define $\pi : Z(R^4) \rightarrow p^{-1}([l_0])$ by

$$\pi((l, p)) := \int_{S^2} R_{i^*y, x}(\pi_x((l, p))) d(i^*y) \quad (4)$$

for all $(l, p) \in Z(R^4)$. As explained in the Appendix, the expression on the right hand side as an algebraic-function-field-valued Lebesgue integral over S^2 exists moreover the map $(l, p) \mapsto \pi((l, p))$ in (4) is holomorphic and is independent of x ; these are proved in Lemma 6.3. In particular we can think of $\int_{S^2} R_{i^*y, x}(\pi_x(\ast)) d(i^*y) \in \mathbb{C}(\pi_x)$ as a rational function in the variable π_x and changing the reference point x just corresponds to using different coordinatizations in the abstract function field $\mathbb{C}(z)$.

Thirdly we extend the map (4) over the whole X_M . Let $y_0 \in K \subset R^4$ be the attaching point used to glue R^4 with the rest of X_M as in Lemma 3.1; we suppose $y_0 \neq [l_0] \in R^4$. Let $j : R^4 \setminus \{y_0\} \rightarrow X_M$ be a smooth embedding which identifies R^4 with the exotic \mathbb{R}^4 -end of X_M in its decomposition (1) such that $j([l_0]) = x_0$ where $x_0 \in X_M$ is the distinguished point of the map (2) to be constructed. Also write $J : Z(R^4 \setminus \{y_0\}) \rightarrow Z$ for the induced inclusion of the twistor space into that of X_M . Then $\pi' := (J^{-1})^*(\pi|_{Z(R^4 \setminus \{y_0\})}) : V \rightarrow p^{-1}(x_0)$ is a partially defined holomorphic map on a connected open subset $V := p^{-1}(j(R^4 \setminus \{y_0\})) \subset Z$ of the twistor space of X_M . We now extend π' holomorphically over the whole Z to be the map (2) as follows. Consider an open covering $X_M = \cup_k U_k$ giving rise to an open covering $Z = \cup_k p^{-1}(U_k)$ of the twistor space, too. If $x \in X_M$ is an inner point of the exotic \mathbb{R}^4 -end $j(R^4 \setminus \{y_0\}) \subset X_M$ (but different from the base point x_0) such that an open subset $x \in U_k$ from the covering satisfies $U_k \subset j(R^4 \setminus \{y_0\}) \subset X_M$ then we get a neighbourhood $p^{-1}(U_k) \subset V \subset Z$ of $p^{-1}(x)$, too. We know that the holomorphic map $\pi'|_{p^{-1}(x)} : p^{-1}(x) \rightarrow p^{-1}(x_0)$ extends to a holomorphic map $\pi'|_{p^{-1}(U_k)} : p^{-1}(U_k) \rightarrow p^{-1}(x_0)$. However, by referring at this step to an important extendibility result of Griffiths [12, Proposition 1.3], this extendibility depends only on two holomorphic data: the pullback tangent bundle $(\pi'|_{p^{-1}(x)})^*(T p^{-1}(x_0))$ over $p^{-1}(x)$ and the normal bundle of it as a complex submanifold $p^{-1}(x) \subset Z$. But the former bundle cannot locally depend on x because holomorphic line bundles over $p^{-1}(x) \cong \mathbb{C}P^1$ form a discrete set. Regarding the latter bundle, $p^{-1}(x) \subset Z$ as a submanifold is a *twistor line* in Z and all twistor lines in the twistor fibration have isomorphic normal bundles (see the

Remark on twistor theory above). Since these twistor lines fulfill the whole Z these arguments convince us that using the open covering $\cup_k p^{-1}(U_k)$ of Z and exploiting the simply connectedness of Z provided by that of X_M we can analytically continue the partial map π' above from the connected open subset $V \subset Z$ to a holomorphic map (2) over the whole Z in a unique way as desired. \diamond

It also follows that $\pi : Z \rightarrow \mathbb{C}P^1$ i.e., the map (2) constructed in Lemma 3.2 is compatible with the real structure $\tau : Z \rightarrow Z$ already fixed by the self-dual structure in Theorem 3.2 therefore twistor theory provides us with a Ricci-flat (and self-dual) Riemannian metric g_0 on X_M . We proceed further and demonstrate that, unlike (X_M, g_1) , the space (X_M, g_0) is complete.

Lemma 3.3. *The Ricci-flat Riemannian manifold (X_M, g_0) is complete.*

Remark. Moreover (X_M, g_0) is simply connected and self-dual i.e., as a by-product, is in fact a hyper-Kähler space. In particular taking $M := S^4$ then $\hat{X}_{S^4} = \mathbb{C}P^2$ so $X_{S^4} = R^4$, the largest member of the Gompf–Taubes radial family, carries a complete hyper-Kähler metric. Hence these spaces are *gravitational instantons* with dominant contribution to the Euclidean quantum gravity path integral [1, 4].

Proof of Lemma 3.3. Since both g_1 and this Ricci-flat metric g_0 stem from the same complex structure on the same twistor space Z we know from twistor theory that these metrics are in fact conformally equivalent. That is, there exists a smooth non-constant strictly positive function $\varphi : X_M \rightarrow \mathbb{R}^+$ such that $\varphi^{-2} \cdot g_1 = g_0$. Our strategy to prove completeness is to follow Gordon [11] i.e., to demonstrate that an appropriate real-valued function on X_M , in our case $\log \varphi^{-1} : X_M \rightarrow \mathbb{R}$, is proper (i.e., the preimages of compact subsets are compact) with bounded gradient in modulus with respect to g_0 implying the completeness.

Referring to (1) the open space X_M arises by deleting $\mathbb{C}P^2 \setminus R^4$ from a $\mathbb{C}P^2$ -factor of the closed space \hat{X}_M . First we observe that $\varphi^{-1} : X_M \rightarrow \mathbb{R}^+$ is uniformly divergent along $\mathbb{C}P^2 \setminus R^4$ as follows. It is clear that the potential singularities of φ^{-1} stem from those of the map (2). The map π_x constructed in (3) has an obvious singularity at $x \in \mathbb{C}P^2 \setminus R^4$ and π itself has been constructed in (4) by integrating together all the π_x 's along $\mathbb{C}P^2 \setminus R^4$ consequently π is singular along the whole $\mathbb{C}P^2 \setminus R^4$. Consequently φ^{-1} is expected to be somehow singular along the whole $\mathbb{C}P^2 \setminus R^4$, too. Moreover, this part of the construction of π in Lemma 3.2 deals with a single $\mathbb{C}P^2$ -factor in (1) only hence is universal in the sense that it is independent of the M -factor in (1). In other words, for all X_M the map (2) arises by analytically continuing the *same* π on R^4 constructed in the first two steps in Lemma 3.2. So we anticipate $\varphi^{-1} : X_M \rightarrow \mathbb{R}^+$ with $X_M \subset \hat{X}_M$ to possess a uniform and universal singular behaviour along $\mathbb{C}P^2 \setminus R^4 \subset \hat{X}_M$ what we analyze now further.

