Exotica or the failure of the strong cosmic censorship in four dimensions

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Abstract

In this letter a generic counterexample to the strong cosmic censor conjecture is exhibited. More precisely—taking into account that the conjecture lacks any precise formulation yet—first we make sense of what one would mean by a "generic counterexample" by introducing the mathematically unambigous and logically stronger concept of a "robust counterexample". Then making use of Penrose' nonlinear graviton construction (i.e., twistor theory) and a Wick rotation trick we construct a smooth Ricci-flat but not flat Lorentzian metric on the largest member of the Gompf–Taubes uncountable radial family of large exotic $\mathbb{R}^4$'s. We observe that this solution of the Lorentzian vacuum Einstein's equations with vanishing cosmological constant provides us with a sort of counterexample which is weaker than a "robust counterexample" but still reasonable to consider as a "generic counterexample". It is interesting that this kind of counterexample exists only in four dimensions.

1 Introduction

Certainly one of the deepest open problems of contemporary classical general relativity is the validity or invalidity of the *strong cosmic censor conjecture* [19]. This is not only a technical conjecture of a particular branch of current theoretical physics: it deals with the very foundations of our rational description of Nature. Indeed, Penrose' original aim in the 1960-70's with formulating this conjecture was to protect causality in generic gravitational situations. We have the strong conviction that in the *classical* physical world at least, every physical event (possibly except the initial Big Bang) has a physical cause which is another and preceding physical event. Since mathematically speaking space-times having this property are called *globally hyperbolic* our requirement can be formulated roughly as follows (cf. e.g. [21, p. 304]):

SCCC. *A generic (i.e., stable), physically relevant (i.e., obeying some energy condition) space-time is globally hyperbolic.*

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We do not make an attempt here to survey the vast physical and mathematical literature triggered by the **SCCC** instead we refer to surveys [14, 18, 17]. Rather we may summarize the current situation as follows. During the course of time the originally single **SCCC** has fallen apart into several mathematical or physical versions, variants, formulations. For example there exists a generally working, mathematically meaningful but from a physical viewpoint rather weak version formulated in [21, p. 305] and proved in [4]. In another approach to the **SCCC** based on initial value formulation [21, Chapter 10], on the one hand, there are certain specific classes of space-times in which the **SCCC** allows a mathematically rigorous as well as physically contentful formulation whose validity can be established [18]; on the other hand counterexamples to the **SCCC** in this formulation also regularly appear in the literature however they are apparently too special, not "generic". In spite of these sporadic counterexamples the overall confidence in the physicist community is that an appropriate form of the **SCCC** must hold true hence causality is saved.

However we claim to exhibit a *generic counterexample to the* **SCCC**. What is then the resolution of the apparent contradiction between the well-known affirmative solutions and our negative result here? No compact smoothable topological 4-manifold is known carrying only one smooth structure. In fact in every well-understood case they admit not only more than one but countably infinitely many different smooth structures [10]. In the case of non-compact (relevant for physics) topological 4-manifolds there is even no obstruction against smooth structure and they typically accommodate an uncountable family of them [9]. The astonishing discovery of exotic (or fake) $\mathbb{R}^4$'s (i.e., smooth 4-manifolds which are homeomorphic but not diffeomorphic to the usual $\mathbb{R}^4$) by Donaldson, Freedman, Taubes and others in the 1980's is just the first example of the general situation completely absent in other dimensions. The cases for which the validity of the **SCCC** has been verified so far [14, 18] seem therefore to be atypical hence essentially negligible ones; on the contrary, our counterexample rests on a typical fake $\mathbb{R}^4$.

The only way to refute the general position adopted here when dealing with the **SCCC** was if one could somehow argue that general smooth 4-manifolds are too "exotic", "fake" or "weird" from the aspect of physical general relativity. However from the physical viewpoint if the "summing over everything" approach to quantum gravity is correct then very general unconventional but still physical space-times should be considered, too [3]; from the mathematical perspective non-linear partial differential equations like Einstein's equations are typically also solvable. Consequently it seems that both physically and mathematically speaking the true properties of general relativity cannot be revealed by understanding it only on simple atypical manifolds; the division of smooth 4-manifolds into "usual" and "unusual" ones can be justified only by conventionalism i.e., one has to evoke historical (and technical) arguments to pick up "usual" spaces from the bottomless sea of smooth 4-manifolds and abandon others. But looking at things optimistically, if it is true that the nature of (quantum) general relativity is genuinely not deterministic (as our result suggests) then this may open up the exciting possibility that the indeterministic character of quantum physics has a quantum gravitational origin.

Our notational convention throughout the text is that $\mathbb{R}^4$ will denote the four dimensional real vector space equipped with its standard differentiable manifold structure whilst R^4 or R_t^4 will denote various exotic (or fake) variants. The notation "$\cong$" will always mean "diffeomorphic to" whilst homeomorphism always will be spelled out as "homeomorphic to". Finally we note that all set theoretical or topological operations (i.e., $\subseteq$, $\cap$, $\cup$, taking open or closed subsets, closures, complements, etc.) will be taken in a manifold M with its well-defined standard manifold topology throughout the text. In particular for $\mathbb{R}^4$ or the R_t^4's this topology is the unique underlying manifold topology.

