

# A proof of the Geroch–Horowitz–Penrose formulation of the strong cosmic censor conjecture motivated by computability theory

Gábor Etesi

*Department of Geometry, Mathematical Institute, Faculty of Science,  
Budapest University of Technology and Economics,  
Egry J. u. 1, H ép., H-1111 Budapest, Hungary \**

November 4, 2021

## Abstract

In this paper we present a proof of a mathematical version of the strong cosmic censor conjecture attributed to Geroch–Horowitz and Penrose but formulated explicitly by Wald. The proof is based on the existence of future-inextendible causal curves in causal pasts of events on the future Cauchy horizon in a non-globally hyperbolic space-time.

By examining explicit non-globally hyperbolic space-times we find that in case of several physically relevant solutions these future-inextendible curves have in fact infinite length. This way we recognize a close relationship between asymptotically flat or anti-de Sitter, physically relevant extendible space-times and the so-called Malament–Hogarth space-times which play a central role in recent investigations in the theory of “gravitational computers”. This motivates us to exhibit a more sharp, more geometric formulation of the strong cosmic censor conjecture, namely “all physically relevant, asymptotically flat or anti-de Sitter but non-globally hyperbolic space-times are Malament–Hogarth ones”.

Our observations may indicate a natural but hidden connection between the strong cosmic censorship scenario and the Church–Turing thesis revealing an unexpected conceptual depth beneath both conjectures.

PACS numbers: 04.20.Dw, 89.20.Ff

Keywords: *Strong cosmic censorship; Malament–Hogarth space-times; Church–Turing thesis*

## 1 Introduction

Certainly the deepest conceptual question of classical general relativity theory is the so-called *cosmic censor conjecture* first formulated by R. Penrose four decades ago [34]. Roughly speaking the conjecture claims that predictability, probably one of the most fundamental concepts of classical physics, remains valid in the realm of classical general relativity i.e., all “physically relevant” space-times admit well-posed initial value formulation akin to other field theories.

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\*[etesi@math.bme.hu](mailto:etesi@math.bme.hu)

Meanwhile there has been a remarkable progress which culminated in a general satisfactory solution of the problem of existence and behaviour of *short-time* solutions to the Einstein's constraint equations [31] the cosmic censor conjecture deals with the existence and properties of *long-time* solutions [11] and is still “very much open” as Penrose says in [37]. One may then wonder what is the reason of this? Is the cosmic censor conjecture merely a technically more difficult question or is rather a conceptually deeper problem? On the contrary of its expected unified solution the cosmic censor conjecture has rather split up into several rigorous or less rigorous formulations, versions during the course of time. Therefore we can say that nowadays there are several “front lines” where “battles” for settling or violating the cosmic censor conjecture are going on. Far from being complete we can mention the following results on the subjectmatter.

The so-called *weak cosmic censor conjecture* in simple terms postulates:

**WCCC** *In a generic (i.e., stable), physically relevant (i.e., obeying some energy condition), asymptotically flat space-time singularities are hidden behind event horizons of black holes.*

The weak version can be formulated rigorously as a Cauchy problem for general relativity and the aim is to prove or disprove that for “generic” or “stable” (in some functional analytic sense) initial values at least, event horizons do form around singularities in an asymptotically flat space-time (where the notion of a black hole exists).

The first arguments in favour to this weak form came from studying the stability of the Schwarzschild event horizon under simple, linear perturbations of the metric. An early attempt to violate the weak version was the following. As it is well-known, a static, electrically charged black hole has only two parameters, namely its mass and charge. However if its charge is too high compared with its mass, event horizon do not occur hence the singularity could be visible by a distant observer. Consequently we may try to overcharge a static black hole in order to destroy its event horizon (we may argue in the same fashion in case of rotating black holes). However this is impossible as it was pointed out by Wald [43] in 1974. Another, more general but still indirect, argument for the validity of weak cosmic censorship is the so-called Riemannian Penrose inequality [35] proved by Bray [5] and Huisken–Ilmanen [28] in 1997. As an important step, the validity of the weak version in case of spherical collapse of a scalar field was established by Christodoulou [8, 9] in 1999.

The *strong cosmic censor conjecture* proposes more generally that all events have cause that is, there exist events chronologically preceding them and these events form a spacelike initial surface in any reasonable space-time. This also implies that singularities, except a possible initial “big bang” singularity, are invisible for observers:

**SCCC** *A generic (i.e., stable), physically relevant (i.e., obeying some energy condition) space-time is globally hyperbolic.*

Therefore this strong version also can be formulated in terms of a Cauchy problem but in this case we want to prove the inextendibility of maximal Cauchy developments of “generic” or “stable” (again in some functional analytic sense) initial data. Apparently this problem requires different techniques compared with the weak version.

Concerning the strong censorship we have partial important results, too. On the one hand its validity was proved by Chruściel–Isenberg–Moncrief [12] and Ringström for certain Gowdy space-times (for a recent survey cf. [41]) while by Chruściel–Rendall [13] in 1995 in the case of spatially compact and locally homogeneous space-times such as the Taub–NUT geometry. On the other hand one may also seek counterexamples to understand the meaning of “generic” in both the weak and strong versions. Many authors (e.g. [6, 8, 23, 27]) found hints in several

physically relevant situations for the violation of the weak or strong versions. A thin class of Gowdy space-times [12, 41] also lacks global hyperbolicity.

We may however find a kind of “compromise” between the two extremal approaches: seeking a general proof or hunting for particular counterexamples. This is the following. As it is well-known, the strong version is false in its simplest intuitive form. This means that there are several physically relevant space-times what is more: basic solutions to the Einstein’s equation which lack global hyperbolicity i.e., the maximal Cauchy development of the corresponding initial data set is extendible. The Taub–NUT space-time is extendible and in this case global hyperbolicity fails in such a way that strong causality breaks down on the future Cauchy horizon in any extension. The Reissner–Nordström, Kerr, (universal covering of) anti-de Sitter space-times are also extendible but in these cases global hyperbolicity is lost in a different way: from the future Cauchy horizons of their extensions a non-compact, infinite portion of their initial surfaces is observable. Therefore we have to indeed allow a collection of counterexamples consisting of apparently “non-generic” i.e., “unstable” space-times. Indeed, there are indirect hints that these extendible solutions are exceptional and atypical in some sense: small generic perturbations of them turn their Cauchy horizons into real curvature singularity thereby destroying extendibility and saving strong cosmic censorship [14, 15, 26, 33, 38].