The conformal scaling function satisfies with respect to g_1 the following equations on X_M :

$$\begin{cases} \Delta \varphi^{-1} + \frac{1}{6} \varphi^{-1} \text{Scal}_1 & = 0 \text{ (vanishing of the scalar curvature of } g_0 \text{ on } X_M); \\ \nabla^2 \varphi - \frac{1}{4} \Delta \varphi \cdot g_1 + \frac{1}{2} \varphi \cdot \text{Ric}_1^0 & = 0 \text{ (vanishing of the traceless Ricci tensor of } g_0 \text{ on } X_M). \end{cases} \quad (5)$$

The Ricci tensor Ric_1 of g_1 extends smoothly over \hat{X}_M because it is just the restriction of the Ricci tensor of the self-dual metric \hat{g}_1 on \hat{X}_M . Therefore both its scalar curvature Scal_1 and traceless Ricci part Ric_1^0 extend. Consequently from the first equation of (5) we can see that $\varphi \Delta \varphi^{-1}$ extends smoothly over \hat{X}_M . Likewise, adding the tracial part to the second equation of (5) we get $\varphi^{-1} \nabla^2 \varphi = -\frac{1}{2} \text{Ric}_1$ hence we conclude that the symmetric tensor field $\varphi^{-1} \nabla^2 \varphi$ extends smoothly over \hat{X}_M so its trace $\varphi^{-1} \Delta \varphi$ as well. The equation $\Delta(\varphi \cdot \varphi^{-1}) = 0$ gives the standard identity $0 = (\Delta \varphi) \varphi^{-1} + 2g_1(d\varphi, d\varphi^{-1}) + \varphi \Delta \varphi^{-1}$ and adjusting this a bit we get

$$\varphi^2 |d\varphi^{-1}|_{g_1}^2 = \frac{1}{2} (\varphi \Delta \varphi^{-1} + \varphi^{-1} \Delta \varphi) \quad (6)$$

consequently the function $\varphi|d\varphi^{-1}|_{g_1}$ extends smoothly over \hat{X}_M , too. Assume now that φ^{-1} is extendible over \hat{X}_M at least continuously. Then, taking into the aforementioned universal behaviour of φ^{-1} around its interesting part, we can take $\hat{X}_{S^4} = \mathbb{C}P^2$ and $\hat{g}_1 = \text{Fubini–Study metric}$. However this metric has constant scalar curvature consequently, by the aid of the first equation of (5) and the maximum principle, we could conclude that φ^{-1} is constant on $\mathbb{C}P^2$, a contradiction. Assume now that φ^{-1} does not extend continuously over \hat{X}_M but $|\varphi^{-1}|$ is bounded. Then its gradient $d\varphi^{-1}$ gets diverge along $\mathbb{C}P^2 \setminus R^4$ hence from the extendibility of $\varphi|d\varphi^{-1}|_{g_1}$ we obtain that φ vanishes along $\mathbb{C}P^2 \setminus R^4$, a contradiction again. Therefore $\varphi^{-1} : X_M \rightarrow \mathbb{R}^+$ with $X_M \subset \hat{X}_M$ is uniformly divergent along $\mathbb{C}P^2 \setminus R^4 \subset \hat{X}_M$ yielding, on the one hand, that $\log \varphi^{-1} : X_M \rightarrow \mathbb{R}$ is a proper function.

As a by-product the inverse function φ is bounded on X_M i.e., $|\varphi| \leq c_1$ with a finite constant. We already know that $|\varphi\Delta\varphi^{-1}| \leq c_2$ and $|\varphi^{-1}\Delta\varphi| \leq c_3$ with other finite constants as well. Since $\varphi|d\varphi^{-1}|_{g_1} = |d(\log \varphi^{-1})|_{g_1}$ and carefully noticing that $|\xi|_{g_0} = \varphi|\xi|_{g_1}$ on 1-forms we can use (6) and the estimates above to come up with

$$|d(\log \varphi^{-1})|_{g_0}^2 \leq c_1^2 |d(\log \varphi^{-1})|_{g_1}^2 \leq c_1^2 (|\varphi\Delta\varphi^{-1}| + |\varphi^{-1}\Delta\varphi|) \leq c_1^2(c_2 + c_3) < +\infty$$

and conclude, on the other hand, that $\log \varphi^{-1} : X_M \rightarrow \mathbb{R}$ has bounded gradient in modulus with respect to g_0 . Therefore, in light of Gordon’s theorem [11], the Ricci-flat space (X_M, g_0) is complete. \diamond

Proof of Theorem 3.1. The proof now readily follows by putting together Lemmata 3.1, 3.2 and 3.3. \diamond

4 Lorentzian considerations

Having established the existence of an abundance of spaces, it is worth summarizing the situation before we proceed further. In Section 3 we have constructed certain non-compact complete Ricci-flat Riemannian 4-manifolds. These geometries are hyper-Kähler as a by-product however, more important to us, they have the odd feature that—although they are non-compact—surely do not split smoothly into the product of any 3-manifold and the real line (with their unique smooth structures) because they contain a “creased” asymptotical region, more precisely a single end diffeomorphic to an exotic \mathbb{R}^4 (see Figure 1). Taking into account [2, 3] this non-splitting phenomenon offers a good starting point to violate the **SCCC** in a generic way. However, our solutions of the vacuum Einstein equation are still *Riemannian* hence we have to work on them further to obtain solutions of the *Lorentzian* vacuum Einstein equation on the same “creased” manifolds. In this section we will prove the following theorem whose proof again needs some preparations and will be presented at the end of this section.

Theorem 4.1. *Consider the Riemannian 4-manifold (X_M, g_0) as in Theorem 3.1. Then out of this space one can construct an oriented smooth Lorentzian 4-manifold*

$$(X_M, g)$$

with the following properties.

The metric g is a Ricci-flat, probably null and-or timelike geodesically incomplete, but surely not globally hyperbolic metric on X_M . Furthermore if $(S, h) \subset (X_M, g)$ is any connected, oriented, complete spacelike sub-3-manifold with corresponding (necessarily partial) initial data set (S, h, k) , then any sufficiently large perturbation (X'_M, g') of (X_M, g) relative to (S, h, k) in the sense of Definition 2.1 is not globally hyperbolic. Here “sufficiently large” means that X'_M , satisfying $S \subset X'_M \subseteq X_M$, contains the image, present in the R^4 -factor of X_M in its decomposition (1), of the compact subset $K \subset R^4$ of part (i) in Theorem 3.3.