2   Definition and construction of a counterexample

In agreement with the common belief in the physicist and mathematician community, formulating the strong cosmic censor conjecture in a mathematically rigorous way is obstacled by lacking an overall satisfactory concept of "genericity". Consequently the main difficulty to find a "generic counterexample" to the **SCCC** lies not in its actual finding (indeed, most of the well-known basic solutions of Einstein's equations provide violations of it) but rather in proving that the particular counterexample is "generic". In this section we outnavigate this problem by mathematically formulating the concept of a certain counterexample which is logically stronger than a "generic counterexample" to the **SCCC**. Then we search for a counterexample of this kind making use of uncountably many large exotic $\mathbb{R}^4$'s.

   A standard reference here is [21, Chapters 8,10]. By a *space-time* we mean a connected, four dimensional, smooth, time-oriented Lorentzian manifold without boundary. By a *(continuous) Lorentzian manifold* we mean the same thing except that the metric is allowed to be a continuous tensor field only.

Definition 2.1. *Let (S,h,k) be an initial data set for Einstein's equations with (S,h) a connected complete Riemannian 3-manifold and with a fundamental matter represented by a stress-energy tensor T obeying the dominant energy condition. Let $(D(S),g|_{D(S)})$ be the unique maximal Cauchy development of this initial data set. Let (M,g) be a further maximal extension of $(D(S),g|_{D(S)})$ as a (continuous) Lorentzian manifold if exists. That is, $(D(S),g|_{D(S)}) \subseteqq (M,g)$ is a (continuous) isometric embedding which is proper if $(D(S),g|_{D(S)})$ is still extendible and (M,g) does not admit any further proper isometric embedding. (If the maximal Cauchy development is inextendible then put simply $(M,g) := (D(S),g|_{D(S)})$ for definiteness.)*
   The (continuous) Lorentzian manifold (M',g') is a perturbation *of (M,g) relative to (S,h,k) if*

(i) *M' has the structure*

$$M' := \text{the connected component of } M \setminus \mathscr{H} \text{ containing } S$$

   where, for a connected open subset $S \subset U \subseteqq M$ containing the initial surface, $\emptyset \subseteqq \mathscr{H} \subseteqq \partial U$ is a closed subset in the boundary of U (consequently M' is open in M hence inherits a differentiable manifold structure);

(ii) *g' is a solution of Einstein's equations at least in a neighbourhood of $S \subset M'$ with a fundamental matter represented by a stress-energy tensor T' obeying the dominant energy condition at least in a neighbourhood of $S \subset M'$;*

(iii) *(M',g') does not admit further proper isometric embeddings and $(S,h') \subset (M',g')$ with $h' := g'|_S$ is a spacelike complete sub-3-manifold.*

Remark. 1. It is crucial that in the spirit of relativity theory we consider metric perturbations of the *four* dimensional space-time (whilst keeping its underlying smooth structure fixed)—and not those of a *three* dimensional initial data set. This natural class of perturbations is therefore immense: it contains all connected manifolds containing the initial surface but perhaps being topologically different from the original manifold. The perturbed metric is a physically relevant solution of Einstein's equations at least in the vicinity of $S \subset M'$ such that (M',g') is inextendible and $(S,h') \subset (M',g')$ is still spacelike and complete. In other words these perturbations are physical solutions allowed to blow up along closed " boundary subsets" $\emptyset \subsetneqq \mathscr{H} \subset M$; the notation $\mathscr{H}$ for these subsets indicates that among them the (closure of the) Cauchy horizon $H(S)$ of (S,h,k) may also appear. Beyond the non-singular perturbations with $\mathscr{H} = \emptyset$ of any space-time a prototypical example with $\mathscr{H} \neq \emptyset$ is the physical perturbation

(M',g') of the (maximally extended) undercharged Reissner–Nordström space-time (M,g) by taking into account the full backreaction of a pointlike particle or any classical field put onto the originally pure electro-vacuum space-time ("mass inflation"). In this case the singularity subset $\mathscr{H}$ is expected to coincide with the (closure of the) full inner event horizon of the Reissner–Nordström black hole which is the Cauchy horizon for the standard initial data set inside the maximally extended space-time [19]. A similar perturbation of the Kerr–Newman space-time is another example with $\mathscr{H} \neq \emptyset$.

2. Accordingly, notice that in the above definition of perturbation none of the terms "generic" or "small" have been used. This indicates that if such types of perturbations can be somehow specified then one should be able to recognize them among the very general but still physical perturbations of a space-time as formulated in Definition 2.1.

Now we are in a position to formulate in a mathematically precise way what we mean by a "robust counterexample" to the **SCCC** as formulated roughly in the Introduction.

Definition 2.2. *Let (S,h,k) be an initial data set for Einstein's equations with (S,h) a connected complete Riemannian 3-manifold and with a fundamental matter represented by a stress-energy tensor T obeying the dominant energy condition. Assume that the maximal Cauchy development of this initial data set is extendible i.e., admits a (continuous) isometric embedding as a proper open submanifold into an inextendible (continuous) Lorentzian manifold (M,g).*

Then (M,g) is a robust counterexample to the **SCCC** *if it is very stably non-globally hyperbolic i.e., all of its perturbations (M',g') relative to (S,h,k) are not globally hyperbolic.*

Remark. 1. Concerning its logical status it is reasonable to consider this as a *generic* counterexample because the perturbation class of Definition 2.1 is expected to contain all "generic perturbations" whatever they are. Consequently in Definition 2.2 we are dealing with a stronger statement than the logical negation of the affirmative sentence in **SCCC**.

