# Computability: the hidden face of gravity*

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**Abstract**

In this essay we shall look at the old problem of the strong cosmic censorship from a new angle allowing us to make a contact with an apparently very different discipline of science namely computability theory and the Church–Turing thesis. This leads on the one hand to a simple proof of a variant of the strong cosmic censor conjecture attributed to Geroch–Horowitz and Penrose but first formulated by Wald as well as on the other hand to a natural but sofar hidden conceptual equivalence of the strong cosmic censor conjecture and the physical Church–Turing thesis.

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## 1

Without doubt the deepest open question of classical general relativity theory is the so-called *cosmic censor conjecture* first formulated by R. Penrose four decades ago [29]. Roughly speaking the conjecture claims that predictability, one of the most fundamental concepts of classical physics, remains valid in the realm of classical general relativity i.e., all "physically relevant" space-times admit well-posed initial value formulation akin to other field theories. Meanwhile there has been a remarkable progress which culminated in a general satisfactory solution of the problem of existence and behaviour of *short-time* solutions to the Einstein's constraint equations [27] the cosmic censor conjecture deals with the existence and properties of *long-time* solutions [9] and is still "very much open" as Penrose says in [32]. One may then wonder what is the reason of this? Is the cosmic censor conjecture merely a technically more difficult question or is rather a conceptually deeper problem? On the contrary of its expected unified solution the cosmic censor conjecture has rather split up into a bunch of rigorous or less rigorous formulations, versions during the course of time. Therefore we can say that nowadays there are several "front lines" where "battles" for settling or violating the cosmic censor conjecture are going on.

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Far from being complete we can mention the following results on the subjectmatter. The so-called *weak cosmic censor conjecture* in simple terms postulates:

**WCCC** *In a generic (i.e., stable), physically relevant (i.e., obeying some energy condition), asymptotically flat space-time singularities are hidden behind event horizons of black holes.*

The weak version can be formulated rigorously as a Cauchy problem for general relativity and the aim is to prove or disprove that for "generic" or "stable" (in some functional analytic sense) initial values at least, event horizons do form around singularities in an asymptotically flat space-time (where the notion of a black hole exists).

The first arguments in favour to this weak form came from studying the stability of the Schwarzschild event horizon under simple, linear perturbations of the metric. An early attempt to violate the weak version was the following. As it is well-known, a static, electrically charged black hole has only two parameters, namly its mass and charge. However if its charge is too high compared with its mass, event horizon do not occur hence the singularity could be visible by a distant observer. Consequently we may try to overcharge a static black hole in order to destroy its event horizon (we may argue in the same fashion in case of rotating black holes). However this is impossible as it was pointed out by Wald [36] in 1974. Another, more general but still indirect, argument for the validity of weak cosmic censorship is the so-called Riemannian Penrose inequality [30] proved by Bray [3] and Huisken–Ilmanen [24] in 1997. As an important step, the validity of the weak version in case of spherical collapse of a scalar field was established by Christodoulou [6, 7] in 1999.

The *strong cosmic censor conjecture* proposes more generally that all events have cause that is, there exist events chronologically preceding them and these events form a spacelike initial surface in any reasonable space-time. This also implies that singularities, except a possible initial "big bang" singularity, are invisible for observers:

**SCCC** *A generic (i.e., stable), physically relevant (i.e., obeying some energy condition) space-time is globally hyperbolic.*

Therefore this strong version also can be formulated in terms of a Cauchy problem but in this case we want to prove the inextendibility of maximal Cauchy developments of "generic" or "stable" (again in some functional analytic sense) initial data. Apparently this problem requires different techniques compared with the weak version.

