Gravitational interpretation of the Hitchin equations

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Abstract

By referring to theorems of Donaldson and Hitchin, we exhibit a rigorous AdS/CFT-type correspondence between classical $2+1$ dimensional vacuum general relativity theory on $\Sigma \times \mathbb{R}$ and SO(3) Hitchin theory (regarded as a classical conformal field theory) on the spacelike past boundary Σ, a compact, oriented Riemann surface of genus greater than one. Within this framework we can interpret the $2+1$ dimensional vacuum Einstein equation as a decoupled "dual" version of the 2 dimensional SO(3) Hitchin equations.

More precisely, we prove that if over Σ with a fixed conformal class a real solution of the SO(3) Hitchin equations with induced flat $\mathrm{SO}(2,1)$ connection is given, then there exists a certain cohomology class of non-isometric, singular, flat Lorentzian metrics on $\Sigma \times \mathbb{R}$ whose Levi–Civita connections are precisely the lifts of this induced flat connection and the conformal class on Σ, induced by this cohomology class, agrees with the fixed one.

Conversely, given a singular, flat Lorentzian metric on $\Sigma \times \mathbb{R}$, the restriction of its Levi–Civita connection gives rise to a real solution of the SO(3) Hitchin equations on Σ with respect to the conformal class, induced by the corresponding cohomology class of the Lorentzian metric.

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1 Introduction

The aim of this paper is to offer a new physical interpretation of the 2 dimensional SO(3) Hitchin equations as the coupled "dual" version of the $2+1$ dimensional vacuum Einstein equation. This interpretation emerges within a rigorous AdS/CFT-type correspondence between Hitchin's

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theory of Higgs bundles over Riemann surfaces, regarded as a 2 dimensional conformal field theory on the boundary and classical $2+1$ dimensional vacuum general relativity on a bulk space-time. Our work may be viewed as an arch spanned by Witten's ideas [14] between two massive bearers: a theorem of Hitchin [6] and another one by Donaldson [4] respectively, as follows.

The relationship between three dimensional general relativity and gauge theory is not new. The main motivation for studying their common features comes from the effort to formulate a satisfactory quantum theory underlying three dimensional general relativity (we cannot make the attempt to survey the vast literature of the quantization issue; for a survey cf. [11] or a more recent excellent one is [3]). Just to mention one trial, Witten argued that Lorentzian vacuum general relativity theory should be equivalent to an $\mathrm{ISO}(2,1)$ Chern–Simons theory at the full quantum level; thereby general relativity in $2+1$ dimensions turned out to be not only exactly soluble at the classical level but also renormalizable as a quantum field theory [14][13]. The key technical tool in Witten's approach is formulating general relativity in terms of a connection and a "dreibein", instead of a metric. This approach is remarkable because establishing any conventional relationship between general relativity and Yang–Mills theory apparently fails in any other dimensions, despite the efforts made over the past thirty years.

More recently there has been interest among physicists in understanding the celebrated Maldacena conjecture or AdS/CFT correspondence [8] which sheds new light onto the gauge theory-gravity duality. Broadly speaking, this conjecture states the existence of a duality equivalence between some quantum gravitational theories on an anti-de Sitter space M and quantum conformal field theories on the boundary at conformal infinity ∂M. At the semi-classical level and using a (Wick rotated) pure gravity theory in the bulk the correspondence was formulated by Witten [12] and states that

$$Z_{CFT}([\gamma]) = \sum \mathrm{e}^{-I(g)} \tag{1}$$

where Z_{CFT} is the partition function of some conformal field theory attached to a conformal structure $[\gamma]$ on ∂M and I is the (renormalized) Einstein–Hilbert action of an Einstein metric on M with conformal infinity $[\gamma]$. The formal sum is taken over all manifolds and Einstein metrics (M, g) with given boundary data $(\partial M, [\gamma])$. As we mentioned, three dimensionality is distinguished in the conventional understanding of the gauge theory-gravity relationship [14], in the holographic approach however, Chern–Simons theory may play a role in various dimensions [2].