2. The trivial perturbation i.e., the extension (M,g) itself in Definition 2.2 cannot be globally hyperbolic as observed already in [4, Remark after Theorem 2.1].

Strongly influenced by [1, 2] we take now an excursion into the weird world of four dimensional exotic *ménagerie* (or rather *plethora*) in order to attack the **SCCC**. A standard reference here is [10, Chapter 9]. Our construction is based on a specific fake $\mathbb{R}^4$ whose proof of existence is very involved: it is based on the works of Gompf [7, 8] and Taubes [20]. Those properties of this exotic $\mathbb{R}^4$ which will be used here are summarized as follows (cf. [10, Lemma 9.4.2, Addendum 9.4.4 and Theorem 9.4.10]):

Theorem 2.1. *There exists a pair (R^4, K) consisting of a differentiable 4-manifold R^4 homeomorphic but not diffeomorphic to the standard $\mathbb{R}^4$ and a compact oriented smooth 4-manifold $K \subset R^4$ such that*

(i) *R^4 cannot be smoothly embedded into the standard $\mathbb{R}^4$ i.e., $R^4 \nsubseteq \mathbb{R}^4$ but it can be smoothly embedded as a proper open subset into the complex projective plane i.e., $R^4 \subsetneqq_{\neq} \mathbb{C}P^2$;*

(ii) *Take a homeomorphism $f : \mathbb{R}^4 \to R^4$, let $0 \in B_t^4 \subset \mathbb{R}^4$ be the standard open 4-ball of radius $t \in \mathbb{R}^+$ centered at the origin and put $R_t^4 := f(B_t^4)$ and $R_{+\infty}^4 := R^4$. Then*

$$\left\{ R_t^4 \mid r \leqq t \leqq +\infty \text{ such that } 0 < r < +\infty \text{ satisfies } K \subset R_r^4 \right\}$$

is an uncountable family of nondiffeomorphic exotic $\mathbb{R}^4$'s none of them admitting a smooth embedding into $\mathbb{R}^4$ i.e., $R_t^4 \nsubseteq \mathbb{R}^4$ for all $r \leqq t \leqq +\infty$.

In what follows this family will be referred to as the radial family of large exotic $\mathbb{R}^4$'s. ◇

Remark. From Theorem 2.1 we deduce that for all $r < t < +\infty$ there is a sequence of smooth proper embeddings

$$R_r^4 \subsetneqq R_t^4 \subsetneqq R_{+\infty}^4 = R^4 \subsetneqq \mathbb{C}P^2$$

which are very wild: the complement $\mathbb{C}P^2 \setminus R^4$ is homeomorphic to S^2 (regarded as an only continuously embedded projective line in the projective plane) consequently it does not contain any open 4-ball in $\mathbb{C}P^2$; hence in particular if $\mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1$ is any holomorphic decomposition then $R^4 \cap \mathbb{C}P^1 \neq \emptyset$ (because otherwise $R^4 \subseteqq \mathbb{C}^2 \cong \mathbb{R}^4$ would hold, a contradiction). This demonstrates that the members of the large radial family live "somewhere between" $\mathbb{C}^2$ and its projective closure $\mathbb{C}P^2$. However a more precise identification or location of them is a difficult task because these large exotic $\mathbb{R}^4$'s—although being honest differentiable 4-manifolds—are very transcendental objects [10, p. 366]: they require infinitely many 3-handles in any handle decomoposition (like any other known large exotic $\mathbb{R}^4$) and there is presently[1] no clue as how one might draw explicit handle diagrams of them (even after removing their 3-handles). We note that the structure of small exotic $\mathbb{R}^4$'s i.e., which admit smooth embeddings into $\mathbb{R}^4$, is better understood, cf. [10, Chapter 9].

We proceed further and construct a solution of the Lorentzian vacuum Einstein's equations (with vanishing cosmological constant) on R^4 by the aid of Penrose' nonlinear graviton construction [16].

Theorem 2.2. *The space R^4 from Theorem 2.1 carries a smooth Lorentzian Ricci-flat metric g. Moreover there exists an open (i.e., non-compact without boundary) contractible (hence connected) spacelike and complete sub-3-manifold $(S, h) \subset (R^4, g)$ in it such that $h = g|_S$.*
The Ricci-flat Lorentzian manifold (R^4, g) might be timelike and (or) null geodesically incomplete.

Proof. The proof consists of two steps: (i) we construct a *Riemannian* Ricci-flat metric on R^4 via twistor theory by exploiting the embedding $R^4 \subset \mathbb{C}P^2$; (ii) "Wick rotate" this solution into a *Lorentzian* one by exploiting the contractibility of R^4.

(i) The original nonlinear graviton construction of Penrose [16], as summarized very clearly in [12] or [13, §4], consists of the following data:

* A complex 3-manifold Z, the total space of a holomorphic fibration $\pi : Z \to \mathbb{C}P^1$;

* A complex 4-paremeter family of sections $Y \subset Z$, each with normal bundle $H \oplus H$ (here H is the dual of the tautological bundle i.e., the unique holomorphic line bundle on $Y \cong \mathbb{C}P^1$ with $\langle c_1(H), [Y] \rangle = 1$);

* A non-vanishing holomorphic section s of $K_Z \otimes \pi^* H^4$ (here K_Z is the canonical bundle of Z);

* A real structure $\tau : Z \to Z$ such that π and s are compatible and Z is fibered by the real sections of the family (here $\mathbb{C}P^1$ is given the real structure of the antipodal map $u \mapsto -\bar{u}^{-1}$).