Concerning the strong censorship we have partial important results, too. On the one hand its validity was proved by Chruściel–Isenberg–Moncrief [10] and Ringström for certain Gowdy space-times (for a recent survey cf. [34]) while by Chruściel–Rendall [11] in 1995 in the case of spatially compact and locally homogeneous space-times such as the Taub–NUT geometry. On the other hand one may also seek counterexamples to understand the meaning of "generic" in both the weak and strong versions. Many authors (e.g. [4, 6, 19, 23]) found hints in several physically relevant situations for the violation of the weak or strong versions. A thin class of Gowdy space-times [10, 34] also lacks global hyperbolicity.

We may however find a kind of "compromise" between two extremal approaches: seeking a general proof or hunting for particular counterexamples. This is the following. As it is well-known, the strong version is false in its simplest intuitive form. That is, there are several physically relevant space-times what is more: basic solutions to the Einstein's equation which lack global hyperbolicity i.e., the maximal Cauchy development of the corresponding initial data set is extendible. The Taub–NUT space-time is extendible and in this case global hyperbolicity fails in such a way that strong causality breaks down on the future Cauchy horizon in any extension. The Reissner–Nordström, Kerr, (universal covering of) anti-de Sitter space-times are also extendible but in these cases global hyperbolicity is lost differently: from the future Cauchy horizons of their extensions a non-compact, infinite portion of their initial surfaces is observable. Therefore we have to allow a collection of counterexamples consisting of apparently "non-generic" i.e., "unstable" space-times. Indeed, there are indirect hints that these extendible solutions are exceptional and atypical in some sense: small generic perturbations of them turn their Cauchy horizons into real curvature singularity thereby destroying extendibilty and saving strong cosmic censorship [12, 13, 22, 28, 33].

Since nowadays we do not know any other type of violation we may roughly formulate the strong version as follows due to Geroch–Horowitz [18] and Penrose [31] from 1979 but explicitly formulated by Wald [37, 305p.]:

**SCCC-GHP** *If a physically relevant (i.e., obeying some energy condition) space-time is not globally hyperbolic then its Cauchy horizon looks like either that of the Taub–NUT or that of the Kerr space-time.*

Observe that compared to **SCCC** in this formulation the highly complex question of "genericity" or "stability" has been suppressed and incorporated into that of Taub–NUT-like [22] and Kerr-like space-times [1]. Since this version focuses only on the causal character of extendible space-times instead of their non-genericity, we may expect a proof of **SCCC-GHP** using causal set theoretic methods only (instead of heavy functional analytic ones). It is quite surprising that this version indeed can be proved [15, Theorem 2.1 and 2.2] using ideas motivated by recent advances in an interdisciplinary field connecting computability and general relativity theory.

## 2

Recently there has been a remarkable interest in the physical foundations of computability theory and the Church–Turing thesis. It turned out that algorithm theory, previously considered as a very mathematical field, has a deep link with basic concepts of physics.

On the one hand we realized that our apparently pure mathematical notion of a Turing machine involves indirect preconceptions on space, time, motion, state and measurement. Hence it is reasonable to ask whether different choices of physical theories put for modeling these things have some effect on our notions of computability or not. At the recent stage of affairs it seems there are striking changes on the whole structure of complexity and even computability theory if we pass from classical physics to quantum or relativistic theories. Even certain variants of the Church–Turing thesis cease to be valid in some cases.

For instance taking *quantum mechanics* as our background theory the famous Chaitin's omega number, a typical non-computable real number, becomes enumerable via an advanced quantum computer [5]. An adiabatic quantum algorithm also exists to attack Hilbert's tenth problem [25, 26]. Chern–Simons *topological quantum field theory* can be used to calculate the Jones polynomial of knots known to belong to a higher class of computational complexity [17]. In the same fashion if we use *general relativity theory*, powerful "gravitational computers" can be constructed, also capable of breaking Turing's barrier [38]: Hogarth proposed a class of space-times in 1994, now called as *Malament–Hogarth space-times* allowing non-Turing computations [20, 21]. Hogarth' construction uses anti-de Sitter space-time which is also in the focal point of recent investigations in high energy physics. In the same vein, in 2001 the author and Németi constructed another example by exploiting properties of the Kerr geometry [16]. This space-time is also relevant as being the only candidate in general relativity for the final state of a collapsed, massive, slowly rotating star. A general introduction to the topic is Chapter 4 of Earman's book [14].