The AdS/CFT correspondence at least in its strict classical form has attracted some attention from the mathematician's side as well and led to nice geometrical results (cf. [1] for a survey and references therein). This paper can also be regarded as an attempt to extend further its mathematical understanding by a natural generalization of the very core of the correspondence as well as link three dimensional gravity with Hitchin's theory of Higgs bundles over Riemannian surfaces. Since this is an integrable system, the relationship provides a further explanation, different from Witten's, why three dimensional gravity is exactly soluble.

This link is probably not surprising because in our opinion three dimensional *classical* gravity in *vacuum* is a two rather than three dimensional theory in its nature as can be seen by a simple topological argument. In three dimensions a Ricci flat space is flat. Although every compact, orientable three-manifold has zero Euler characteristic i.e., admits Lorentzian structures, only a few of them are flat and are not interesting examples of three-manifolds because typically are just finitely covered by the three-torus. Consequently we have to seek non-compact spaces carrying solutions of the classical vacuum Einstein equation like the annulus $\Sigma \times \mathbb{R}$; however this is rather a two dimensional object from a topological viewpoint. Nevertheless three dimensionality enters at the full *quantum* level as it was pointed out by Witten [13].

Our paper is organized as follows. In Section 2 we prove that a real solution of the SO(3) Hitchin equations over a compact oriented Riemann surface Σ of genus $g > 1$ induces a certain cohomology class of singular solutions of the Lorentzian vacuum Einstein equation over the annulus $\Sigma\times\mathbb{R}$. "Singular" in this context means that the metrics may degenerate. Such solutions appear naturally if the metric is expressed via a connection and an independent dreibein, as was pointed out by Witten [13]. Our construction is based on a theorem of Hitchin [6] (cf. Theorem 2.1 here) on the relationship between real SO(3) Hitchin pairs and flat SO(2, 1) connections.

Conversely, in Section 3, by referring to a theorem of Donaldson [4] (cf. Theorem 3.1 here) which states the equivalence between flat $\mathrm{PSL}(2,\mathbb{C})$ connections and SO(3) Hitchin pairs, we present the reversed construction, namely, starting from a cohomology class of flat singular Lorentzian metrics on $\Sigma\times\mathbb{R}$ one can recover a unique real solution of the SO(3) Hitchin equations on Σ, regarded as the past boundary of the annulus. The conformal structure of Σ also emerges by the cohomology class.

We present our main results in Section 4. On the one hand we exhibit the correspondence in a precise form using the field equations (cf. Theorem 4.1). This presentation has the remarkable feature that it exhibits the vacuum Einstein equation as a sort of "untwisted", or "decoupled" variant of the Hitchin equations and vica versa if we interpret the flat SO(2, 1) connection on the gravitational side as being dual to the (non-flat) SO(3) connection on the gauge theoretic side and simlarly, we regard the dreibein as being dual to the Higgs field.

Then we rephrase the correspondence in terms of the metric on the gravitational side. This way we can see that this correspondence looks like a generalized geometric AdS/CFT correspondence. Here "generalized" means that the conformal geometry on the boundary emerges in a rather abstract way compared with the usual AdS/CFT correspondence, namely, by exploiting the conformal properties of an equation which is essentially the massless Dirac equation as explained by Propositions 2.3 and 2.4.

Finally, we conclude with some speculations on the subject in Section 5.