These data allow one to construct a Ricci-flat and self-dual (i.e., the Ricci tensor and the antiself-dual part of the Weyl tensor vanishes) solution (M, g) of the *Riemannian* Einstein's vacuum equations (with vanishing cosmological constant) in a well-known way. The holomorphic lines $Y \subset Z$ form a locally complete family and fit together into a complex 4-manifold $M^{\mathbb{C}}$. This space carries a natural complex conformal structure by declaring two nearby points $y', y'' \in M^{\mathbb{C}}$ to be null-separated if the corresponding lines intersect i.e., $Y' \cap Y'' \neq \emptyset$ in Z. Infinitesimally this means that on every tangent space $T_y M^{\mathbb{C}} = \mathbb{C}^4$ a null cone is specified. Restricting this to the real lines parameterized by an embedded real 4-manifold

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[1]More precisely in the year 1999, cf. [10].

$M \subset M^{\mathbb{C}}$ we obtain the real conformal class $[g]$ of a Riemannian metric on M. The isomorphism $s : K_Z \cong \pi^* H^{-4}$ is essentially uniquely fixed by its holomorphic and reality properties and gives rise to a volume form on M this way fixing the metric g in the conformal class. Since Z can be identified with the projective negative chiral spinor bundle $P\Sigma^-$ over M we obtain a smooth twistor fibration $p : Z \to M$ whose fibers are $\mathbb{C}P^1$'s hence $\pi : Z \to \mathbb{C}P^1$ can be regarded as a parallel translation with respect to a flat connection which is nothing but the induced negative spin connection of g on Σ^-. This partial flatness implies that g is Ricci-flat and self-dual. For more details cf. [12, 13].

In the case of our large exotic $\mathbb{R}^4$ i.e., R^4 from Theorem 2.1 these data arise as follows. Putting $Z := P(T^*\mathbb{C}P^2)$ to be the projective cotangent bundle we obtain the twistor fibration $p : Z \to \mathbb{C}P^2$ of the complex projective space. More classically Z can be viewed as the flag manifold $F_{12}(\mathbb{C}^3)$ consisting of pairs $(\mathfrak{l}, \mathfrak{p})$ where $0 \in \mathfrak{l} \subset \mathbb{C}^3$ is a line and $\mathfrak{l} \subset \mathfrak{p} \subset \mathbb{C}^3$ is a plane containing the line. The projection sends $(\mathfrak{l}, \mathfrak{p}) \in F_{12}(\mathbb{C}^3) \cong Z$ into $[\mathfrak{l}] \in \mathbb{C}P^2$. This is a smooth $\mathbb{C}P^1$-fibration over $\mathbb{C}P^2$. Part (i) of Theorem 2.1 tells us that $R^4 \subset \mathbb{C}P^2$. Writing $Z' := Z|_{R^4}$ and $p' := p|_{Z'}$ the restricted twistor fibration $p' : Z' \to R^4$ is topologically trivial i.e., Z' is homeomorphic to $R^4 \times S^2 \cong \mathbb{R}^4 \times S^2$ because R^4 is contractible.[2] Take an $(\mathfrak{l}, \mathfrak{p}) \in Z'$ and choose a local complex coordinate $z \in \mathbb{C}$ on the corresponding projective line $[\mathfrak{p}] \subset \mathbb{C}P^2$ such that the point $[\mathfrak{l}] \in [\mathfrak{p}]$ satisfies $z([\mathfrak{l}]) = 0$. Define the unique element $[\mathfrak{l}_\infty] \in [\mathfrak{p}]$ by the two infima

$$\left| z([\mathfrak{l}'']) \right| := \inf_{[\mathfrak{l}'] \in [\mathfrak{p}] \cap (\mathbb{C}P^2 \setminus R^4)} \left| z([\mathfrak{l}']) \right| \in [0, +\infty) \ , \qquad \arg z([\mathfrak{l}_\infty]) := \inf_{[\mathfrak{l}''] \in [\mathfrak{p}] \cap (\mathbb{C}P^2 \setminus R^4)} \arg z([\mathfrak{l}'']) \in [0, 2\pi) \ .$$