On the other hand it is conjectured that these generalized computational methods are not only significant from a computational viewpoint but, in case of quantum computers at least, they are also in connection with our most fundamental physical concepts such as the standard model and string theory [2, 35].

The natural question therefore arises if the same is true for "gravitational computers" i.e., is there any pure physical characterization of Malament–Hogarth space-times? The aforementioned proof [15, Theorem 2.1 and 2.2] of **SCCC-GHP** uses standard causal set theory only with the simple but key observation that if a space-time admits a non-globally hyperbolic extension then in the causal pasts of events on its future Cauchy horizon future-inextendible, non-spacelike curves appear. These curves also play a crucial role in the theory of "gravitational computers": Malament–Hogarth space-times are exactly those for which such curves exist, are timelike and complete.

Let us recall here what a Malament–Hogarth space time is (cf. [14, 15, 16, 20, 21, 38]):

**Definition.** $(M, g)$ *is called a* Malament–Hogarth space-time *if there is a future-directed timelike half-curve* $\gamma_C : \mathbb{R}^+ \to M$ *such that* $\|\gamma_C\| = +\infty$ *and a point* $q \in M$ *satisfying* $\gamma_C(\mathbb{R}^+) \subset J^-(q)$. *The event* $q \in M$ *is called a* Malament–Hogarth event.

If $(M, g)$ is a Malament–Hogarth space-time then there exists a future-directed timelike curve $\gamma_O : [a, b] \to M$ joining $p \in J^-(q)$ with $q$ satisfying $\|\gamma_O\| < +\infty$. The point $p \in M$ can be chosen to lie in the causal future of the past endpoint of $\gamma_C$. Moreover (cf. [15, Lemma 3.1]) a Malament–Hogarth space-time cannot be globally hyperbolic; if $q \in M$ is a Malament–Hogarth event and $S \subset M$ is a connected spacelike hypersurface such that $\gamma_C(\mathbb{R}^+) \subset J^+(S)$ then $q$ is on or beyond the future Cauchy horizon $H^+(S)$ of $S$.

The motivation is the following (for details we refer to [16]). Consider a Turing machine realized by a physical computer $C$ moving along the curve $\gamma_C$ of *infinite* proper time. Hence the physical computer (identified with $\gamma_C$) can perform arbitrarily long calculations in the ordinary sense. In addition there exists an observer $O$ following the curve $\gamma_O$ (hence denoted by $\gamma_O$) of *finite* length such that he hits the Malament–Hogarth event $q \in M$ in *finite* proper time. But by definition $\gamma_C(\mathbb{R}^+) \subset J^-(q)$ therefore in $q$ he can receive the answer for a *yes or no question* as the result of an *arbitrarily long* calculation carried out by the physical computer $\gamma_C$. This is because $\gamma_C$ can send a light beam at arbitrarily late proper time to $\gamma_O$. Clearly the pair $(\gamma_C, \gamma_O)$ in $(M, g)$ with a Malament–Hogarth event $q$ is an artificial computing system i.e., a generalized computer in the sense of [16].