2 From Hitchin pairs to flat metrics

The embedding $\mathrm{SL}(2,\mathbb{R})\subset \mathrm{SL}(2,\mathbb{C})$ induces the factorized embedding $\mathrm{PSL}(2,\mathbb{R})\subset\mathrm{PSL}(2,\mathbb{C})$ and we will write $\mathrm{PSL}(2,\mathbb{R})\cong\mathrm{SO}(2,1)$. Let $(\Sigma,[\gamma])$ be a compact, oriented Riemann surface of genus $g > 1$ with the conformal equivalence class of a smooth Riemannian metric γ that is, a complex structure on it. Moreover let P be an SO(3) principal bundle over Σ with either $w_2(P) = 0$ or $w_2(P) = 1$ and denote by $P^{\mathbb{C}}$ the corresponding complexified $\mathrm{PSL}(2,\mathbb{C})$ principal bundle. Regarding SO(3) as a real form of $\mathrm{PSL}(2,\mathbb{C})$ there is an associated anti-involution $*$ of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. If ∇_A is an SO(3) connection with curvature F_A on P and $\Phi\in\Omega^{1,0}(\Sigma,\mathrm{ad}(P^{\mathbb{C}}))$ is a complex Higgs field then the Hitchin equations over $(\Sigma,[\gamma])$ read as follows [6]:

$$\begin{cases} F_A+\frac{1}{2}[\Phi,\Phi^*] &= 0\\ \overline{\partial}_A\Phi &= 0.\end{cases}\tag{2}$$

Recall that these equations are the dimensional reduction of the four dimensional SO(3) self-duality equations hence are conformally invariant and exactly soluble.

Consider a solution (∇_A, Φ) of (2) associated to a fixed SO(3) principal bundle P. If $\mathcal{A}(P)$ is the affine space of SO(3) connections over P then a map $\alpha : \mathcal{A}(P) \times \Omega^{1,0}(\Sigma, \mathrm{ad}(P^{\mathbb{C}})) \to \mathcal{A}(P^{\mathbb{C}})$ is defined as

$$\alpha(\nabla_A, \Phi) = \nabla_A + \Phi + \Phi^*. \tag{3}$$

Clearly the map descends to the gauge equivalence classes. Locally, on an open subset the resulting $\mathrm{PSL}(2,\mathbb{C})$ connection ∇_B looks like $\nabla_B|_U = \mathrm{d} + B_U$ with $B_U = A_U + \Phi|_U + \Phi^*|_U$. It is easy to see that ∇_B is flat. Indeed, one quickly calculates

$$F_B = \mathrm{d}B + \frac{1}{2}[B,B] = F_A + \frac{1}{2}[\Phi,\Phi^*] + \overline{\partial}_A\Phi + \partial_A\Phi^* = 0$$

via (2). One may raise the question of ∇_B is moreover real valued i.e., wether takes its value in $\mathrm{SO}(2,1)$. This is answered by a theorem of Hitchin (cf. Proposition 10.2 and Theorem 10.8 in [6]) which is the starting point of our discussion:

Theorem 2.1. (Hitchin, 1987) *Let $(\Sigma, [\gamma])$ be a compact, oriented Riemann surface of genus $g > 1$ endowed with a conformal equivalence class $[\gamma]$ of a smooth Riemannian metric γ. Let P be a principal SO(3) bundle over Σ satisfying either $w_2(P) = 0$ or $w_2(P) = 1$. Denote by $\mathcal{M}(P)$ the moduli space of gauge equivalence classes of smooth solutions to the SO(3) Hitchin equations over P with respect to $[\gamma]$. Consider a map $\sigma : \mathcal{M}(P) \to \mathcal{M}(P)$ given by*

$$\sigma([(\nabla_A, \Phi)]) := [(\nabla_A, -\Phi)].$$

The fixed point set of σ has connected components $\mathcal{M}_0$, $\mathcal{M}_2$, $\mathcal{M}_4, \dots, \mathcal{M}_{2g-2}$ for $w_2(P) = 0$ and $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_3, \dots, \mathcal{M}_{2g-3}$ for $w_2(P) = 1$. The subset $\mathcal{M}_0$ is the space of flat SO(3) connections on P while $\mathcal{M}_k$ with $k > 0$ can be identified with the $6g - 6$ dimensional moduli of smooth, flat, irreducible SO(2,1) connections of the form (3) on certain principal SO(2,1) bundles Q_k of Euler class k over Σ. $\diamond$