The assignment $[\mathfrak{p}] \mapsto [\mathfrak{l}_\infty] \in \mathbb{C}P^2 \setminus R^4$ always satisfies $[\mathfrak{l}_\infty] \neq [\mathfrak{l}]$ because $[\mathfrak{l}] \in R^4$ is an inner point. Pick any projective line $\ell \subset \mathbb{C}P^2 \setminus \{[\mathfrak{l}]\}$ and identify $(\mathbb{C}P^2 \setminus \{[\mathfrak{l}]\}, \ell)$ with the line bundle $(H, \mathbb{C}P^1)$ such that $\mathbb{C}P^2 \setminus \{[\mathfrak{l}]\}$ is the total space H, ℓ is the zero section hence the base $\mathbb{C}P^1$ and the punctured projective lines $[\mathfrak{p}] \setminus \{[\mathfrak{l}]\} \subset \mathbb{C}P^2 \setminus \{[\mathfrak{l}]\}$ through $[\mathfrak{l}] \in \mathbb{C}P^2$ represent the fibers of the line bundle. This way we can regard the assignment $[\mathfrak{p}] \mapsto [\mathfrak{l}_\infty]$ as a section $s : \ell \to \mathbb{C}P^2 \setminus \{[\mathfrak{l}]\}$. The Fubini–Study metric gives rise to a fiberwise Hermitian metric on $\mathbb{C}P^2 \setminus \{[\mathfrak{l}]\}$ turning s into an L^2-section. Its L^2-orthogonal projection onto the 2 dimensional subspace of holomorphic sections gives rise to a unique holomorphic section what we continue to write as $s : \ell \to \mathbb{C}P^2 \setminus \{[\mathfrak{l}]\}$. Consequently this holomorphic section yields a well-defined *holomorphic assignment* what we denote by $[\mathfrak{p}] \mapsto [\mathfrak{l}_\infty]$ as before.[3] Fix a point $[\mathfrak{l}_0] \in R^4$ and using this assignment define $\pi' : Z' \to p'^{-1}([\mathfrak{l}_0]) \cong \mathbb{C}P^1$ by putting

$$\pi'((\mathfrak{l}, \mathfrak{p})) := (\mathfrak{l}_0, \mathfrak{p}_0) \text{ where } \mathfrak{p}_0 \supset \mathfrak{l}_0 \text{ in } \mathbb{C}^3 \text{ satisfies that } [\mathfrak{p}_0] \subset \mathbb{C}P^2 \text{ connects } [\mathfrak{l}_0] \text{ with } [\mathfrak{l}_\infty] \text{ assigned to } [\mathfrak{p}]$$

(see Fig. 1.). By construction this is a holomorphic map whose fibers are diffeomorphic to R^4. Then restricting everything onto $R^4 \subset \mathbb{C}P^2$ it is immediate that Z' contains the complex 4-parameter family $(R^4)^{\mathbb{C}}$ of holomorphic lines and the corresponding real lines parameterize R^4. The map π' is compatible with the real structure. A non-vanishing holomorphic real section s of $K_{Z'} \otimes \pi'^* H^4$ then fixes the *Riemannian* metric g_1 on R^4 which is Ricci-flat.
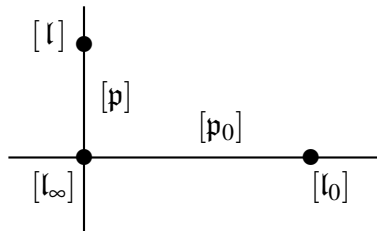

Figure 1. Construction of the map π'.

We proceed further and demonstrate that (R^4, g_1) is complete. Since both the Fubini–Study metric g_0 and this Ricci-flat metric g_1 stem from the same complex structure on the twistor space we know from twistor theory that these metrics are in fact conformally equivalent. Therefore there exists a smooth non-constant positive function $\varphi : R^4 \to \mathbb{R}^+$ such that $\varphi^{-2} \cdot (g_0|_{R^4}) = g_1$ and satisfying with respect to g_0 the following equations:

$$\begin{cases} \Delta\varphi^{-1} + \frac{1}{6}\mathrm{scal}(g_0|_{R^4})\varphi^{-1} & = 0 \text{ (vanishing of the scalar curvature of } g_1); \\ \nabla^2\varphi + \frac{1}{4}\Delta\varphi \cdot (g_0|_{R^4}) & = 0 \text{ (vanishing of the traceless Ricci tensor of } g_1). \end{cases} \tag{1}$$

Taking into account that the scalar curvature of the Fubini–Study metric is constant it follows from the first equation of (1) and the maximum principle that φ^{-1} diverges along $\mathbb{C}P^2 \setminus R^4$ hence on the one hand the function $\log\varphi^{-1} : R^4 \to \mathbb{R}$ is a proper function on R^4.

Denoting by X the dual vector field to $d\varphi$ with respect to g_0 the decomposition of the $(0,2)$-type symmetric tensor field $\nabla^2\varphi$ into trace and traceless symmetric parts gives

$$\nabla^2\varphi + \frac{1}{4}\Delta\varphi \cdot (g_0|_{R^4}) = \frac{1}{2}\left(L_X(g_0|_{R^4}) + \frac{1}{2}\Delta\varphi \cdot (g_0|_{R^4}) \right)$$

hence the second equation of (1) says that X (or φ) satisfies the *conformal Killing equation* with respect to the (restricted) Fubini–Study metric: $L_X(g_0|_{R^4}) + \frac{1}{2}\Delta\varphi \cdot (g_0|_{R^4}) = 0$. The conformal Killing equation for X can be prolongated in a well-known way i.e., can be re-written in terms of the conformal Killing data $(d\varphi, d^2\varphi = 0, -\Delta\varphi, -d(\Delta\varphi))$ for X as a system of differential equations (cf. [6, Eqn. B.3]). The relevant equation for us deals with the fourth data and on the Einstein manifold $(R^4, g_0|_{R^4})$ with constant scalar curvature takes the shape $\nabla(d(\Delta\varphi)) = -\frac{1}{12}\mathrm{scal}(g_0|_{R^4})\Delta\varphi \cdot (g_0|_{R^4})$. Combining this with the second equation of (1) leads to

$$\nabla\left(d\left(\Delta\varphi - \frac{1}{3}\mathrm{scal}(g_0|_{R^4})\varphi \right) \right) = 0 \ .$$