Take a Riemannian 4-manifold (X_M, g_0) as in Theorem 3.1 and let $S \subset X_M$ be any smoothly embedded, connected and orientable (with induced orientation) sub-3-manifold in it such that with the restricted Riemannian metric $h := g_0|_S$ is complete i.e., $(S, h) \subset (X_M, g_0)$ is a complete Riemannian sub-3-manifold. Of course any compact $S \subset X_M$ works but S can be non-compact, too.

Remark. Complete examples $(S, h) \subset (X_M, g_0)$ such that $S \subset X_M$ is non-compact can be constructed if $S \subset \mathbb{R}^4 \subset \mathbb{C}P^2$ i.e., it fully belongs to the exotic \mathbb{R}^4 -factor in the decomposition (1) of X_M as follows. The boundary of the unit disk bundle inside the total space of the tautological line bundle H over $\mathbb{C}P^1$ is a circle bundle over its zero section $\mathbb{C}P^1$ more precisely a Hopf fibration; hence it is a 3-manifold homeomorphic to S^3 . Fixing an ideal point $x \in \mathbb{C}P^2 \setminus \mathbb{R}^4$ we can identify the total space H with $\mathbb{C}P^2 \setminus \{x\}$ and denote by $N \subset \mathbb{C}P^2 \setminus \{x\}$ the image of the aforementioned boundary of the unit disk bundle. Define

$$S := \text{one connected component of } N \cap \mathbb{R}^4 .$$

Every exotic \mathbb{R}^4 in general hence our \mathbb{R}^4 in particular, has the property that it contains a compact subset $C \subset \mathbb{R}^4$ which cannot be surrounded by a smoothly embedded $S^3 \subset \mathbb{R}^4$ [10, Exercise 9.4.1]. Taking the radii of the constituent circles of N sufficiently large we can suppose by the compactness of C that $C \cap S = \emptyset$ i.e., S could surround C if S was homeomorphic to S^3 . This would be a contradiction hence $S \subset \mathbb{R}^4$ is an open (i.e., non-compact without boundary) and connected sub-3-manifold of \mathbb{R}^4 . Therefore, exploiting the contractibility of \mathbb{R}^4 we conclude that S is an open contractible sub-3-manifold within \mathbb{R}^4 . Putting $h := g_0|_S$ we therefore obtain an open contractible Riemannian sub-3-manifold $(S, h) \subset (X_M, g_0)$ which is complete by construction, as the reader may verify.

Lemma 4.1. *Consider the Riemannian 4-manifold (X_M, g_0) as in Theorem 3.1 and let $(S, h) \subset (X_M, g_0)$ be a connected, oriented and complete Riemannian sub-3-manifold in it.*

Then there exists a real line sub-bundle $L \subset TX_M$ of the tangent bundle such that there exists a smooth Whitney-sum decomposition $TX_M = L \oplus L^\perp$ with the property that the orthogonal complement (with respect to g_0) bundle $L^\perp \subset TX_M$ satisfies $L^\perp|_S \cong TS$ i.e., its restriction is isomorphic to the tangent bundle of S .

Remark. Before we embark upon the proof we clarify that the existence of the smooth Whitney-sum decomposition $TX_M \cong L \oplus L^\perp$ of the tangent bundle of X_M should not be confused with any smooth splitting $X_M \cong \mathbb{R} \times S$ of X_M itself. Indeed, this latter splitting was excluded already in Lemma 3.1 once and for all. In fact this non-splitting of X_M is a key property of these spaces and is the reason we use them throughout the paper.

Proof of Lemma 4.1. Good references here are [16, 26]. For an oriented Riemannian 4-manifold standard obstruction theory says that the obstruction characteristic classes against its tangent bundle being trivial live in the cohomology groups $H^i(X_M; \pi_{i-1}(\text{SO}(4)))$, $i = 1, \dots, 4$. We know that $\pi_0(\text{SO}(4)) \cong 0$, $\pi_1(\text{SO}(4)) \cong \mathbb{Z}_2$, $\pi_2(\text{SO}(4)) \cong 0$ and $\pi_3(\text{SO}(4)) \cong \mathbb{Z}$ but X_M is open and oriented hence $H^4(X_M; \mathbb{Z}) \cong 0$. Hence the only obstruction is

$$w_2(X_M) \in H^2(X_M; \pi_1(\text{SO}(4))) \cong H^2(X_M; \mathbb{Z}_2) ,$$

the so-called 2nd Stiefel–Whitney class of X_M . Consequently if X_M is a spin manifold which by definition means that $w_2(X_M) = 0$ then its tangent bundle is already trivial hence admits a nowhere vanishing smooth section i.e., a non-zero vector field $v : X_M \rightarrow TX_M$. Assume X_M is not spin therefore having non-trivial tangent bundle. Then exploiting its simply connectivity and openness, X_M is homotopic to its 2-skeleton $X_M(2)$ hence isomorphism classes of vector bundles over X_M are in one-to-one correspondence with those over $X_M(2)$. However $X_M(2)$ as a topological space is a 2 dimensional CW-complex

therefore any real rank-4 topological vector bundle E over it splits, more precisely is isomorphic to $F \oplus \mathbb{R}^2$ where F is a real rank-2 vector bundle and \mathbb{R}^2 is the trivial real rank-2 vector bundle. Consequently the tangent bundle TX_M also splits. This of course again means that TX_M admits a nowhere vanishing smooth section $v : X_M \rightarrow TX_M$ (in fact TX_M admits at least two linearly independent sections).

We construct a section as follows. Taking into account that $S \subset X_M$ is orientable, its normal bundle is trivial which means that a small tubular neighbourhood $N_\varepsilon(S) \subset X_M$ of S diffeomorphic to $S \times (-\varepsilon, \varepsilon)$. This induces a splitting $TX_M|_S \cong TS \oplus \mathbb{R}$ of the restricted tangent bundle. Without loosing generality we can assume that this local splitting is orthogonal for the Riemannian metric g_0 . Let $v_S : S \rightarrow \mathbb{R}$ be a nowhere vanishing section of this local orthogonal line bundle i.e., $v_S \neq 0$ but $g_0(v_S, TS) = 0$. Obstruction theory says that v_S can be extended continuously to a section $v : X_M \rightarrow TX_M$. Of course this extension is not unique and we can arrange it to be smooth and nowhere vanishing because the only obstruction class against this latter requirement lives in $H^4(X_M; \pi_3(\mathbb{R}^4 \setminus \{0\})) \cong H^4(X_M; \mathbb{Z}) \cong 0$ hence is trivial. The image of this nowhere vanishing smooth section v within TX_M then gives rise to a line bundle $L \subset TX_M$ and an orthogonal splitting $L \oplus L^\perp = TX_M$ with respect to g_0 over the whole X_M . This splitting satisfies $L^\perp|_S \cong TS$ by construction, as claimed. \diamond

Lemma 4.2. *Take the Ricci-flat Riemannian 4-manifold (X_M, g_0) as in Theorem 3.1 and let $(S, h) \subset (X_M, g_0)$ be a connected, oriented and complete Riemannian sub-3-manifold.*

There exists a smooth Lorentzian metric g on X_M such that (X_M, g) is a Ricci-flat Lorentzian manifold (probably null and-or timelike incomplete) and $(S, h) \subset (X_M, g)$ is a connected, complete spacelike sub-3-manifold.