Since nowadays we do not know any other type of violation we may roughly formulate the strong version as follows due to Geroch–Horowitz [21] and Penrose [36] from 1979 but explicitly formulated by Wald [44, 305p] (also cf. Theorem 2.1 here):

**SCCC-GHP** *If a physically relevant (i.e., obeying some energy condition) space-time is not globally hyperbolic then its Cauchy horizon looks like either that of the Taub–NUT or that of the Kerr space-time.*

In this formulation the highly complex question of “genericity” or “stability” has been suppressed and incorporated into that of Taub–NUT-like [26] and Kerr-like space-times [2]. Since this version focuses only on the causal character of extendible space-times instead of their non-genericity, we may expect a proof of this form using causal theoretic methods only (instead of heavy functional analytic ones). One aim of this paper is to rigorously prove this version using ideas motivated by recent advances in an interdisciplinary field connecting computability and general relativity theory.

Recently there has been a remarkable interest in the physical foundations of computability theory and the Church–Turing thesis. It turned out that algorithm theory, previously considered as a very mathematical field, has a deep link with basic concepts of physics.

On the one hand nowadays we can see that our apparently pure mathematical notion of a Turing machine involves indirect preconceptions on space, time, motion, state and measurement. Hence it is reasonable to ask whether different choices of physical theories put for modeling these things have some effect on our notions of computability or not. At the recent stage of affairs it seems there are striking changes on the whole structure of complexity and even computability theory if we pass from classical physics to quantum or relativistic theories. Even certain variants of the Church–Turing thesis cease to be valid in some cases.

For instance taking *quantum mechanics* as our background theory the famous Chaitin’s omega number, a typical non-computable real number, becomes enumerable via an advanced quantum computer [7]. An adiabatic quantum algorithm exists to attack Hilbert’s tenth problem [29, 30]. Chern–Simons *topological quantum field theory* can also be used to calculate the Jones polynomial of knots [19]. In the same fashion if we use *general relativity theory*, powerful “gravitational computers” can be constructed, also capable to break Turing’s barrier [45]: Hogarth proposed a

class of space-times in 1994, now called as *Malament–Hogarth space-times* allowing non-Turing computations [24, 25]. Hogarth’s construction uses anti-de Sitter space-time which is also in the focal point of recent investigations in high energy physics. In the same vein, in 2001 the author and N emeti constructed another example by exploiting properties of the Kerr geometry [18]. This space-time is also relevant as being the only candidate in general relativity for the final state of a collapsed, massive, slowly rotating star. A general introduction to the topic is Chapter 4 of Earman’s book [16].

On the other hand it is conjectured that these generalized computational methods are not only significant from a computational viewpoint but they are also in connection with our most fundamental physical concepts such as the standard model and string theory [4, 42]. A relation between computability and gravity also has been examined in [1, 20].

The natural question arises if the same is true for “gravitational computers” i.e., is there any pure physical characterization of Malament–Hogarth space-times? In our previous letter we tried to argue that these space-times also appear naturally in the strong cosmic censorship scenario [17]. Namely we claimed that space-times possessing powerful “gravitational computers” form the unstable borderline separating the allowed and not-allowed space-times by the strong cosmic censor (but these space-times are still considered as “physically relevant”).

In this paper we try to push this analogy further and claim that using a concept emerging from the theory of “gravitational computers” we can prove the Geroch–Horowitz–Penrose form of the strong cosmic censor conjecture. The proof uses standard causal set theory only with the simple but key observation that if a globally hyperbolic space-time is future-extendible then in the causal pasts of events on its future Cauchy horizon future-inextendible, non-spacelike curves appear. These curves also play a crucial role in the theory of non-Turing computers in general relativity: Malament–Hogarth space-times are exactly those for which the aforementioned curves exist, are timelike and complete. But checking case-by-case several physically relevant maximally extended examples lacking global hyperbolicity like Kerr, Reissner–Nordstr om, (universal cover of) anti-de Sitter we find that these future-inextendible curves are indeed timelike and have infinite length. But on the contrary the Taub–NUT and certain extendible polarized Gowdy space-times with toroidal spatial topology lack this property: the corresponding inextendible curves are incomplete.

This motivates us to sharpen the Geroch–Horowitz–Penrose version of the strong cosmic censor conjecture recalled above like this (cf. Conjecture 3.1 here):

**SCCC-MH** *If a physically relevant (i.e., obeying some energy condition), asymptotically flat or asymptotically hyperbolic (i.e., anti-de Sitter) space-time is not globally hyperbolic then it is a Malament–Hogarth space-time.*

Note that this formulation continues to avoid the question of “genericity” or “stability”. We cannot prove or disprove this version but call attention that this formulation sheds some light onto a possible deep link between the cosmic censorship scenario and computability theory as follows. Consider the following physical reformulation of the Church–Turing thesis:<sup>1</sup>

**Ph-ChT** *An artificial computing system based on a generic (i.e., stable), relevant (i.e., obeying some energy condition) classical physical system realizes Turing-computable functions.*

Note that this formulation—in contrast to versions like [18, Thesis 2 and 2’]—is a quite demo-

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<sup>1</sup>For the concept of an “artificial computing system” and of a “Turing computable function” in particular and for further details in general, we refer to [18, Chapter 2] and references therein.

cratic one because it does not *a priori* excludes the existence of too powerful computational devices; it just says that they must in one or another way be unstable (which is apparently true for the various devices in [7, 29, 30, 18, 24, 25]). Accepting that all artificial computing systems based on classical physics can be modeled by “gravitational computers” as will be argued in Section 3 here as well as accepting **SCCC-MH** we can see that **SCCC** and **Ph-ChT** are roughly equivalent hence involve the same depth. This might serve as an explanation for the permanent difficulty present in all approaches to the strong cosmic censor conjecture scenario.

The paper is organized as follows. In Section 2 we prove rigorously the Geroch–Horowitz–Penrose version of the conjecture using methods mentioned above (cf. Theorem 2.1 here).

Then in Section 3 we introduce all the notions required by computability theory. These are the concept of a *Malament–Hogarth space-time* (cf. Definition 3.1 here) and that of a *gravitational computer*. Then a case-by-case study of explicit examples helps us to select those non-globally hyperbolic space-times which also possess the Malament–Hogarth property. In this framework we can then offer a sharper Geroch–Horowitz–Penrose-type version of the strong cosmic censor conjecture which relates it with the aforementioned physical formulation of the Church–Turing thesis (cf. Conjecture 3.1 here).

In Section 4 we conclude our paper and speculate what has actually been proved.

## 2 The strong cosmic censor conjecture

Basic references for this section are [22, 44] but we will also frequently use [3]. Let  $(M, g)$  be a connected, four dimensional, smooth, time-oriented Lorentzian manifold-without-boundary i.e., a space-time.