The restricted Fubini–Study geometry is still irreducible in the sense that its holonomy group acts irreducibly on the tangent spaces hence we conclude that in fact $d\left(\Delta\varphi - \frac{1}{3}\mathrm{scal}(g_0|_{R^4})\varphi \right) = 0$ i.e., there exists $c_1 \in \mathbb{R}$ such that

$$\Delta\varphi = \frac{1}{3}\mathrm{scal}(g_0|_{R^4})\varphi + c_1 \tag{2}$$

holds. It again follows from the maximum principle that surely $c_1 \neq 0$. Adjusting the standard identity $0 = (\Delta\varphi)\varphi^{-1} + 2g_0(d\varphi, d\varphi^{-1}) + \varphi\Delta\varphi^{-1}$ as $\varphi^2|d\varphi^{-1}|_{g_0}^2 = \frac{1}{2}(\varphi\Delta\varphi^{-1} + \varphi^{-1}\Delta\varphi)$, plugging the first equation of (1) as well as (2) into it and carefully writing $|\xi|_{g_1} = \varphi|\xi|_{g_0}$ on 1-forms we obtain on the other hand the estimate

$$|d(\log\varphi^{-1})|_{g_1}^2 = \varphi^4|d\varphi^{-1}|_{g_0}^2 = \frac{1}{2}\left(\varphi^3\Delta\varphi^{-1} + \varphi\Delta\varphi \right) = \frac{1}{12}\mathrm{scal}(g_0|_{R^4})\varphi^2 + \frac{1}{2}c_1\varphi \leqq c_2$$

with some $c_2 \in \mathbb{R}^+$ because $\mathrm{scal}(g_0|_{R^4})$ is constant and φ is bounded. Recalling a classical result of Gordon [11] a Riemannian manifold is complete if and only if it admits an at least C^3 proper function whose gradient is bounded in modulus. Since $\log\varphi^{-1} : R^4 \to \mathbb{R}$ satisfies these conditions we conclude that the Ricci flat space (R^4, g_1) is moreover complete.[4]

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[4]The metric g_1 is additionally self-dual and R^4 is simply connected hence (R^4, g_1) is in fact a *hyper-Kähler gravitational instanton*. Therefore this geometry is expected to make a dominant contribution to the Euclidean quantum gravitational partition function, cf. [3].

(ii) Next we "Wick rotate" this Riemannian solution into a Lorentzian one. We begin with the construction of a Riemannian sub-3-manifold $(S,h) \subset (R^4,g_1)$. The boundary of the unit disk bundle inside the total space of the line bundle H over $\mathbb{C}P^1$ is a circle bundle over its zero section $\mathbb{C}P^1$ more precisely a Hopf fibration hence is a 3-manifold homeomorphic to S^3. Fixing an $[\mathfrak{l}_\infty] \in \mathbb{C}P^2 \setminus R^4$ we identify again the total space H with $\mathbb{C}P^2 \setminus \{[\mathfrak{l}_\infty]\}$ and denote by $N \subset \mathbb{C}P^2 \setminus \{[\mathfrak{l}_\infty]\}$ the image of this 3-manifold. Define

$$S := \text{one connected component of } N \cap R^4 \ .$$

Every exotic $\mathbb{R}^4$ in general hence our R^4 in particular, has the property that it contains a compact subset $C \subset R^4$ which cannot be surrounded by a smoothly embedded $S^3 \subset R^4$ [10, Exercise 9.4.1]. Taking the radii of the constituent circles of N sufficiently large we can suppose by the compactness of C that $C \cap S = \emptyset$ i.e., S could surround C if S was homeomorphic to S^3. This would be a contradiction hence $S \subset R^4$ is an open (i.e., non-compact without boundary) and connected sub-3-manifold of R^4. Therefore, exploiting the contractibility of R^4 we conclude that S is an open contractible sub-3-manifold within R^4. Putting $h := g_1|_S$ we therefore obtain an open contractible Riemannian sub-3-manifold $(S,h) \subset (R^4,g_1)$ which is complete.

Consider $(S,h) \subset (R^4,g_1)$ constructed above. Pick a real line bundle L over R^4 such that $L \subset TR^4$ and its orthogonal complement $L^\perp \subset TR^4$ within TR^4 satisfies $L^\perp|_S \cong TS$. Moreover take the complex tangent bundle $T(R^4)^{\mathbb{C}}$ of the complexification $(R^4)^{\mathbb{C}}$ and restrict it onto $R^4 \subset (R^4)^{\mathbb{C}}$. This $T(R^4)^{\mathbb{C}}|_{R^4}$ is a trivial smooth rank-4 complex vector bundle over R^4 and obviously contains the real tangent bundle TR^4. Consider the imaginary line bundle $\mathbf{i}L \subset T(R^4)^{\mathbb{C}}|_{R^4}$. We claim that L can be fiberwisely rotated into $\mathbf{i}L$ within $T(R^4)^{\mathbb{C}}|_{R^4}$ in a continuous manner over the whole R^4. To see this let $\mathscr{G}$ denote the gauge group of the complex vector bundle $T(R^4)^{\mathbb{C}}|_{R^4}$ consisting of smooth fiberwise $\mathrm{SO}(4,\mathbb{C})$-transformations (provided by the complexification of the metric constructed on R^4 by twistor theory). Assume that an element $\alpha \in \mathscr{G}$ satisfies