Remark. We emphasize that g is not the result of an analytic continuation of g_0 within some complex manifold hence the procedure described in Lemmata 4.1 and 4.2 is not a ‘‘Wick rotation’’ in any sense of e.g. [13] and the references therein. Accordingly, g is not uniquely determined by g_0 , it depends on the chosen subspace $S \subset X_M$ and more generally, the line bundle $L \subset TX_M$. The main reason for not using the standard approach, beyond its rigidity, is that we do not want to loose the subtle smoothness properties of X_M by replacing it with another manifold within its complexification $X_M^{\mathbb{C}}$ which, by twistor theory, exists (cf. the *Remark* on twistor theory in Section 3).

Proof of Lemma 4.2. Take the complexification $T^{\mathbb{C}}X_M := TX_M \otimes_{\mathbb{R}} \mathbb{C}$ of the real tangent bundle as well as the complex linear extension of the Riemannian Ricci-flat metric g_0 on TX_M to a complex Ricci-flat metric $g_0^{\mathbb{C}}$ on $T^{\mathbb{C}}X_M$. This means that if $v^{\mathbb{C}}$ is a complex tangent vector then both $v^{\mathbb{C}} \mapsto g_0^{\mathbb{C}}(v^{\mathbb{C}}, \cdot)$ and $v^{\mathbb{C}} \mapsto g_0^{\mathbb{C}}(\cdot, v^{\mathbb{C}})$ are \mathbb{C} -linear maps and $\text{Ric}^{\mathbb{C}} = 0$. Then the real splitting $TX_M = L \oplus L^\perp$ of Lemma 4.1, satisfying $L^\perp|_S \cong TS$ with the chosen $(S, h) \subset (X_M, g_0)$, induces a splitting

$$T^{\mathbb{C}}X_M = L \oplus L^\perp \oplus \sqrt{-1}L \oplus \sqrt{-1}L^\perp \quad (7)$$

over \mathbb{R} i.e., if $T^{\mathbb{C}}X_M$ considered as a real rank-8 bundle over X_M . Define a metric on the real rank-4 sub-bundle $L^\perp \oplus \sqrt{-1}L \subset T^{\mathbb{C}}X_M$ by taking the restriction $g_0^{\mathbb{C}}|_{L^\perp \oplus \sqrt{-1}L}$. It readily follows from the orthogonality of the splitting that this is a non-degenerate real bilinear form of Lorentzian type on this real sub-bundle. To see this, we simply have to observe that with real vector fields $v_1, v_2 : X_M \rightarrow L$ and $w_1, w_2 : X_M \rightarrow L^\perp$

$$g_0^{\mathbb{C}}|_{L^\perp \oplus \sqrt{-1}L}(\sqrt{-1}v_1, \sqrt{-1}v_1) = g_0^{\mathbb{C}}(\sqrt{-1}v_1, \sqrt{-1}v_1) = -g_0^{\mathbb{C}}(v_1, v_1) = -g_0(v_1, v_1)$$

and

$$g_0^{\mathbb{C}}|_{L^\perp \oplus \sqrt{-1}L}(\sqrt{-1}v_1, w_1) = g_0^{\mathbb{C}}(\sqrt{-1}v_1, w_1) = \sqrt{-1}g_0^{\mathbb{C}}(v_1, w_1) = \sqrt{-1}g_0(v_1, w_1) = 0$$

and finally

$$g_0^{\mathbb{C}}|_{L^\perp \oplus \sqrt{-1}L}(w_1, w_2) = g_0^{\mathbb{C}}(w_1, w_2) = g_0(w_1, w_2) .$$

Consider the \mathbb{R} -linear bundle isomorphism $W_L : T^{\mathbb{C}}X_M \rightarrow T^{\mathbb{C}}X_M$ of the complexified tangent bundle defined by, with respect to the splitting (7), as $W_L(v_1, w_1, \sqrt{-1}v_2, \sqrt{-1}w_2) := (v_2, w_1, \sqrt{-1}v_1, \sqrt{-1}w_2)$. It maps the real tangent bundle $TX_M = L \oplus L^\perp \subset T^{\mathbb{C}}X_M$ onto the real bundle $L^\perp \oplus \sqrt{-1}L \subset T^{\mathbb{C}}X_M$ and *vice versa* making the diagram

$$\begin{array}{ccc} T^{\mathbb{C}}X_M & \xrightarrow{W_L} & T^{\mathbb{C}}X_M \\ \downarrow & & \downarrow \\ X_M & \xrightarrow{\text{Id}_{X_M}} & X_M \end{array}$$

commutative. In fact W_L is a real *reflection* satisfying $W_L^2 = \text{Id}_{T^{\mathbb{C}}X_M}$. Then with arbitrary two tangent vectors $v, w : X_M \rightarrow TX_M$ putting

$$g(v, w) := g_0^{\mathbb{C}}(W_L v, W_L w)$$

we obtain a Lorentzian metric g on TX_M such that $(S, h) \subset (X_M, g)$ is a connected, complete spacelike sub-3-manifold.

Concerning its Ricci tensor, the Levi-Civita connection ∇ of g and $\nabla^{\mathbb{C}}$ of $g_0^{\mathbb{C}}$ are related by

$$\begin{aligned} g(\nabla_u v, w) + g(v, \nabla_u w) &= dg(v, w)u = dg_0^{\mathbb{C}}(W_L v, W_L w)u \\ &= g_0^{\mathbb{C}}(\nabla_u^{\mathbb{C}}(W_L v), W_L w) + g_0^{\mathbb{C}}(W_L v, \nabla_u^{\mathbb{C}}(W_L w)) \end{aligned}$$

yielding $\nabla = W_L \nabla^{\mathbb{C}} W_L$ (as an \mathbb{R} -linear operator) consequently the curvature Riem of g takes the shape

$$\text{Riem}(v, w)u = [\nabla_v, \nabla_w]u - \nabla_{[v, w]}u = W_L(\text{Riem}^{\mathbb{C}}(v, w)W_L u) .$$