In this paper a class of space-times will be considered whose members  $(M, g)$  can be carefully constructed as follows. Take a smooth, globally hyperbolic space-time  $(D(S), g'|_{D(S)})$  which is supposed to be a solution to the coupled Einstein’s equation

$$r - \frac{1}{2}sg'|_{D(S)} = 8\pi T + \Lambda g'|_{D(S)}$$

with cosmological constant  $\Lambda$  and matter content represented by a stress energy-tensor  $T$  obeying the *dominant energy condition*. Suppose that this matter field is *fundamental* in the sense that the associated Einstein’s equation can be adjusted into the form of a quasilinear, diagonal, second order system of hyperbolic partial differential equations. In this case  $(D(S), g'|_{D(S)})$  admits a well-posed initial value formulation as follows. There exists an initial data set  $(S, h, k)$  where  $(S, h)$  is a smooth Riemannian three-manifold which is supposed to be complete and  $k$  is a smooth  $(0, 2)$ -type tensor field satisfying the usual constraint equations and this initial data set is related to the original space-time  $(D(S), g'|_{D(S)})$  in the usual way [22, 44]: the unique maximal Cauchy development of  $(S, h, k)$  is  $(D(S), g'|_{D(S)})$  i.e. this is the largest space-time in which  $S$  is a connected, spacelike Cauchy surface and  $h$  and  $k$  are the induced metric and extrinsic curvature of  $S$  as embedded into  $(D(S), g'|_{D(S)})$ , respectively.

Let  $(M', g')$  be a (not necessarily unique) at least continuous extension of  $(D(S), g'|_{D(S)})$  if exists. If  $(D(S), g'|_{D(S)})$  is inextendible in this sense then we put  $(M', g') := (D(S), g'|_{D(S)})$ . Thus  $(D(S), g'|_{D(S)}) \subseteq (M', g')$  is a proper isometric embedding and  $D(S)$  is open in  $M'$ . Then  $S$  is a *Cauchy surface* in  $(M', g')$  which is *partial* if  $(D(S), g'|_{D(S)})$  is not identical to  $(M', g')$ . We shall also suppose that  $(M', g')$  is a maximal extension in the sense that if  $(M, g)$  is another Lorentzian manifold of the same dimension such that  $(M', g') \subseteq (M, g)$  is a continuous isometric embedding

then  $(M', g') = (M, g)$ . It is of course clear that for any given initial data set its unique maximal Cauchy development is maximal as a Cauchy development hence it is reasonable to demand its extension to be maximal as a continuous extension as well in order to get rid of very artificial extensions constructed by e.g. unphysical puncturing, etc. Thus throughout the text we shall suppose that given any initial data set  $(S, h, k)$  as above whose maximal Cauchy development  $(D(S), g|_{D(S)})$  satisfies an Einstein equation with fundamental matter obeying the dominant energy condition then its further extension  $(M, g)$ , if exists, is also maximal as an extension (and maybe satisfies an extended Einstein equation as well).

Finally we note that all causal or set-theoretical operations (i.e.,  $J^\pm(\cdot)$ ,  $\subseteq$ ,  $\cap$ ,  $\cup$ , taking complement, closure, etc.) will be taken in the space  $M$  with its standard manifold topology throughout the text.

**Lemma 2.1.** *Let  $(M, g)$  be a space-time as above with a strictly partial Cauchy surface  $S \subset M$  i.e., the corresponding maximal Cauchy development satisfies  $(D(S), g|_{D(S)}) \subsetneq (M, g)$ .*

*Then for any  $q \in M \setminus D(S)$  either strong causality is violated in  $J^\pm(S) \cap J^\mp(q)$  or  $J^\pm(S) \cap J^\mp(q)$  is non-compact (or both can happen).*

*Proof.* Fix a point  $q \in M \setminus D(S)$ . Then by definition there exists at least one non-spacelike curve in  $M$  from  $q$  such that: its image lies in  $J^\mp(q)$ , it is past/future-inextendible and having no intersection with  $S$ . Furthermore in any extension i.e., isometric embedding  $(D(S), g|_{D(S)}) \subset (M, g)$  if  $S' \subset M$  is a spacelike submanifold such that  $S \subseteq S'$  then writing  $h' := g|_{S'}$  we obtain an induced embedding  $(S, h) \subseteq (S', h')$  yielding that actually  $S = S'$  taking into account that  $(S, h)$  is a complete Riemannian manifold by assumption. This implies that the aforementioned curve is contained within  $J^\pm(S) \cap J^\mp(q) \subset M$  i.e., this subset contains a non-spacelike curve of which at least one endpoint is missing. Hence referring to [44, Lemma 8.2.1] either strong causality is violated in the subset  $J^\pm(S) \cap J^\mp(q)$  or it cannot be compact as claimed.  $\diamond$

Our next goal is to find future inextendible non-spacelike curves in the causal pasts of events “on” or “beyond” the future Cauchy horizon in non-globally hyperbolic space-times.<sup>2</sup> We will be frequently using various limit curve theorems from [3, 22] however a more compact approach also exists based on an improved theorem [32, Theorem 3.1] of the same kind.

**Lemma 2.2.** *Let  $(M, g)$  be a space-time as above with a strictly partial Cauchy surface  $S \subset M$  i.e., the corresponding maximal Cauchy development satisfies  $(D(S), g|_{D(S)}) \subsetneq (M, g)$ .*

*If  $q \in J^+(S) \cap (M \setminus D(S))$  then there exists a non-spacelike future-directed parameterized half-curve  $\lambda : \mathbb{R}^+ \rightarrow J^+(S) \cap J^-(q)$  or  $\lambda : [0, a) \rightarrow J^+(S) \cap J^-(q)$  with  $a < +\infty$  such that  $\lambda(0) \in S \cap J^-(q)$  and  $\lambda$  has no future endpoint in  $J^+(S) \cap J^-(q)$ .*

*Proof.* By virtue of the previous lemma we can see that either (i)  $J^+(S) \cap J^-(q)$  fails to be strongly causal or (ii)  $J^+(S) \cap J^-(q)$  is non-compact (or both cases can happen).