$$\alpha L = \mathbf{i}L \ .$$

Then if $\beta_L \in \mathscr{G}$ and $\beta_{\mathbf{i}L} \in \mathscr{G}$ are rotations fixing L and $\mathbf{i}L$ within $T(R^4)^{\mathbb{C}}|_{R^4}$ respectively then

$$(\beta_{\mathbf{i}L}^{-1} \alpha \beta_L)L = \mathbf{i}L$$

holds as well. The fiberwise stabilizers of both L and $\mathbf{i}L$ are isomorphic to $\mathbb{Z}_2 \subset \mathrm{SO}(4,\mathbb{C})$ therefore the existence of an $\alpha \in \mathscr{G}$ implies that a principal $\mathbb{Z}_2 \times \mathbb{Z}_2$-bundle $P_{L,\mathbf{i}L}$ over R^4 (within the $\mathrm{SO}(4,\mathbb{C})$-bundle providing the gauge group $\mathscr{G}$) given by the relative positions of L and $\mathbf{i}L$ within $T(R^4)^{\mathbb{C}}|_{R^4}$ admits a continuous section $(\beta_L, \beta_{\mathbf{i}L}) \in C^0(R^4; P_{L,\mathbf{i}L})$. Consequently this principal bundle, the "obstruction bundle" of the rotation, must be trivial otherwise $\alpha \in \mathscr{G}$ cannot exists. Standard obstruction theory says that the only obstruction class against $P_{L,\mathbf{i}L}$ to be trivial lives in the cohomology group

$$H^1(R^4 \,;\, \pi_0(\mathbb{Z}_2 \times \mathbb{Z}_2)) \cong H^1(R^4 \,;\, \mathbb{Z}_2) \times H^1(R^4 \,;\, \mathbb{Z}_2) \ .$$

However referring to the contractibility of R^4 once again we conclude that $H^1(R^4 \,;\, \mathbb{Z}_2) = 0$ hence the continuous "Wick rotation" of $L \subset TR^4$ into $\mathbf{i}L \subset T(R^4)^{\mathbb{C}}|_{R^4}$ can be performed.

The real structure on Z' cuts out $R^4 \subset (R^4)^{\mathbb{C}}$ and its infinitesimal form at $x \in R^4 \subset (R^4)^{\mathbb{C}}$ gives rise to a real subspace $\mathbb{R}^4 = T_x R^4 \subset T_x(R^4)^{\mathbb{C}} = \mathbb{C}^4$; in addition twistor theory equips $T_x R^4$ with a real scalar product $(g_1)_x$, too. Taking the complex linear extension of this real scalar product we obtain a complex scalar product on $T_x(R^4)^{\mathbb{C}}$ yielding an inclusion of the corresponding spin groups

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}(4,\mathbb{C}) \ .$$

Since $T_x R^4 = L_x \oplus L_x^\perp$ the complex scalar product restricted to $\mathbb{R}^4 = \mathbf{i}L_x \oplus L_x^\perp \subset T_x(R^4)^{\mathbb{C}} = \mathbb{C}^4$ gives an indefinite real scalar product with its associated real spin group

$$\mathrm{Spin}(3,1) \cong \mathrm{SL}(2,\mathbb{C}) \subset \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}(4,\mathbb{C})$$

being diagonally embedded into the complex spin group. Therefore, on the one hand, the real rank-4 bundle $\mathbf{i}L \oplus L^\perp \subset T(R^4)^{\mathbb{C}}|_{R^4}$ over R^4 carries a metric g which is Lorentzian and continues to be Ricci-flat but not flat because it follows from the above analysis of the spin groups that its full Weyl tensor is not zero. On the other hand $\mathbf{i}L \oplus L^\perp$ can be identified with TR^4 because both bundles are trivial. Consequently we obtain a *Lorentzian* Ricci-flat manifold what we call (R^4, g). It also possesses a complete spacelike $(S,h) \subset (R^4, g)$ which is nothing but the previously constructed $(S,h) \subset (R^4, g_1)$.

We conclude that R^4 admits a solution of the Lorentzian vacuum Einstein's equations as desired. However this solution might be incomplete in the non-spacelike directions. ◇

After this technical warm-up we inspect (R^4, g) concerning its global hyperbolicity.

Lemma 2.1. *Consider the pair (R^4, K) from Theorem 2.1 and the Ricci-flat Lorentzian manifold (R^4, g) of Theorem 2.2 with its open contractible spacelike and complete sub-3-manifold $(S,h) \subset (R^4, g)$. Let (S,h,k) be the initial data set inside (R^4, g) induced by (S,h) and let (M',g') be a perturbation of (R^4,g) relative to (S,h,k) as in Definition 2.1.*

Assume that $K \subset M'$ holds. Then (M',g') is not globally hyperbolic.

Proof. First we prove that the trivial perturbation i.e., (R^4, g) itself is not globally hyperbolic. Since R^4 is an exotic $\mathbb{R}^4$ then by a result of McMillen [15] it does not admit a smooth splitting like $R^4 \cong W \times \mathbb{R}$ where W is an open contractible 3-manifold. Hence it follows from the smooth splitting theorem for globally hyperbolic space-times [1] that (R^4, g) cannot be globally hyperbolic.[5] Consequently the initial data set (S,h,k) induced by $(S,h) \subset (R^4, g)$ is only a partial initial data set inside (R^4, g).