Let $\{e_0, e_1, e_2, e_3\}$ be a real orthonormal frame for g at $T_p X_M$ satisfying $g(e_0, e_0) = -1$ and $+1$ for the rest, then $W_L e_0 = \sqrt{-1}e_0$ and $W_L e_j = e_j$ for $j = 1, 2, 3$ together with the definition of g imply first that

$$g(\text{Riem}(e_0, v)w, e_0) = g_0^{\mathbb{C}}(W_L(\text{Riem}(e_0, v)w), W_L e_0) = g_0^{\mathbb{C}}(\text{Riem}^{\mathbb{C}}(e_0, v)W_L w, \sqrt{-1}e_0)$$

and likewise

$$g(\text{Riem}(e_j, v)w, e_j) = g_0^{\mathbb{C}}(W_L(\text{Riem}(e_j, v)w), W_L e_j) = g_0^{\mathbb{C}}(\text{Riem}^{\mathbb{C}}(e_j, v)W_L w, e_j) .$$

The Ricci tensor in any signature looks like $\text{Ric}(v, w) = \sum_{k=1}^m g(e_k, e_k)g(\text{Riem}(e_k, v)w, e_k)$; hence

$$\begin{aligned} \text{Ric}(v, w) &= g(e_0, e_0)g(\text{Riem}(e_0, v)w, e_0) + \sum_{j=1}^3 g(e_j, e_j)g(\text{Riem}(e_j, v)w, e_j) \\ &= g_0^{\mathbb{C}}(\sqrt{-1}e_0, \sqrt{-1}e_0)g_0^{\mathbb{C}}(\text{Riem}^{\mathbb{C}}(e_0, v)W_L w, \sqrt{-1}e_0) + \sum_{j=1}^3 g_0^{\mathbb{C}}(e_j, e_j)g_0^{\mathbb{C}}(\text{Riem}^{\mathbb{C}}(e_j, v)W_L w, e_j) \\ &= -(\sqrt{-1} + 1)g_0^{\mathbb{C}}(e_0, e_0)g_0^{\mathbb{C}}(\text{Riem}^{\mathbb{C}}(e_0, v)W_L w, e_0) + \text{Ric}^{\mathbb{C}}(v, W_L w) \\ &= (\sqrt{-1} - 1)g(\text{Riem}(e_0, v)w, e_0) \end{aligned}$$

and we also used $\{e_0, e_1, e_2, e_3\}$ as a complex basis to obtain $\sum_{j=0}^3 g_0^{\mathbb{C}}(e_j, e_j)g_0^{\mathbb{C}}(\text{Riem}^{\mathbb{C}}(e_j, v)W_L w, e_j) = \text{Ric}^{\mathbb{C}}(v, W_L w) = 0$. However the last expression can be real for all $v, w \in T_p X_M$ if and only if it vanishes. This demonstrates that g is indeed Ricci-flat. \diamond

After this very long technical journey through Sections 3 and 4 we are now ready to inspect (X_M, g) concerning its global hyperbolicity.

Lemma 4.3. *Consider the Ricci-flat Lorentzian 4-manifold (X_M, g) of Lemma 4.2 with any spacelike and complete sub-3-manifold $(S, h) \subset (X_M, g)$ in it (Lemma 4.2 also provides us that non-empty sub-manifolds of this sort exist). Let (S, h, k) be the initial data set inside (X_M, g) induced by (S, h) and let (X'_M, g') be a perturbation of (X_M, g) relative to (S, h, k) as in Definition 2.1. Consider the pair (R^4, K) from Theorem 3.3. Assume that X'_M contains the image, present in the R^4 -factor of X_M in its decomposition (1), of the compact subset K . Then (X'_M, g') is not globally hyperbolic.*

Proof. First we prove that the trivial perturbation i.e., (X_M, g) itself is not globally hyperbolic. To see this observe that X_M is not a product of any 3-manifold S and \mathbb{R} due to its creased end (cf. Lemma 3.1); hence it follows from the smooth splitting theorem for globally hyperbolic space-times [2] that (X_M, g) cannot be globally hyperbolic.

Let us secondly consider its non-trivial perturbations (X'_M, g') relative to (S, h, k) as in Definition 2.1. Suppose that (X'_M, g') is globally hyperbolic. Referring to Definition 2.1 we know that $(S, h') \subset (X'_M, g')$ is a complete spacelike submanifold hence we can use it to obtain an initial data set (S, h', k') for (X'_M, g') . Again by [2] we find $X'_M \cong S \times \mathbb{R}$. But by our Definition 2.1 the perturbed space always satisfies $S \subset X'_M \subseteq X_M$ and, by our assumption in the present lemma, X'_M still contains the image of the compact subset $K \subset R^4$. This means that there exists a connected smooth 4-manifold M' satisfying

$$S \subset M' \subseteq M \# \underbrace{\mathbb{C}P^2 \# \dots \# \mathbb{C}P^2}_{k-1}$$

and an exotic R_t^4 with $0 < r \leq t \leq +\infty$ from the family in part (ii) of Theorem 3.3 such that X'_M has a decomposition $X'_M \cong M' \#_K R_t^4$, too. However this is in a contradiction with the splitting of X'_M above. This demonstrates that our supposition was wrong hence (X'_M, g') cannot be globally hyperbolic. \diamond

Remark. The condition that the perturbed space X'_M should contain the compact subset $K \subset R^4$ can be interpreted as follows. Decomposition (1) shows that X_M has only one asymptotic region namely its creased end from its exotic \mathbb{R}^4 -component (see Figure 1). Therefore the Lorentzian manifold (X_M, g) can be regarded as a vacuum space-time describing some topologically non-trivial “inner” region corresponding to $M \# \mathbb{C}P^2 \# \dots \# \mathbb{C}P^2$ and a contractible surrounding “outer” region described by R^4 in the decomposition (1) of X_M . The condition that the perturbation satisfying $S \subset X'_M \subseteq X_M$ should contain the compact subset $K \subset R^4$ present in the original space-time X_M means, taking into account the precise glueing descriptions in Lemma 3.1, that X'_M yet contains a “sufficiently large part” of the original space X_M i.e., cannot be simply e.g. a small tubular neighbourhood $S \subset N_\varepsilon(S) \subset X_M$ of the initial surface. Therefore this simple assumption says that the perturbation about $S \subset X_M$ is large enough in the topological sense hence is capable to “scan” the exotic regime of X_M . More on the physical interpretation of (X_M, g) we refer to Section 5.

In fact this condition is effectively necessary to exclude globally hyperbolic perturbations of the original space (X_M, g) . Let $M := S^4$ then $X_{S^4} = R^4$ and let $S \subset X_{S^4}$ be any connected open sub-3-manifold in it; then putting $X'_{S^4} := N_\varepsilon(S) \subset X_{S^4}$ to be a small tubular neighbourhood of $S \subset X_{S^4}$ the contractibility of S implies $N_\varepsilon(S) \cong S \times \mathbb{R}$ hence again by [18] we know that $N_\varepsilon(S) \cong \mathbb{R}^4$. Therefore putting g' just to be the standard Minkowski metric on X'_{S^4} we obtain (X'_{S^4}, g') is the usual Minkowski space-time hence is a globally hyperbolic perturbation of (X_{S^4}, g) relative to (S, h, k) . This perturbation is “small” in the topological sense above however might be “large” in any analytical sense i.e., the corresponding (S, h', k') might significantly deviate from the original (S, h, k) .