Remark. Putting a suitable complex structure J onto $\mathcal{M}(P)$ the map σ can be regarded as an anti-holomorphic involution i.e., a *real structure* on $(\mathcal{M}(P), J)$. This explains why complex flat connections of the form (3) corresponding to the fixed point set of σ inherit a real nature in the sense above (cf. [6], Section 10 for details). Notice that *all* irreducible, flat $\mathrm{SO}(2,1)$ connections with non-zero Euler class over Σ arise this way.

We wish to use these real, flat, irreducible connections to construct certain flat Lorentzian metrics over $\Sigma \times \mathbb{R}$ with a fixed orientation induced by the orientation of Σ. Consider the standard 3 dimensional real, irreducible representation $\varrho : \mathrm{SO}(2,1) \times \mathbb{R}^3 \to \mathbb{R}^3$ and take the associated real rank 3 vector bundles $E_k := Q_k \times_\varrho \mathbb{R}^3$. We restrict attention to the bundle E_{2g-2} for which we have an isomorphism $E_{2g-2} \cong T\Sigma \oplus \underline{\mathbb{R}}$. For simplicity we shall denote this bundle as E and the associated irreducible, flat $\mathrm{SO}(2,1)$ connections of Theorem 2.1 on E as ∇_B.

Let $\pi : \Sigma \times \mathbb{R} \to \Sigma$ be the obvious projection and consider the pullback bundle π^*E. This bundle admits an irreducible, flat $\mathrm{SO}(2,1)$ connection $\pi^*\nabla_B$ by pulling back ∇_B from E. In the remaining part of the paper we will study the complex

$$0 \to \Omega^0(\Sigma \times \mathbb{R},\, \pi^*E) \xrightarrow{\pi^*\nabla_B} \Omega^1(\Sigma \times \mathbb{R},\, \pi^*E) \xrightarrow{\pi^*\nabla'_B} \Omega^2(\Sigma \times \mathbb{R},\, \pi^*E) \to 0 \tag{4}$$

where ∇'_B is the induced connection, and in particular its first cohomology

$$H^1(\pi^*\nabla_B) = \frac{\mathrm{Ker}\,(\pi^*\nabla'_B)}{\mathrm{Im}\,(\pi^*\nabla_B)}$$

which will turn out to be of central importance to us.

Let $\widehat{\pi^*E}$ be an affine vector bundle over $\Sigma\times\mathbb{R}$ whose underlying vector bundle is π^*E. Fix an element $\xi\in\Omega^1(\Sigma\times\mathbb{R},\,\pi^*E)$. We can regard ξ as a fiberwise translation in $\widehat{\pi^*E}$ by writing $\xi_X(\hat{v}):=\hat{v}+\xi(X)$ with X a vector field on $\Sigma\times\mathbb{R}$ and $\hat{v}\in\Omega^0(\Sigma\times\mathbb{R},\,\widehat{\pi^*E})$. Therefore we have an embedding

$$\Omega^1(\Sigma\times\mathbb{R},\,\pi^*E)\subset\Omega^1(\Sigma\times\mathbb{R},\,\mathrm{End}(\widehat{\pi^*E}))\tag{5}$$

into the Lie algebra of the affine transformation group of $\widehat{\pi^*E}$. Consequently if $\pi^*\nabla_B$ is a flat $\mathrm{SO}(2,1)$ connection on π^*E then

$$\hat{\nabla}_{B,\xi}:=\pi^*\nabla_B+\xi\tag{6}$$

is an $\mathrm{ISO}(2,1)$ connection on $\widehat{\pi^*E}$ where $\mathrm{ISO}(2,1)$ denotes the $2+1$ dimensional Poincaré group. Its curvature is