First assume case (i) is valid and let  $x \in J^+(S) \cap J^-(q)$  be such that strong causality fails in this point. If  $x$  is an interior point then this failure means that there is a neighbourhood  $x \in U \subset J^+(S) \cap J^-(q)$  and a countable collection  $U = V_1 \supset \dots \supset V_n \supset \dots$  of open sets satisfying  $x \in V_n$  and corresponding non-spacelike parameterized half-curves  $\lambda_n : [0, a_n] \rightarrow M$  or  $\lambda_n : \mathbb{R}^+ \rightarrow M$  such that  $\lambda_n$  intersects  $V_n$  more than once for all  $n \in \mathbb{N}$ . Without loss of generality we can suppose that  $\lambda_n(0) = p \in S \cap J^-(q)$  for all  $n \in \mathbb{N}$ . For a given  $n \in \mathbb{N}$  it

<sup>2</sup>The issue of finding past-inextendible non-spacelike curves in the causal futures of points chronologically preceding the globally hyperbolic regime is similar hence we restrict our attention to the former case.

can happen that  $\lambda_n$  has a future endpoint  $q_n \in M$ ; if not then put  $q_n := \emptyset$ . Then if we remove these points from  $M$  then we get a sequence  $\{\lambda_n\}$  of non-spacelike future-inextendible curves with past accumulation point  $\lambda_n(0) = p \in M$  for all  $n \in \mathbb{N}$ . Consequently via [3, Proposition 3.31] the sequence  $\{\lambda_n\}$  has a limit curve (in the pointwise sense, cf. [3, Definition 3.28]) i.e., a future-inextendible non-spacelike parameterized curve

$$\lambda : \left[0, \sup_n a_n\right) \longrightarrow M \setminus \bigcup_{n \in \mathbb{N}} \{q_n\}$$

with  $\lambda(0) = p$ . It is clear that this non-spacelike curve considered as a curve in  $M \supseteq M \setminus \bigcup_{n \in \mathbb{N}} \{q_n\}$  cannot have future endpoint in  $M$ . Furthermore since  $\lambda(0) \in S \cap J^-(q)$  we can suppose that its image is contained within  $J^+(S) \cap J^-(q)$ . This is because if the non-spacelike  $\lambda$  happens to exit  $J^-(q)$  then it cannot return into it anymore contradicting the fact that it intersects each  $V_n \ni x$  more than once.

Now assume  $x \in \partial J^-(q)$ . Then, since  $(M, g)$  is open, there is a point  $q' \in M$  such that  $q \in J^-(q')$  is an interior point therefore  $J^-(q) \subset J^-(q')$  and  $x$  is an interior point of  $J^-(q')$ . Then we can find a curve  $\lambda$  as above for  $q'$ . Taking into account that the image of  $\lambda$  is contained within  $J^+(S) \cap J^-(q')$  for all points  $q'$  with  $q \in J^-(q')$  we obtain that actually  $\lambda$  lies within  $J^+(S) \cap J^-(q)$  as desired.

Secondly suppose (ii) is valid. Then there is at least one sequence  $\{q_n\}$  in  $J^+(S) \cap J^-(q)$  such that no subsequence of it converges in  $J^+(S) \cap J^-(q)$ . Exploiting the completeness of  $(S, h)$  we can pick this sequence such that  $q \notin \{q_n\}$  and there is a point  $p \in D(S)$  satisfying  $\bigcup_{n \in \mathbb{N}} \{q_n\} \subset J^+(p)$ . Let  $\lambda_n : [0, a_n] \rightarrow J^+(p) \cap J^-(q)$  be a non-spacelike parametrized curve connecting  $p$  with  $q_n$ . If we remove the points of  $\{q_n\}$  from  $J^+(p) \cap J^-(q)$  then  $\lambda_n$ 's are future-inextendible non-spacelike half-curves in the punctured set with a past accumulation point  $\lambda_n(0) = p$ . Consequently again via [3, Proposition 3.31] the sequence  $\{\lambda_n\}$  has a limit curve (in the pointwise sense, cf. [3, Definition 3.28]) i.e., a future-inextendible non-spacelike parameterized curve

$$\lambda : \left[0, \sup_n a_n\right) \longrightarrow (J^+(p) \cap J^-(q)) \setminus \bigcup_{n \in \mathbb{N}} \{q_n\}$$

with  $\lambda(0) = p$ . Taking into account that  $\{q_n\}$  has no convergent subsequence,  $\lambda$  remains inextendible even in  $J^+(p) \cap J^-(q)$  that is, has no future endpoint. Extending  $\lambda$  in the past of  $p$  back to  $S$  or taking its intersection with  $S$  and reparameterizing if necessary, we obtain a non-spacelike future-directed half-curve with  $\lambda(0) \in S \cap J^-(q)$  but without future endpoint as claimed.  $\diamond$

Our next step is to understand how strong causality or compactness of intersections of causal sets break down if we exit the globally hyperbolic region of a space-time.

We need a technical tool namely a convenient topology on the set of non-spacelike parameterized curves as follows (cf. e.g. [3, Chapter 3] or [44, 206p]). Let  $p \in J^+(S)$  and consider the set  $P(S, p)$  of continuous non-spacelike parameterized curves starting on  $S$  and terminating at  $p$ . Let  $K \subset \mathbb{R}^+$  be compact and  $V \subseteq M$  be open and define  $O_{K,V} \subseteq P(S, p)$  by

$$O_{K,V} := \{\lambda \in P(S, p) \mid \lambda(K) \subset V\}.$$

That is,  $O_{K,V}$  consists of all non-spacelike parameterized curves from  $S$  to  $p$  whose portions  $\lambda(K)$  for a fixed  $K \subset \mathbb{R}^+$  lie entirely within a fixed  $V \subseteq M$ . The *compact-open topology* on  $P(S, p)$  is generated by the sets  $\{O_{K,V} \mid K \subset \mathbb{R}^+ \text{ compact, } V \subseteq M \text{ open}\}$ . In particular if some

$O \subseteq P(S, p)$  has the form  $O = \bigcup O_{K,V}$  then it belongs to this topology hence is open but not all open subsets are of this form.<sup>3</sup> The notion of convergence in the compact-open topology is the following. The sequence  $\{\lambda_n\}$  in  $P(S, p)$  converges to  $\lambda \in P(S, p)$  in the compact-open topology if for every open  $\lambda \in O \subseteq P(S, p)$  there exists an integer  $n_O \in \mathbb{N}$  such that  $\lambda_n \in O$  for all  $n > n_O$ . This implies that if  $\lambda_n \rightarrow \lambda$  ( $n \rightarrow +\infty$ ) in the compact-open topology then for all  $K \subset \mathbb{R}^+$  compact and  $V \subseteq M$  open such that  $\lambda(K) \subset V$  there exists an integer  $n_{K,V} \in \mathbb{N}$  such that  $\lambda_n(K) \subset V$  for all  $n > n_{K,V}$ .

The following proposition is taken from [17] where its proof was sketched only hence we present a detailed proof here.