Proof of Theorem 4.1. Putting together Lemmata 4.1, 4.2 and 4.3 we obtain the result. \diamond

Finally we are in a position to draw the main conclusion of our efforts so far, namely to put the immense class of vacuum space-times we found in the context of the **SCCC**. Taking into account the non-trivial condition regarding $K \subset R^4$ in Lemma 4.3, the space (X_M, g) is not a robust counterexample to the **SCCC** in the strict sense of Definition 2.2. However knowing that we can start with *any* closed and simply connected M to construct open spaces like X_M with a creased end carrying a solution g of the Lorentzian vacuum Einstein equation, and the class of non-globally-hyperbolic perturbations (X'_M, g') of (X_M, g) are subject only to this mild topological condition, the corresponding perturbation class is certainly still enormously vast. Therefore in our opinion it is reasonable to say that *all the members of these immense family of Lorentzian vacuum space-times (X_M, g) give rise to generic counterexamples to the **SCCC*** as formulated in the Introduction (recall that being generic is not a well-defined concept). This is the content of the informal statement $\overline{\text{SCCC}}$, also formulated in the Introduction. In other words, in light of our considerations so far: *the **SCCC** typically fails in four dimensions!*

5 Conclusion and outlook

From the viewpoint of low dimensional differential topology it is not surprising that confining ourselves into the initial value approach when thinking about the **SCCC** typically brings affirmative while more global techniques might yield negative answers: the initial value formulation of Einstein's equation likely just explores the vicinity of 3 dimensional smooth spacelike submanifolds inside the full 4 dimensional space-time. It is well-known that an embedded smooth submanifold of an ambient space always admits a tubular neighbourhood which is an open disk bundle over the submanifold i.e., has a locally product smooth structure. However exotic 4 dimensional smooth structures never arise as products of lower dimensional ones consequently the *four* dimensional exotica i.e., the general structure of space-time never can be detected from a *three* dimensional perspective such as the initial value formulation. There is a *qualitative leap* between these dimensions.

The physical interpretation of the vacuum solutions (X_M, g) is a challenging question because, as we have seen in the *Remark* after Lemma 3.3 the corresponding Riemannian spaces (X_M, g_0) are all gravitational instantons hence are dominantly present if some quantum theory lurks behind; i.e., although they might play no role in classical general relativity, the interpretation of these solutions is unavoidable in a broader quantum context. As we already observed in the *Remark* after Lemma 4.3, (X_M, g) is a vacuum space-time such that X_M is simply connected consisting of a topologically non-trivial interior part and a topologically trivial asymptotic region; however the metric g on this space surely cannot decay to the flat metric because this asymptotic region is creased and g still has a Weyl tensor. Consequently (X_M, g) is not the geometry of a “compact gravitating system” or anything like that. On the contrary, its peculiarity is its asymptotical structure on its creased end. Exotic phenomena are genuinely non-local in the sense that all 4-manifolds are locally the same therefore, in our understanding at the current state of art, these vacuum solutions with their distant creased properties rather correspond to (quantum)cosmological solutions. The inherent non-global-hyperbolicity of them very likely stems from the violation of strong causality along their creased end probably caused by the fractal-like behaviour of distant spacelike submanifolds. Indeed, we already mentioned in the *Remark* before Lemma 4.1 that exotic \mathbb{R}^4 's have the property that sufficiently large compact subsets of them cannot be surrounded by smoothly embedded S^3 's. The non-deterministic character of these “(quantum)cosmological solutions” towards their infinity could perhaps be physically understood as the manifestation of the quantum properties of the Big Bang singularity.

6 Appendix: Lebesgue integration in algebraic function fields

Here we work out how to integrate functions on manifolds but taking values in the algebraic function field of complex rational functions. The construction of this integral is straightforward and is fully based on the by-now classical approach of Riesz–Szőkefalvi-Nagy (cf. [22, §16, §17]).

Let $P(z), Q(z)$ be two complex polynomials in the single variable $z \in \mathbb{C}$ and let $\mathbb{C}(z)$ denote the (commutative) algebraic function field of fractions $R(z) := \frac{P(z)}{Q(z)}$. A norm of R is defined by the formula $|R|_{c,z_0} := c^{\text{ord}_{z_0}(R)}$ where $c \in (0, 1)$ is a fixed real number and $\text{ord}_{z_0}(R) \in \mathbb{Z}$ is the lowest one among the indices $k \in \mathbb{Z}$ of the non-zero coefficients $a_k \in \mathbb{C}$ in the Laurent expansion

$$R(z) = \sum_{k \gg -\infty}^{+\infty} a_k (z - z_0)^k \quad (8)$$

of R about a fixed point $z_0 \in \mathbb{C}$; note that the number $\text{ord}_{z_0}(R)$ is independent of the particular coordinate system used for the expansion hence is well-defined and this definition makes sense for $R = 0$ if we put $\text{ord}_{z_0}(0) := +\infty$ and of course yields $|0|_{c,z_0} = 0$. It is known [27, Theorem 1.11] that, being \mathbb{C} algebraically closed, $|\cdot|_{c,z_0}$ with $c \in (0, 1)$ and $z_0 \in \mathbb{C} \cup \{\infty\}$ is the complete list of norms on $\mathbb{C}(z)$ which are trivial on \mathbb{C} . Then $\mathbb{C}(z)$ can be completed with respect to $|\cdot|_{c,z_0}$ which is $\mathbb{C}((z - z_0))$, the field of formal Laurent series in $z - z_0$. There is an embedding of fields $\mathbb{C}(z) \subset \mathbb{C}((z - z_0))$ for all c, z_0 but up to isomorphisms of topological fields, these completions are independent of the norms used [27].

Remark. Unlike the usual norms on \mathbb{R} or \mathbb{C} , all the norms of this kind on $\mathbb{C}(z)$ are *non-Archimedean* hence $\mathbb{C}(z)$ does not admit any norm-compatible embedding into \mathbb{C} consequently its analytical properties are quite different from those of the real or complex numbers. Moreover the spectra of our norms here are *discrete*, more precisely $|\mathbb{C}((z - z_0))|_{c,z_0} = c^{\mathbb{Z}} \subset [0, +\infty]$ consequently the spectrum of $|\cdot|_{c,z_0}$ for all c, z_0 has only one accumulation point $0 \in \mathbb{R}$. Another essential difference is that, unlike \mathbb{R} , the algebraic function field $\mathbb{C}(z)$ is *not ordered*.