$$\hat{F}_{B,\xi}=\pi^*F_B+(\pi^*\nabla'_B)\xi+\frac{1}{2}[\xi,\xi]=(\pi^*\nabla'_B)\xi$$

since translations commute. We obtain that $\hat{\nabla}_{B,\xi}$ is flat if and only if

$$(\pi^*\nabla'_B)\xi=0.\tag{7}$$

Observe that *all* irreducible, flat $\mathrm{ISO}(2,1)$ connections arise this way. Clearly the gauge equivalence class of $\hat{\nabla}_{B,\xi}$ is unchanged if $\pi^*\nabla_B$ is replaced by an $\mathrm{SO}(2,1)$ gauge equivalent connection and ξ by $\xi+(\pi^*\nabla_B)v$ with an arbitrary section $v\in\Omega^0(\Sigma\times\mathbb{R},\,\pi^*E)$. Therefore on the one hand we identify the space $H^1(\pi^*\nabla_B)$ with the underlying vector space of gauge equivalence classes of smooth flat $\mathrm{ISO}(2,1)$ connections on $\widehat{\pi^*E}$ with fixed $\mathrm{SO}(2,1)$ part $\pi^*\nabla_B$.

On the other hand out of a flat $\mathrm{ISO}(2,1)$ connection one can construct a singular Lorentzian metric on $\Sigma\times\mathbb{R}$ as follows [14]. Fix once and for all a smooth $\mathrm{SO}(2,1)$ metric h on π^*E (notice that this bundle is an $\mathrm{SO}(2,1)$ bundle) and pick up a flat connection of the form (6) on $\widehat{\pi^*E}$. Via the isomorphism

$$\Omega^1(\Sigma\times\mathbb{R},\,\pi^*E)\cong\Gamma(\mathrm{Hom}(T(\Sigma\times\mathbb{R}),\,\pi^*E))$$

we can interpret ξ as a "dreibein" $\xi:T(\Sigma\times\mathbb{R})\to\pi^*E$ (since π^*E and $T(\Sigma\times\mathbb{R})$ are isomorphic bundles). Assume for a moment that $\xi^{-1}_x:(\pi^*E)_x\to T_x(\Sigma\times\mathbb{R})$ exists for all $x\in\Sigma\times\mathbb{R}$ that is, ξ is invertible as a bundle map. Using ξ we can construct a smooth Lorentzian metric $g_\xi:=h\circ(\xi\times\xi)$ on $T(\Sigma\times\mathbb{R})$. Locally $(g_\xi)_{ij}=\xi^p_i\xi^q_jh_{pq}$. We can suppose that the metric constructed this way is inextensible. The connection $\xi^{-1}\circ(\pi^*\nabla_B)\circ\xi$ is compatible with g_ξ hence it represents the Levi–Civita connection of g_ξ if it is torsion free. However this is provided by (7) since this equation is just the Cartan equation for the smooth metric g_ξ and the connection $\xi^{-1}\circ(\pi^*\nabla_B)\circ\xi$. This shows that g_ξ is flat. We obtain that a flat connection (6) gives rise to the pair of a smooth Lorentzian metric on $\Sigma\times\mathbb{R}$ and its smooth Levi–Civita connection

$$g_\xi=h\circ(\xi\times\xi),\qquad\nabla_{B,\xi}=\xi^{-1}\circ(\pi^*\nabla_B)\circ\xi.\tag{8}$$

If ξ is not invertible everywhere, the associated metric and Levi–Civita connection suffers from singularities. Our construction however requires to allow such singular metrics as well hence we will do that in what follows. Let us say that two, not necessarily isometric, singular metrics are *equivalent* if they dreibeins differ only by a transformation $\xi\mapsto\xi+(\pi^*\nabla_B)v$. In other words we assign a metric to the cohomology class $[\xi]$ of ξ only. Identifying metrics this way has the advantage that although a particular metric (8) can be singular, within the equivalence class however we can always pass to a smooth representative describing an ordinary metric on $\Sigma\times\mathbb{R}$.