Let us secondly consider its non-trivial perturbations (M',g') relative to (S,h,k) satisfying $K \subset M'$. Suppose that (M',g') is globally hyperbolic. Referring to Definition 2.1 we know that $(S,h') \subset (M',g')$ is a complete spacelike submanifold hence we can use it to obtain an initial data set (S,h',k') for (M',g'). Again by [1] we find $M' \cong S \times \mathbb{R}$ and taking into account that S is an open contractible manifold we can refer again to [15] to conclude that $M' \cong \mathbb{R}^4$. Essentially by Uryshon's lemma we can find in part (ii) of Theorem 2.1 a homeomorphism $f : \mathbb{R}^4 \to R^4$ and a value $r \leqq t_0 \leqq +\infty$ such that with the corresponding exotic space $R^4_{t_0} = f(B^4_{t_0})$ a sequence of smooth embeddings

$$K \subsetneqq R^4_{t_0} \subseteqq M' \subseteqq R^4_{+\infty} = R^4$$
$$\quad\quad\quad \wr\| $$
$$\quad\quad\quad \mathbb{R}^4$$

exists. However this is a contradiction because $R^4_{t_0}$ is a member of the radial family of large exotic $\mathbb{R}^4$'s of Theorem 2.1 consequently it cannot be smoothly embedded into $M' \cong \mathbb{R}^4$. This demonstrates that our supposition was wrong hence (M',g') cannot be globally hyperbolic as well. ◇

Remark. The simple assumption $K \subset M'$ says that the perturbation about $S \subset R^4$ is large enough in the topological sense hence is capable to "scan" the exotic regime of R^4. In fact this condition is effectively necessary to exclude globally hyperbolic perturbations of (R^4, g). Taking $M' := N_\varepsilon(S) \subset R^4$ to be a small tubular neighbourhood of $S \subset R^4$ then the contractibility of S implies $N_\varepsilon(S) \cong S \times \mathbb{R}$ hence again

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[5]Or simply we can refer to [2, Theorem A] to get this result.

by [15] we know that $N_\varepsilon(S) \cong \mathbb{R}^4$. Therefore putting g' just to be the standard Minkowski metric on M' then (M', g') is the usual Minkowski space-time hence is a globally hyperbolic perturbation of (R^4, g) relative to (S, h, k). This perturbation is "small" in the topological sense above however might be "large" in any analytical sense i.e., the corresponding (S, h', k') might siginificantly deviate from the original (S, h, k).

Taking into account that the class of perturbations (M', g') of (R^4, g) relative to (S, h, k) has to satisfy a non-trivial condition $K \subset M'$ in order to be non-globally hyperbolic the space (R^4, g) is not a robust counterexample to the **SCCC** in the strict sense of Definition 2.2. However this condition is just a mild topological one hence the corresponding perturbation class is certainly still enormously vast. Therefore in our opinion it is reasonable to say that *the Ricci-flat Lorentzian space-time* (R^4, g) *is a generic counterexample to the* **SCCC** as formulated in the Introduction (recall that being generic is not a well-defined concept). We also have the suspicion that this particular case sheds light onto the general situation i.e., when the space-time is modelled on a general non-compact smooth 4-manifold [9, 10]. That is, we suspect that the **SCCC** typically fails in four dimensions!

3   Conclusion and outlook

From the viewpoint of low dimensional differential topology it is not surprising that confining ourselves into the initial value approach when thinking about the **SCCC** typically brings affirmative while more global techniques might yield negative answers: the initial value formulation of Einstein's equations likely just explores the vicinity of 3 dimensional smooth spacelike submanifolds inside the full 4 dimensional space-time. It is well-known that an embedded smooth submanifold of an ambient space always admits a tubular neighbourhood which is an open disk bundle over the submanifold i.e., has a locally product smooth structure. However exotic 4 dimensional smooth structures never arise as products of lower dimensional ones consequently the *four* dimensional exotica i.e., the general structure of space-time never can be detected from a *three* dimensional perspective such as the initial value formulation. There is a *qualitative leap* between the two dimensions.

Finally we make a comment here on Malament–Hogarth space-times and "gravitational computers" as formulated for instance in [5, 4]. Following the terminology introduced in [4, Definition 3.1] if the maximal Cauchy development of an initial data set is extendible in the sense of Definition 2.1 then this (necessarily non-globally hyperbolic) extension is an example of a *generalized Malament–Hogarth space-time*; a space-time of this kind is essentially conformally equivalent to a *Malament–Hogarth space-time* (cf. [4, Remark after Definition 3.1]). However members of this latter class can in principle be used for powerful computations beyond the theoretical Turing barrier as explained for instance in [5] and the references therein. Since these space-times are never globally hyperbolic (this is well-known, cf. e.g. [4, Lemma 3.1]) the **SCCC**, if holds, forbids the existence of both physically relevant and stable Malament–Hogarth space-times. But our results here demonstrate that stable and physically relevant at least *generalized* Malament–Hogarth space-times exist because the **SCCC** can fail in a generic way. Therefore we ask ourselves whether or not our results can be sharpened to prove the existence of physically relevant stable Malament–Hogarth space-times: if yes then the theoretical possibility of building physically relevant as well as stable powerful "gravitational computers" would open up [5, 4].

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