Let (M, g) be an oriented Riemannian m -manifold. Then M is equipped with a measure vol_g coming from the volume-form $d\text{vol}_g := *_g 1$ provided by the orientation and the metric; the corresponding measure of a measurable subset $\emptyset \subseteq A \subseteq M$ is $\text{vol}_g(A) := \int_M \chi_A d\text{vol}_g = \int_A d\text{vol}_g$ where $\chi_B : M \rightarrow \{0, 1\}$ is the characteristic function of any subset $\emptyset \subseteq B \subseteq M$. Clearly $0 \leq \text{vol}_g(A) \leq +\infty$ is a non-negative (extended) real number. Take finitely many measurable subsets $U_1, \dots, U_n \subset M$ whose closures are coordinate balls of finite volume but are pairwise almost non-intersecting; that is, $U_i \subset M$ has the property that \bar{U}_i is diffeomorphic to the standard closed ball $\bar{B}^m \subset \mathbb{R}^m$ moreover $0 < \text{vol}_g(U_i) < +\infty$ for all i but $\text{vol}_g(U_i \cap U_j) = 0$ for all $i \neq j$. Also take elements $R_1, \dots, R_n \in \mathbb{C}((z - z_0))$. A function $\varphi : M \rightarrow \mathbb{C}((z - z_0))$ of the form

$$\varphi := \sum_{j=1}^n R_j \chi_{U_j}$$

is called an *elementary step function*. This definition makes sense since \mathbb{R} acts on $\mathbb{C}((z - z_0))$ by multiplication; nevertheless these functions might be ill-defined in boundary points however, as we already anticipate from Lebesgue theory, ambiguities of this sort will be negligible concerning their integrals. The *integral* of an elementary step function against the measure induced by $d\text{vol}_g$ is defined as

$$\int_M \varphi d\text{vol}_g := \sum_{j=1}^n R_j \text{vol}_g(U_j) \in \mathbb{C}((z - z_0))$$

(φ is written sometimes as $R : M \rightarrow \mathbb{C}((z - z_0))$ and its integral as $\int_M R_x dx$, too) in full analogy with the usual case.

Next let us recall the two elementary but fundamental lemmata from [22] what we state here in appropriately modified forms and prove as follows.

Lemma 6.1. (cf. [22, Lemme A, p. 30]) *Let $\{\varphi_i\}_{i \in \mathbb{N}}$ be a sequence of elementary step functions from a compact oriented Riemannian manifold (M, g) into $\mathbb{C}((z - z_0))$. If $\{|\varphi_i|_{c, z_0}\}_{i \in \mathbb{N}}$ is strictly decreasing almost everywhere, then the integrals of these functions converge to zero in $\mathbb{C}((z - z_0))$.*

Proof. As mentioned above the spectrum of $|\cdot|_{c, z_0}$ has only one limit point $0 \in \mathbb{R}$ therefore if $\{|\varphi_i|_{c, z_0}\}_{i \in \mathbb{N}}$ strictly decreases almost everywhere then in fact $|\varphi_i(x)|_{c, z_0} \rightarrow 0$ if $x \in M \setminus B$ as $i \rightarrow +\infty$ where B is a subset of measure zero i.e., for any $\delta \geq 0$ there exist open subsets $V_\delta \subset M$ satisfying $\text{vol}_g(V_\delta) \leq \delta$ such that $\emptyset \subseteq B \subset V_\delta$. This means on the one hand that for every $\varepsilon \geq 0$ there exists an index i_ε such that for all $i \geq i_\varepsilon$ one finds

$$0 \leq \left| \int_{M \setminus B} \varphi_i d\text{vol}_g \right|_{c, z_0} \leq \sup_{x \in M \setminus B} |\varphi_i(x)|_{c, z_0} \leq \varepsilon .$$

On the other hand, if for any fixed i and $x \in B$ the lowest non-zero coefficient of $\varphi_i(x)$ in (8) is $a_{iK}(x) \in \mathbb{C}$ then the same coefficient of $\int_B \varphi_i d\text{vol}_g$ can be estimated from above by

$$\sup_{x \in B} |a_{iK}(x)| \text{vol}_g(V_\delta) \leq \sup_{x \in B} |a_{iK}(x)| \delta$$

and exploiting the compactness of M we can assume that the number of the different leading coefficients $a_{iK}(x)$ is finite as x runs over B hence surely $\sup_{x \in B} |a_{iK}(x)| < +\infty$. It then follows that the leading coefficient of the integral is arbitrary small hence $|\int_B \varphi_i d\text{vol}_g|_{c, z_0} = 0$ i.e., the integral over B vanishes for every fixed index i . Consequently $0 \leq |\int_M \varphi_i d\text{vol}_g|_{c, z_0} \leq \varepsilon$ for all $i \geq i_\varepsilon$. That is, the sequence of integrals converges to zero as stated. \diamond

Lemma 6.2. (cf. [22, Lemme B, p. 30]) *Let $\{\varphi_i\}_{i \in \mathbb{N}}$ be a sequence of elementary step functions from an oriented Riemannian manifold (M, g) into $\mathbb{C}((z - z_0))$. If $\{|\varphi_i|_{c, z_0}\}_{i \in \mathbb{N}}$ is increasing and the sequence $\{\int_M \varphi_i d\text{vol}_g\}_{i \in \mathbb{N}}$ of the corresponding integrals converges to an element in $\mathbb{C}((z - z_0))$, then $\{\varphi_i\}_{i \in \mathbb{N}}$ converges to a finite limit in $\mathbb{C}((z - z_0))$ almost everywhere.*

Proof. Let $\emptyset \subseteq B \subseteq M$ denote the collection of all of those points $x \in M$ where $\varphi_i(x)$ is divergent in $\mathbb{C}((z - z_0))$ as $i \rightarrow +\infty$. This can mean two (not necessarily mutually exclusive) things: either a sequence $\{a_{iK_i}(x)\}_{i \in \mathbb{N}}$ of coefficients in the expansions (8) of the $\varphi_i(x)$'s is divergent or $\{|\varphi_i(x)|_{c, z_0}\}_{i \in \mathbb{N}}$ is divergent i.e., the index set $\{K_i\}_{i \in \mathbb{N}}$ of the lowest non-zero $a_{iK_i}(x)$'s in the expansions of the $\varphi_i(x)$'s with $x \in B$ is unbounded from below. In either cases, since the sequence of the corresponding integrals $\int_M \varphi_i d\text{vol}_g$ converges to a well-defined element $R \in \mathbb{C}((z - z_0))$ with a well-defined expansion (8) whose coefficients are of the form $a_k \text{vol}_g(U)$, these divergences can be absent from the integral if and only if for every $\delta \geq 0$ there exist open subsets $\emptyset \subseteq B \subset V_\delta \subset M$ such that $\text{vol}_g(V_\delta) \leq \delta$. In other words B is of measure zero as stated. \diamond