Imagine the following exciting situation as an example. $\gamma_C$ is asked to check all theorems of our usual set theory (ZFC) in order to check consistency of mathematics. This task can be carried out by $\gamma_C$ since its world line has infinite proper time. If $\gamma_C$ finds a contradiction, it can send a message (for example an appropriately coded light beam) to $\gamma_O$. Hence if $\gamma_O$ receives a signal from $\gamma_C$ *before* the Malament–Hogarth event $q \in M$ he can be sure that ZFC set theory is not consistent. On the other hand, if $\gamma_O$ does not receive any signal before $q$ then, *after* $q$, $\gamma_O$ can conclude that ZFC set theory is consistent. Note that $\gamma_O$ having finite proper time between the events $\gamma_O(a) = p$ (departure for the experiment) and $\gamma_O(b) = q$ (hitting the Malament–Hogarth event), he can be sure about the consistency of ZFC set theory within finite (possibly very short) time. This shows that certain very general formulations of the Church–Turing thesis (for instance [16, Thesis 2,2' and 3]) cannot be valid in the framework of classical general relativity.

After getting some feeling of Malament–Hogarth space-times we indicate their relationship with the strong cosmic censorship. First one can raise the question if Malament–Hogarth space-times are relevant or not from a physical viewpoint. Checking case-by-case several physically relevant maximally extended examples lacking global hyperbolicity like Kerr, Reissner–Nordström, (universal cover of) anti-de Sitter we find that these space-times indeed possess the Malament–Hogarth property. On the contrary the Taub–NUT and certain extendible polarized Gowdy space-times with toroidal spatial topology lack this: future inextendible curves in the past of events on the future Cauchy horizon exist but are incomplete.

This motivates us to sharpen the easily provable **SCCC-GHP** version of the strong cosmic censor conjecture like this (cf. [15, Conjecture 3.1 and 3.2]):

**SCCC-MH** *If a physically relevant (i.e., obeying some energy condition), asymptotically flat or asymptotically hyperbolic (i.e., anti-de Sitter) space-time is not globally hyperbolic then it is a Malament–Hogarth space-time.*

Note that this formulation—like **SCCC-GHP**—continues to avoid the question of "genericity" or "stability". We cannot prove or disprove this version but a promising attack on it is to study the so-called "radiation problem" [8].

Instead we call attention that **SCCC-MH** sheds some light onto a possible deep link between cosmic censorship and computability theory as follows. Consider the following physical reformulation of the Church–Turing thesis:[1]

**Ph-ChT** *An artificial computing system based on a generic (i.e., stable), relevant (i.e., obeying some energy condition) classical physical system realizes Turing-computable functions.*

This formulation—in contrast to versions like [16, Thesis 2 and 2']—is quite democratic because it does not *a priori* excludes the existence of too powerful computational devices; it just says that they must in one or another way be unstable (which is apparently true for the various devices in [5, 25, 26, 16, 20, 21]).

In general—keeping in mind the definition of a Malament–Hogarth space-time—a quintuple $(M, g, q, \gamma_C, \gamma_O)$ is called a *gravitational computer* if $(M, g)$ is a space-time, $\gamma_C$, $\gamma_O$ are timelike curves and $q \in M$ is an event such that the images of these curves lie within $J^-(q)$. This concept is broad enough to serve as an abstract model for all kind of artificial computing systems based on classical physics so that an artificial computing system can perform non-Turing computations if and only if the corresponding gravitational computer is defined in an ambient space-time $(M, g)$ possessing the Malament–Hogarth property [15].

Accepting that all artificial computing systems based on classical physics can be modeled by gravitational computers as well as accepting **SCCC-MH** we can see that **SCCC** and **Ph-ChT** are roughly equivalent hence involve the same depth. This might serve as an explanation for the permanent difficulty present in all approaches to the strong cosmic censor conjecture. Indeed, in light of our considerations sofar **SCCC-MH** can be read such a way that a non-globally hyperbolic asymptotically flat or anti-de Sitter space-time contains a gravitational computer capable of breaking the Turing barrier.

Therefore the problem of the existence of non-globally hyperbolic space-times is apparently the same as that of artificial computing systems capable of performing non-Turing computations; both expected to be non-generic, unstable phenomena in Nature.

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[1] For the concept of an "artificial computing system" and of a "Turing computable function" in particular and for further details in general, we refer to [16, Chapter 2] and references therein.

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