Notice that $g_{\pm\xi}$ are identical metrics (accordingly, $\nabla_{B,\pm\xi}$ are equivalent). Define an action of $\mathbb{Z}_2$ on $H^1(\pi^*\nabla_B)$ via $[\xi]\mapsto[-\xi]$. Then the quotient $H^1(\pi^*\nabla_B)/\mathbb{Z}_2$ is identified with the space of equivalence classes of flat Lorentzian metrics on $\Sigma\times\mathbb{R}$ of the form (8).

If G is a Lie group, consider the space

$$\mathrm{Hom}_0(\pi_1(\Sigma\times\mathbb{R}), G)/G,$$

where Hom_0 denotes the discrete embeddings of $\pi_1(\Sigma\times\mathbb{R})\cong\pi_1(\Sigma)$ into G. It can be identified with one connected component of the space of gauge equivalence classes of flat G connections on $\Sigma\times\mathbb{R}$ and has real dimension $(2g-2)\dim G$. From our construction it is clear that $H^1(\pi^*\nabla_B)$ can be described as the space of flat $\mathrm{ISO}(2,1)$ connections modulo flat $\mathrm{SO}(2,1)$ connections showing its real dimension is $h^1=12g-12-(6g-6)=6g-6$. Therefore, putting all these things together, we have proved:

Proposition 2.2. *The first cohomology $H^1(\pi^*\nabla_B)$ of the complex (4) admits the following two interpretations.*

First $H^1(\pi^*\nabla_B)$ can be identified with the underlying vector space of gauge equivalence classes of those flat $ISO(2,1)$ connections on $\widehat{\pi^*E}$, an affine vector bundle with underlying vector bundle π^*E, which are of the form (6).

Secondly the quotient $H^1(\pi^*\nabla_B)/\mathbb{Z}_2$ can be identified with the space of equivalence classes of inextensible, flat, singular Lorentzian metrics on $\Sigma\times\mathbb{R}$ of the form (8).

We have $h^1=6g-6$ for the corresponding Betti number. $\Diamond$

In light of Proposition 2.2 we can assign a $6g-6$ dimensional moduli of inequivalent singular Lorentzian metrics on $\Sigma\times\mathbb{R}$, solutions to the $2+1$ dimensional vacuum Einstein equation, to a given flat, irreducible $\mathrm{SO}(2,1)$ connection of maximal Euler class. In order to achieve a more explicit description of these singular metrics, we have to analyze the solutions of the Cartan equation (7) on $\Sigma\times\mathbb{R}$. We can do this by carrying out a suitable $\mathrm{ISO}(2,1)$ gauge transformation on the connections in (6).

The splitting $T^*(\Sigma\times\mathbb{R})\otimes\pi^*E\cong(T^*\Sigma\otimes\pi^*E)\oplus(T^*\mathbb{R}\otimes\pi^*E)$ allows us to decompose a dreibein $\xi\in\Omega^1(\Sigma\times\mathbb{R},\pi^*E)$ as $\xi=\pi^*\xi_t+u_t\mathrm{d}t$ with $\xi_t\in\Omega^1(\Sigma,E)$, $u_t\in\Omega^0(\Sigma,E)$ and $t\in\mathbb{R}$. In the obvious temporal gauge for $\pi^*\nabla_B$ (see next section), we have $\pi^*\nabla'_B=\nabla'_B+\frac{\partial}{\partial t}\mathrm{d}t$ and then (7) reads as

$$\nabla'_B\xi_t+\left(\frac{\partial\xi_t}{\partial t}+\nabla_B u_t\right)\wedge\mathrm{d}t+\frac{\partial u_t}{\partial t}\,\mathrm{d}t\wedge\mathrm{d}t=0$$

or simply

$$\nabla'_B\xi_t=0,\qquad \frac{\partial\xi_t}{\partial t}+\nabla_B u_t=0 \tag{9}$$