**Lemma 2.3.** *Let  $(M, g)$  be a space-time as above. Let  $S$  be a (partial) Cauchy surface in it with maximal Cauchy development  $(D(S), g|_{D(S)}) \subseteq (M, g)$ . If  $x \in J^+(S)$  such that  $\overline{S \cap J^-(x)}$  is compact and  $J^+(S) \cap J^-(x)$  is strongly causal then  $J^+(S) \cap J^-(x)$  is compact.*

*Proof.* Let  $\{q_i\}$  be an arbitrary sequence of points in  $J^+(S) \cap J^-(x)$ . We have to find a subsequence of it converging to a point in  $J^+(S) \cap J^-(x)$ . If there was a subsequence  $\{q_j\} \subset \{q_i\}$  such that  $q_j \rightarrow x$  ( $j \rightarrow +\infty$ ) then we could finish the proof right now. Assume this is not the case. Then there is a neighbourhood  $x \in U \subset M$  such that  $U \cap J^+(S) \cap J^-(x)$  does not contain any point of  $\{q_i\}$ .

First suppose that all but finite points of  $\{q_i\}$  sit in the interior of  $J^+(S) \cap J^-(x)$ . In this case by exploiting (the first and last time here) the starting assumption that  $(M, g)$  is maximal as a continuous extension hence is not punctured, etc. we can choose timelike future-directed parameterized half-curves  $\lambda_i : [0, a_i] \rightarrow J^+(S) \cap J^-(x)$  such that

- (i)  $\lambda_i(a_i) = x$  for all  $i \in \mathbb{N}$  that is, all curves terminate at  $x$ ;
- (ii) there is a  $t_i \in [0, a_i]$  such that  $\lambda_i(t_i) = q_i$  for all  $i$  that is  $\lambda_i$  intersects the point  $q_i$  of the sequence  $\{q_i\}$ ;
- (iii) if  $t_i > 0$  then  $\lambda_i(0) =: p_i \in S \cap J^-(x)$  for all  $i$  that is  $\lambda_i$  departs from  $S$  giving rise to a sequence  $\{p_i\}$  in  $S \cap J^-(x)$  (if  $t_i = 0$  for some  $i$  then  $p_i := q_i$  in this case).

Note that the construction of this curves is highly non-unique. Nevertheless (i) and (iii) imply that  $\{\lambda_i\}$  is a sequence in  $P(S, x)$ . Taking into account that  $\overline{S \cap J^-(x)}$  is compact, there exists a subsequence  $\{p_j\}$  of the induced sequence  $\{p_i\}$  given by (iii) converging to a point  $p \in \overline{S \cap J^-(x)} \subseteq S$  since  $S$  is closed in  $M$ . Consequently the corresponding subsequence  $\{\lambda_j\}$  of our curves has a past accumulation point:  $\lambda_j(0) = p_j \rightarrow p \in S$  ( $j \rightarrow +\infty$ ). Moreover if we remove  $x$  from  $J^+(S) \cap J^-(x)$ , then  $\{p_j\}$  gives rise to a sequence of future-inextendible non-spacelike half-curves  $\{\lambda_j\}$  by property (i) and (iii). Hence again by [3, Proposition 3.31] there exists a future-directed non-spacelike parameterized limit curve (again in the pointwise sense, cf. [3, Definition 3.28])

$$\lambda : \left[ 0, \sup_j a_j \right) \longrightarrow (J^+(S) \cap J^-(x)) \setminus \{x\}$$

satisfying  $\lambda(0) = p \in S$ . Moreover  $\lambda_j(a_j) = x$  for all  $j \in \mathbb{N}$  hence  $x$  is a future accumulation point of  $\{\lambda_j\}$  thus in fact  $a := \sup_j a_j < +\infty$  and  $\lambda(a) = x$ . Thus  $\lambda \in P(S, x)$ . Therefore after

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<sup>3</sup>Note that in general the  $O_{K,V}$ 's do *not* form a basis for the compact-open topology.



reparametrizing we obtain a sequence  $\{\lambda_j\}$  in  $P(S, x)$  such that

$$\begin{cases} \lambda_j : [0, a] \longrightarrow J^+(S) \cap J^-(x) \text{ for all } j \in \mathbb{N} \\ \lambda_j(0) \longrightarrow p \text{ (} j \rightarrow +\infty \text{)} \\ \lambda_j(a) \longrightarrow x \text{ (} j \rightarrow +\infty \text{)} \end{cases}$$

i.e., both endpoints of  $\{\lambda_j\}$  converge. Since  $J^+(S) \cap J^-(x)$  is strongly causal by assumption, there is a subsequence  $\{\lambda_k\} \subseteq \{\lambda_j\}$  such that  $\lambda_k \rightarrow \lambda$  ( $k \rightarrow +\infty$ ) in the compact-open topology on  $P(S, x)$ , too (cf. [3, Proposition 3.34] as well as [32, Theorem 3.1]). Thus for any  $0 \leq \varepsilon$  and  $V \subseteq M$  open such that  $\lambda([\varepsilon, a - \varepsilon]) \subset V$  one finds that  $\lambda_k([\varepsilon, a - \varepsilon]) \subset V$  for all  $k > k_{\varepsilon, V}$ . But  $\lambda$  is a continuous image of the compact interval  $[\varepsilon, a - \varepsilon] \subset \mathbb{R}^+$  consequently  $\lambda([\varepsilon, a - \varepsilon]) \subset M$  is compact hence we can choose  $V \subseteq M$  and  $0 < \varepsilon$  so that  $\bar{V}$  is compact and  $\bar{V} \subset J^+(S) \cap J^-(x)$  holds.<sup>4</sup> Since we already know that  $p_k \rightarrow p \in J^+(S) \cap J^-(x)$  i.e. converges let assume that except finitely many cases  $q_k \neq p_k$ . Then  $q_k \in \lambda_k([\varepsilon, a - \varepsilon]) \subset \bar{V}$  via property (ii) therefore by exploiting the compactness of  $\bar{V}$  there is a convergent subsequence  $\{q_l\} \subseteq \{q_k\} \subset \bar{V} \subset J^+(S) \cap J^-(x)$ . Thus eventually we succeeded to find a subsequence of the original arbitrary sequence  $\{q_i\}$  which is convergent in  $J^+(S) \cap J^-(x)$  demonstrating the compactness of  $J^+(S) \cap J^-(x)$ .