From here we proceed in the standard way (cf. [22, §17]) hence we only quickly summarize the main steps. Let (M, g) be a compact oriented Riemannian manifold. If $C_0(M; \mathbb{C}((z - z_0)))$ is the set of elementary step functions from M to $\mathbb{C}((z - z_0))$ then let $C_1(M; \mathbb{C}((z - z_0)))$ denote the set of those

functions $f : M \rightarrow \mathbb{C}((z - z_0))$ which arise as limits of sequences of functions $\{\varphi_i\}_{i \in \mathbb{N}}$ in Lemma 6.2 i.e., arise almost everywhere as the limits $f(x) := \lim \varphi_i(x)$ of increasing elementary step functions with a convergent sequence of corresponding integrals. Define their *integrals*, which therefore exist, to be

$$\int_M f \, d\text{vol}_g := \lim_{i \rightarrow +\infty} \int_M \varphi_i \, d\text{vol}_g \in \mathbb{C}((z - z_0))$$

(again f is written sometimes as $R : M \rightarrow \mathbb{C}((z - z_0))$ and its integral as $\int_M R_x \, dx$, too). This definition is correct because, by referring to Lemma 6.1, the integral does not depend on the particular choice of the sequence $\{\varphi_i\}_{i \in \mathbb{N}}$ converging almost everywhere to a given f . The set $C_1(M; \mathbb{C}((z - z_0)))$ has already the structure of a vector space over $\mathbb{C}((z - z_0))$ and is closed and complete in an appropriate sense; it is more commonly denoted as $L^1(M; \mathbb{C}((z - z_0)))$ and called the *space of $\mathbb{C}((z - z_0))$ -valued Lebesgue integrable functions on M* (with respect to a measure coming from the orientation and metric on M).

The main purpose of these investigations is to complete the proof of Lemma 3.2 by demonstrating

Lemma 6.3. *Using the notations of Lemma 3.2, the map $\pi : Z(\mathbb{R}^4) \rightarrow p^{-1}([l_0])$ constructed by the integral (4) is well-defined and holomorphic. Consequently for every fixed $x \in \mathbb{C}P^2 \setminus \mathbb{R}^4$ this integral satisfies $\int_{S^2} R_{i^*y,x}(\pi_x(\ast)) \, d(i^*y) \in \mathbb{C}(\pi_x)$ i.e., is again a rational function in the variable π_x .*

Moreover, picking two points $x_1, x_2 \in \mathbb{C}P^2 \setminus \mathbb{R}^4$ and for every $(l, p) \in Z(\mathbb{R}^4)$ we find

$$\int_{S^2} R_{i^*y,x_1}(\pi_{x_1}((l, p))) \, d(i^*y) = \int_{S^2} R_{i^*y,x_2}(\pi_{x_2}((l, p))) \, d(i^*y)$$

hence taking into account the relation

$$\pi_{x_2}((l, p)) = \frac{P_{x_2,x_1}(\pi_{x_1}((l, p)))}{Q_{x_2,x_1}(\pi_{x_1}((l, p)))}$$

*as well, the change of the reference point x in (4) can be regarded as an algebraic change of variables in the coordinatized algebraic function field of rational functions. Therefore one can talk about an integral $\int_{S^2} R_{i^*y} \, d(i^*y) \in \mathbb{C}(z)$ in an abstract sense.*

Proof. First of all it easily follows that π exists since taking any $(l, p) \in Z(\mathbb{R}^4)$ the corresponding extended complex number $\pi((l, p)) \in p^{-1}([l_0]) \cong \mathbb{C}P^1$ is well-defined because it arises as the particular value of a Lebesgue integral; and this Lebesgue integral exists by the appropriate version of Lebesgue's dominated convergence theorem applied to the sequence of step functions converging almost everywhere to the bounded function $i^*y \mapsto R_{i^*y,x}$ on the compact S^2 equipped with its standard orientation and metric providing a measure $d(i^*y)$ on it. Regarding its holomorphicity, observe that the integral in (4) is nothing else than a limit:

$$\int_{S^2} R_{i^*y,x}(\pi_x((l, p))) \, d(i^*y) = \lim_{n \rightarrow +\infty} \sum_{j=1}^n R_{j,x}(\pi_x((l, p))) \, \text{vol}(U_j) = \lim_{n \rightarrow +\infty} \sum_{j=1}^n \frac{P_{j,x}(\pi_x((l, p)))}{Q_{j,x}(\pi_x((l, p)))} \, \text{vol}(U_j)$$

of integrals of elementary step functions. All the terms in these sums hence the sums themselves for any finite n are holomorphic. Holomorphicity here means that if

$$I_{(l,p)} : T_{(l,p)}Z(\mathbb{R}^4) \longrightarrow T_{(l,p)}Z(\mathbb{R}^4)$$

is the induced almost complex operator of $Z(\mathbb{R}^4)$ at (l, p) and

$$\pi_x((l, p))_* : T_{(l,p)}Z(\mathbb{R}^4) \longrightarrow T_{\pi_x((l,p))}p^{-1}([l_0])$$

is the derivative of $\pi_x : Z(R^4) \rightarrow p^{-1}([l_0])$ at (l, p) and

$$J_{\pi_x((l,p))} : T_{\pi_x((l,p))}p^{-1}([l_0]) \longrightarrow T_{\pi_x((l,p))}p^{-1}([l_0])$$

is the induced almost complex operator of $p^{-1}([l_0]) \cong \mathbb{C}P^1$ at $\pi_x((l, p))$ then the derivatives

$$(R_{j,x}(\pi_x((l, p)))\text{vol}(U_j))_* : T_{(l,p)}Z(R^4) \longrightarrow T_{\pi_x((l,p))}p^{-1}([l_0])$$

of the individual terms in the sum above, equal to

$$\left(\frac{P'_{j,x}(\pi_x((l, p)))Q_{j,x}(\pi_x((l, p))) - P_{j,x}(\pi_x((l, p)))Q'_{j,x}(\pi_x((l, p)))}{Q_{j,x}^2(\pi_x((l, p)))} \text{vol}(U_j) \right) \pi_x((l, p))_* ,$$

commute with the almost complex operators i.e.,

$$(R_{j,x}(\pi_x((l, p)))\text{vol}(U_j))_* \circ I_{(l,p)} = J_{\pi_x((l,p))} \circ (R_{j,x}(\pi_x((l, p)))\text{vol}(U_j))_*$$

for each $j = 1, 2, \dots, n$. However this property obviously survives the limit $n \rightarrow +\infty$ to be taken.

Last but not least, for every fixed $y \in \mathbb{C}P^2 \setminus R^4$ the map $\pi_y : Z(R^4) \rightarrow p^{-1}([l_0])$ constructed in (3) is well-defined consequently

$$R_{y,x_1}(\pi_{x_1}((l, p))) = \pi_y((l, p)) = R_{y,x_2}(\pi_{x_2}((l, p)))$$

demonstrating the equality of the corresponding integrals. \diamond

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