over $\Sigma\times\{t\}$. By fixing a "Coulomb gauge" on the inhomogeneous part ξ as a next step, we can adjust the first equation in (9) into an elliptic one as follows. Consider the complex

$$0\to\Omega^0(\Sigma,\,E)\xrightarrow{\nabla_B}\Omega^1(\Sigma,\,E)\xrightarrow{\nabla'_B}\Omega^2(\Sigma,\,E)\to 0, \tag{10}$$

whose pullback is (4). Using the orientation and picking up a metric γ on Σ take the associated elliptic complex

$$0\to\Omega^1(\Sigma,\,E)\xrightarrow{\nabla^*_B\oplus\nabla'_B}\Omega^0(\Sigma,\,E)\oplus\Omega^2(\Sigma,E)\to 0.$$

We claim that

Proposition 2.3. *Consider a compact, oriented Riemann surface of genus $g>1$ and fix a metric γ on it. Let $\pi^*\nabla_B$ be an arbitrary flat $SO(2,1)$ connection on $\pi^*E\cong T(\Sigma\times\mathbb{R})$. Then there is a natural vector space isomorphism*

$$H^1(\pi^*\nabla_B)\cong \mathrm{Ker}(\nabla^*_B\oplus\nabla'_B)$$

depending only on the conformal class $[\gamma]$. That is, for all $[\xi]\in H^1(\pi^*\nabla_B)$ there is a unique gauge transformation $\xi':=\xi+(\pi^*\nabla_B)v$ with $v\in\Omega^0(\Sigma\times\mathbb{R},\pi^*E)$ such that all solutions of (7) take the shape $\xi'=\pi^*\eta_{[\xi]}$ with

$$\eta_{[\xi]}=a_1\eta_1+a_2\eta_2+\dots+a_{6g-6}\eta_{6g-6}$$

where $a_i\in\mathbb{R}$ are constants and $\eta_i\in\Omega^1(\Sigma,E)$ with $i=1,\dots,6g-6$ form a fixed basis for the kernel of the elliptic operator $\nabla^*_B\oplus\nabla'_B$. That is, in this gauge ξ' is independent of time.

Notice that this gauge transformation keeps ξ within its cohomology class therefore indeed all solutions of the original Cartan equation (7) over $\Sigma\times\mathbb{R}$ are of this form up to a gauge transformation.

Proof. For technical reasons we temporarily put an auxiliary Riemannian metric onto E to carry out the calculations in the course of this proof. Therefore, referring to the metric γ on $T\Sigma$, we have Riemannian metric on $\Lambda^1\Sigma\otimes E$. We shall denote by h^0, h^1 and h^2 the corresponding Betti numbers of (10). The index of this complex is equal to

$$\mathrm{Index}(\nabla^*_B\oplus\nabla'_B)=-\int\limits_\Sigma(3+c_1(E^{\mathbb{C}}))\wedge(1+e(T\Sigma))=-3\cdot(2-2g)=6g-6$$

since $E^{\mathbb{C}}$ is an $\mathrm{PSL}(2,\mathbb{C})$ bundle consequently its first Chern class vanishes (in fact $E^{\mathbb{C}}$ is a trivial bundle). On the other hand $\mathrm{Index}(\nabla^*_B\oplus\nabla'_B)=-h^0+h^1-h^2$ and Proposition 2.2 shows that $h^1=6g-6$ hence we find $h^0=0$ that is, $\mathrm{Ker}\nabla_B=\mathrm{Coker}\nabla^*_B=\{0\}$ and $h^2=0$ hence $\mathrm{Coker}\nabla'_B=\{0\}$ showing that actually

$$\mathrm{Index}(\nabla^*_B\oplus\nabla'_B)=\dim\mathrm{Ker}(\nabla^*_B\oplus\nabla'_B)=6g-6.$$