Finally, if the points of  $\{q_i\}$  lie on  $J^+(S) \cap \partial J^-(x)$  then take an  $x' \in M$  such that  $x \in J^-(x')$  is an interior point. Then we have  $J^-(x) \subset J^-(x')$  and the members of  $\{q_i\}$  are interior points of  $J^+(S) \cap J^-(x')$ . Repeating the previous procedure we obtain that  $\{q_i\}$  possesses a convergent subsequence in  $J^+(S) \cap J^-(x')$ . But taking into account that this is true for all  $x' \in M$  with  $x \in J^-(x')$  we find that in fact the accumulation point of  $\{q_i\}$  is contained within  $J^+(S) \cap J^-(x)$  as claimed.  $\diamond$

After these preliminaries we are in a position to prove a variant of the strong cosmic censor conjecture attributed to Geroch–Horowitz [21] and Penrose [36] but formulated explicitly by Wald [44, 305p]). Recall that  $H^+(S) := \overline{\partial D^+(S)}$  is called the *future Cauchy horizon* of  $D(S) \subset M$ .

**Theorem 2.1.** (the Geroch–Horowitz–Penrose version of the strong cosmic censor conjecture) *Let  $(S, h, k)$  be an initial data set for Einstein’s equation with  $(S, h)$  a complete Riemannian three-manifold and with a fundamental matter represented by a stress-energy tensor  $T$  obeying the dominant energy condition.*

*Then, if the maximal Cauchy development of this initial data set is extendible, for each  $x \in H^+(S)$  in any extension<sup>5</sup> either strong causality is violated at  $x$  or  $S \cap J^-(x)$  is non-compact.*

*Remark.* Note that  $\overline{S \cap J^-(x)} = \overline{S \cap I^-(x)}$  hence our statement coincides with [44, 305p]. The theorem also implies that these extensions of course cannot be globally hyperbolic.

*Proof.* Let  $(D(S), g|_{D(S)})$  be the unique maximal Cauchy development of  $(S, h, k)$  and assume it admits a further at least continuous maximal extension  $(M, g)$ . Let  $x \in H^+(S)$  be any point on the future Cauchy horizon in this extension. Then, by Lemma 2.1 the set  $J^+(S) \cap J^-(x)$  cannot be strongly causal and compact. If strong causality is violated in this set, we get the first possibility of the theorem. Indeed, strong causality can fail only in  $(J^+(S) \cap J^-(x)) \cap H^+(S)$  because the remaining portion is globally hyperbolic. But we can repeat the previous procedure

<sup>4</sup>Recall that the topology of  $M$  is generated by the complete metric space  $(M, d_h)$  associated to some auxiliary complete Riemannian metric  $h$  put onto  $M$  thus we can take  $V$  to be an open “narrow sausage” surrounding  $\lambda([\varepsilon, a - \varepsilon]) \subseteq J^+(S) \cap J^-(x)$ .

<sup>5</sup>Cf. again our careful definition of an extension at the beginning of Section 2.

for  $(J^+(S) \cap J^-(x)) \cap U$  where  $U$  is an arbitrary open set in  $M$  containing  $x$  yielding that this point must coincide with  $x$  itself.

If strong causality is valid within  $J^+(S) \cap J^-(x)$  then  $\overline{S \cap J^-(x)}$  cannot be compact because in this case  $J^+(S) \cap J^-(x)$  would be compact, too via Lemma 2.3 contradicting again Lemma 2.1. Hence one obtains the second possibility of the theorem.  $\diamond$

For a future comparison we give a reformulation of the above theorem based on Lemma 2.2.

**Theorem 2.2.** (reformulation of the Geroch–Horowitz–Penrose version). *Let  $(S, h, k)$  be an initial data set as in Theorem 2.1. Then, if the maximal Cauchy development of this initial data is extendible, for each  $x \in H^+(S)$  in any extension,  $J^+(S) \cap J^-(x)$  contains a future-directed non-spacelike curve without future endpoint.*  $\diamond$

*Remark.* 1. Notice that the Geroch–Horowitz–Penrose form of the strong cosmic censor conjecture deals with causal or conformal properties of a space-time only. This is also reflected in the mathematical structure of the proof: we were not forced to use hard analytical techniques to achieve the result. But note again that we assumed that our extensions are maximal and satisfy an Einstein equation at least inside their globally hyperbolic domain consequently are free of removable singularities arising for instance from artificial puncturation of space-time, etc.

2. However in fact we had to use the validity of the Einstein equation with some fundamental matter obeying the dominant energy condition only in an auxiliary way in the proof: it was only necessary to formulate a space-time as the unique solution to a Cauchy problem (implying this globally hyperbolic region being free of removable singularities). Hence our results remain valid for a vast class of space-times which are still subject to the formulation as a Cauchy problem but satisfy more general field equations than Einstein’s equation [39, 40].

3. Of course we also have to pay some price for this approach: although we have been able to conclude that any extension has the desired causal property, we actually do not know whether or not these extensions are generic or unstable in any sense. Indeed, if the counterexamples mentioned in Section 1 turn out to be generic in some strict mathematical sense then extendibility of space-times with the Geroch–Horowitz–Penrose property must be generic, too that is, a stable phenomenon.

### 3 Malament–Hogarth space-times

In this section we turn the coin and introduce the concept of a Malament–Hogarth space-time and that of a “gravitational computer”. As a motivation we mention that in these space-times, at least in principle, one can construct powerful computational devices capable for computations beyond the Turing barrier. A typical example for such a computation is checking the consistency of ZFC set theory [18, 45].

Let us consider the following class of space-times (cf. [16, 18, 24, 25, 45]):

**Definition 3.1.** *Let  $(S, h, k)$  be an initial data set for Einstein’s equation, with  $(S, h)$  a complete Riemannian manifold. Suppose a fundamental matter field is given represented by its stress-energy tensor  $T$  satisfying the dominant energy condition. Let  $(M, g)$  be a maximal continuous extension (if exists) of the unique maximal Cauchy development  $(D(S), g|_{D(S)})$  of the above initial data set.*

- (i) Then  $(M, g)$  is called a Malament–Hogarth space-time if there is a future-directed timelike half-curve  $\gamma_C : \mathbb{R}^+ \rightarrow M$  such that  $\|\gamma_C\| = +\infty$  and there is a point  $q \in M$  satisfying  $\gamma_C(\mathbb{R}^+) \subset J^-(q)$ . The event  $q \in M$  is called a Malament–Hogarth event;
- (ii)  $(M, g)$  is called a generalized Malament–Hogarth space-time if there is a future-directed timelike half-curve  $\gamma_C : \mathbb{R}^+ \rightarrow M$  without future endpoint and there is a point  $q \in M$  satisfying  $\gamma_C(\mathbb{R}^+) \subset J^-(q)$ . The event  $q \in M$  is called a generalized Malament–Hogarth event.

*Remark.* 1. If  $(M, g)$  is a (generalized) Malament–Hogarth space-time then there exists a future-directed timelike curve  $\gamma_O : [a, b] \rightarrow M$  joining  $p \in J^-(q)$  with  $q$  satisfying  $\|\gamma_O\| < +\infty$ . The point  $p \in M$  can be chosen to lie in the causal future of the past endpoint of  $\gamma_C$ .

2. Moreover the reason we require fundamental matter fields obeying the dominant energy condition, geodesically complete initial surfaces, extensions to be maximal etc., is that we want to exclude the very artificial examples of Malament–Hogarth space-times.

3. It follows from Theorem 2.2 here that all non-globally hyperbolic space-times are generalized Malament–Hogarth ones. But a sufficiently nice non-globally hyperbolic space-time is in fact *conformally equivalent* to a Malament–Hogarth-like space-time which is however probably not the solution of the Einstein’s equation with a physically relevant matter content; indeed, any so-called distinguishable space-time can be conformally rescaled to be timelike and null geodesically complete (cf. [3, Theorem 6.5]).

The motivation is the following (for details we refer to [18]). Take any Malament–Hogarth space-time  $(M, g)$ . Consider a Turing machine realized by a physical computer  $C$  moving along the curve  $\gamma_C$  of *infinite* proper time. Hence the physical computer (identified with  $\gamma_C$ ) can perform arbitrarily long calculations in the ordinary sense. In addition there exists an observer  $O$  following the curve  $\gamma_O$  (hence denoted by  $\gamma_O$ ) of *finite* length such that he hits the Malament–Hogarth event  $q \in M$  in *finite* proper time. But by definition  $\gamma_C(\mathbb{R}^+) \subset J^-(q)$  therefore in  $q$  he can receive the answer for a *yes or no question* as the result of an *arbitrarily long* calculation carried out by the physical computer  $\gamma_C$ . This is because  $\gamma_C$  can send a light beam at arbitrarily late proper time to  $\gamma_O$ . Clearly the pair  $(\gamma_C, \gamma_O)$  in  $(M, g)$  with a Malament–Hogarth event  $q$  is an artificial computing system i.e., a generalized computer in the sense of [18].

Imagine the following exciting situation as an example.  $\gamma_C$  is asked to check all theorems of our usual set theory (ZFC) in order to check consistency of mathematics. This task can be carried out by  $\gamma_C$  since its world line has infinite proper time. If  $\gamma_C$  finds a contradiction, it can send a message (for example an appropriately coded light beam) to  $\gamma_O$ . Hence if  $\gamma_O$  receives a signal from  $\gamma_C$  *before* the Malament–Hogarth event  $q \in M$  he can be sure that ZFC set theory is not consistent. On the other hand, if  $\gamma_O$  does not receive any signal before  $q$  then, *after*  $q$ ,  $\gamma_O$  can conclude that ZFC set theory is consistent. Note that  $\gamma_O$  having finite proper time between the events  $\gamma_O(a) = p$  (departure for the experiment) and  $\gamma_O(b) = q$  (hitting the Malament–Hogarth event), he can be sure about the consistency of ZFC set theory within finite (possibly very short) time. This shows that certain very general formulations of the Church–Turing thesis (for instance [18, Thesis 2,2’ and 3]) cannot be valid in the framework of classical general relativity.

In general—keeping in mind the definition of a Malament–Hogarth space-time—a quintuple  $(M, g, q, \gamma_C, \gamma_O)$  is called a *gravitational computer* if  $(M, g)$  is a space-time,  $\gamma_C, \gamma_O$  are timelike curves and  $q \in M$  is an event such that the curves lie within  $J^-(q)$ . This concept is broad enough to serve as an abstract model for all kind of artificial computing systems based on classical

physics so that an artificial computing system can perform non-Turing computations if and only if the corresponding gravitational computer is defined in an ambient space-time possessing the Malament–Hogarth property. Indeed, in the case of modeling a usual (Turing) artificial computing system the ambient space-time  $(M, g)$  can be simply taken to be the Minkowskian or Newtonian one with any event  $q \in M$  and curves  $\gamma_C = \gamma_O$  in its causal past.<sup>6</sup> However if the artificial computing system is expected to perform non-Turing computations then it is equivalent (cf. [18, Chapter 2]) to a usual (Turing) artificial computing system with the only extra property of being able to solve at least once the so-called halting problem; this can be carried out if the ambient space-time  $(M, g)$  is a Malament–Hogarth one with Malament–Hogarth event  $q \in M$  and curves  $\gamma_C$  and  $\gamma_O$  as above.

One can raise the question if Malament–Hogarth space-times are relevant or not from a physical viewpoint. We put off this very important question for a few moments; instead we prove basic properties of Malament–Hogarth space-times by evoking [16, Lemmata 4.1 and 4.3]. These properties are also helpful in seeking realistic examples.

**Lemma 3.1.** *Let  $(M, g)$  be a Malament–Hogarth space-time as above. Then  $(M, g)$  is not globally hyperbolic. Moreover, if  $q \in M$  is a Malament–Hogarth event and  $S \subset M$  is a connected spacelike hypersurface such that  $\gamma_C(\mathbb{R}^+) \subset J^+(S) \cap J^-(q)$  then  $q$  is on or beyond the future Cauchy horizon  $H^+(S)$  of  $S$ .*

*Proof.* Consider a point  $p \in M$  such that  $\gamma_C(0) = p$ . If  $(M, g)$  was globally hyperbolic then  $(M, g)$  would be strongly causal and in particular  $J^+(p) \cap J^-(q) \subset M$  compact. We know that  $\gamma_C(\mathbb{R}^+) \subset J^-(q)$  hence in fact  $\gamma_C(\mathbb{R}^+) \subset J^+(p) \cap J^-(q)$ . Consequently its future (and of course, past) endpoint are contained in  $J^+(p) \cap J^-(q)$  (cf. [44, Lemma 8.2.1]). However  $\gamma_C$  is a causal curve with  $\|\gamma_C\| = +\infty$  hence it is future inextendible i.e., has no future endpoint. But this is impossible hence  $J^+(p) \cap J^-(q)$  cannot be compact or strong causality must be violated within this set leading us to a contradiction.

Secondly, assume  $q \in \overline{D^+(S)} \setminus \partial D^+(S)$  i.e.,  $q$  is an interior point of the future domain of dependence of  $S$ . Then there is an  $r \in D^+(S)$  chronologically preceded by  $q$  (with respect to some time function assigned to the Cauchy foliation of  $D^+(S)$ ). Letting  $N := J^+(S) \cap J^-(r)$  then  $N \subset D^+(S)$  hence  $(N, g|_N)$  is a globally hyperbolic space-time containing the Malament–Hogarth event  $q$  and the curve  $\gamma_C$ . Consequently we can proceed as above to arrive at a contradiction again.  $\diamond$

By the aid of this one can provide a characterization of Malament–Hogarth space-times [17].

**Lemma 3.2.** *Let  $(M, g)$  be a Malament–Hogarth space-time with  $q \in H^+(S) \subset M$  a Malament–Hogarth event. Consider a timelike curve  $\gamma_C$  as above with  $\gamma_C(\mathbb{R}^+) \subset J^+(S) \cap J^-(q)$ . Then either  $\overline{S \cap J^-(q)}$  is non-compact or strong causality is violated at  $q \in M$  (or both cases can happen).*

*Proof.* By definition and construction  $\gamma_C$  is a future-inextendible non-spacelike half-curve in  $J^+(S) \cap J^-(q)$ . Subsequently, by [44, Lemma 8.2.1] and Lemma 2.3 here we get the result.  $\diamond$

Now we can turn our attention to the existence of physically relevant examples of space-times possessing the Malament–Hogarth property. Lemma 3.2 indicates that the class of Malament–Hogarth space-times can be divided into two major subclasses: the first one contains space-times in which an infinite, non-compact portion of a spacelike submanifold is visible from some event.

<sup>6</sup>That is, the computer and the observer “stay together” during the course of the computation along a common worldline  $\gamma_{CO}$  in  $(M, g)$  which is moreover of finite length in practice.

There is an abundance of such examples: any maximal extension of the Reissner–Nordström, Kerr [18], (universal cover of the) anti-de Sitter [24, 25] are examples.

The second subclass consists of those which lack strong causality along the future Cauchy horizon of their maximal extension. A maximally extended Taub–NUT space-time, certain extendible Gowdy space-times possess this property however the corresponding inextendible curves are incomplete i.e., have finite lengths only.<sup>7</sup> In other words these non-globally hyperbolic space-times are generalized Malament–Hogarth space-times only. At this moment we cannot answer the question whether or not this second subclass of Malament–Hogarth space-times is empty.

After getting some feeling of Malament–Hogarth space-times we indicate their relationship with the strong cosmic censorship scenario. The content of Lemma 3.2 is that the Malament–Hogarth property implies the Geroch–Horowitz–Penrose property for non-globally hyperbolic space-times. However, the converse is not necessarily true as we have seen. But the converse seems to be true at least for asymptotically flat or hyperbolic space-times. Guided by these observations we cannot resist the temptation to exhibit a sharper formulation of the strong cosmic censor conjecture as follows [17].

**Conjecture 3.1.** (sharpening of the Geroch–Horowitz–Penrose version of the strong cosmic censor conjecture) *Let  $(S, h, k)$  be an asymptotically flat or asymptotically hyperbolic (i.e., anti-de Sitter) initial data for Einstein’s equation (this implies  $(S, h)$  is geodesically complete). Suppose a fundamental matter field is given represented by its stress-energy tensor  $T$  satisfying the dominant energy condition.*

*Then, if the maximal Cauchy development of this initial data set is extendible, this extension is a Malament–Hogarth space-time and Malament–Hogarth events lie on or beyond the future Cauchy horizon  $H^+(S)$  in the extension.*

In analogy with the reformulated Theorem 2.2 this conjecture may be reformulated as well.

**Conjecture 3.2.** (reformulation of the sharpening) *Let  $(S, h, k)$  be an initial data set for Einstein’s equation as in Conjecture 3.1. Then if the maximal Cauchy development of this space-time is extendible, for each  $x \in H^+(S)$  in any extension,  $J^+(S) \cap J^-(x)$  contains a future-directed timelike curve of infinite length.*

A promising attack on this conjecture, straightforward by this reformulation is to study the so-called “radiation problem” formulated in the introduction of [10]. The authors address the problem of finding points whose causal futures are complete in the Cauchy development of a given asymptotically flat initial data.

In light of our considerations sofar Conjecture 3.2 can be read such a way that a non-globally hyperbolic asymptotically flat or anti-de Sitter space-time contains a gravitational computer capable to break the Turing barrier. Therefore the problem of the existence of such space-times is apparently the same as that of computers capable of performing non-Turing computations.

## 4 Concluding remarks: what has been proved?

One conclusion of our considerations here is that the problem of the *strong cosmic censorship* naturally splits up into two parts: suppressing the problem of genericity or stability in the formulation we obtain a causal or conformal variant **SCCC-GHP** in Section 1 (i.e., Theorems 2.1

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<sup>7</sup>Hereby we acknowledge that [17, Proposition 2.5] is false because it is based on an erroneous calculation.

and 2.2) whose proof is easy (essentially a consequence of the definition of global hyperbolicity). Meanwhile putting the emphasis onto the genericity or stability of extendible space-times we obtain a more geometric version like **SCCC** in Section 1 and run into the well-known technical difficulties. Unlike **SCCC-GHP**, the **SCCC** is “very much open”. But we have learned that **SCCC-GHP** (i.e., Theorems 2.1 and 2.2) has an appropriate geometric modification namely **SCCC-MH** in Section 1 (i.e., Conjectures 3.1 and 3.2).

The other conclusion is that in dealing with the usual formulation **SCCC** of the strong cosmic censor conjecture one also seems to encounter (through the concept of a gravitational computer and **SCCC-MH**) certain very general variants of the Church–Turing thesis namely **Ph-ChT** in Section 1 controlling the computational capacity of a broad class of physical computers (called gravitational computers here). This indicates that the strong cosmic censor conjecture in its full depth might be not only technically but even conceptually an extraordinary difficult problem.

However we have to emphasize again that our speculations require future work: for example it is important to understand if other asymptotically flat or hyperbolic, extendible space-times admit the Malament–Hogarth property or not. It would be also interesting to know if the aforementioned new type of Malament–Hogarth space-times (i.e., which violate the strong causality along their Cauchy horizons) exist or not.

Nevertheless if our considerations turn out to be correct then we can establish an intimate link betwix the strong cosmic censor conjecture, a problem situated in the heart of recent theoretical physics and computability theory, a subject previously considered as a pure mathematical discipline.

**Acknowledgement.** The author is grateful to I. Rácz, E. Minguzzi and L.B. Szabados for the stimulating discussions. The work was partially supported by OTKA grant No. NK81203 (Hungary).

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