A "Coulomb" gauge transformation $\xi'_t:=\xi_t+\nabla_Bv_t$ and $u'_t:=u_t+\frac{\partial v_t}{\partial t}$ with $v_t=v|_{\Sigma\times\{t\}}\in\Omega^0(\Sigma,E)$ such that ξ'_t satisfies the *elliptic* equation

$$(\nabla^*_B\oplus\nabla'_B)\xi'_t=0\tag{11}$$

exists if and only if $\triangle_Bv_t=-\nabla^*_B\xi_t$ for the gauge parameter v_t where $\triangle_B=\nabla^*_B\nabla_B$ is the trace Laplacian of ∇_B. This equation has solution if $\nabla^*_B\xi_t$ is orthogonal to the cokernel of $\triangle_B$ that is, the kernel of $\triangle_B$. However $\mathrm{Ker}\triangle_B\cong\mathrm{Ker}\nabla_B$ which is trivial as we have seen hence $\nabla^*_B\xi_t$ is certainly orthogonal to the trivial cokernel of $\triangle_B$. Moreover this gauge transformation is unique.

Therefore picking up a fixed basis in the kernel of the elliptic operator and observing that $H^1(\pi^*\nabla_B)$ and $\mathrm{Ker}(\nabla^*_B\oplus\nabla'_B)$ are of equal dimensions we can write all solutions of the first equation of (9) as

$$\xi'_t=f(t)(a_1\eta_1+a_2\eta_2+\dots+a_{6g-6}\eta_{6g-6})$$

with a universal function $f(t)$, independent of $\pi^*\nabla_B$. The concrete shape of this function emerges by observing that in this gauge, taking into account the second equation of (9) too, we find

$$\xi'_t\in\mathrm{Ker}\nabla^*_B,\qquad\frac{\partial\xi'_t}{\partial t}\perp_{L^2(\Sigma)}\mathrm{Ker}\nabla^*_B$$

by referring to an L^2 scalar product on $\Omega^1(\Sigma,E)$. Consequently

$$0=\left\langle \xi'_t\ ,\ \frac{\partial\xi'_t}{\partial t}\right\rangle_{L^2(\Sigma)}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi'_t\|^2_{L^2(\Sigma)}$$

implying ξ'_t is independent of time in this gauge therefore we have to set $f(t)=1$. We denote this ξ'_t as $\eta_{[\xi]}$. Furthermore (9) yields $\nabla_B u'_t=0$ hence $u'_t=0$ by uniqueness as claimed. Taking into account the conformal invariance of (11), which is essentially the massless Dirac equation, the result follows. $\Diamond$

Remark. From the viewpoint of ISO(2,1) gauge theory, we interpret this result as the existence of temporal gauge for a connection (6). Indeed, its $\pi^*\nabla_B$ part is time-independent (see next section) as well as the translation ξ' as we have seen. In light of this proposition a generic representative $\xi\in[\xi]\in H^1(\pi^*\nabla_B)$ looks like

$$\xi=\pi^*(\eta_{[\xi]}+\nabla_B v_t)+\frac{\partial v_t}{\partial t}\,\mathrm{d}t \tag{12}$$

and we can suppose that its characteristic part $\eta_{[\xi]}$, a solution of (11), is always smooth by elliptic regularity. Concerning the vector field v_t we only know *a priori* that it somehow diverges as $t\to\pm\infty$ because the corresponding (singular) metric is inextensible by assumption, but otherwise arbitrary. For simplicity we suppose it is smooth (we could relax this regularity).

This shows and we also emphasize that the cohomology class $[\xi]$ is quite immense from a geometric viewpoint: The corresponding non-isometric flat metrics of the form (8) have rich asymptotics, depending on the gauge parameter v in (12).

To illustrate this we check some examples. The zero dreibein $0\in[0]\in H^1(\pi^*\nabla_B)$ corresponds to the totally degenerate "metric" $g_0=0$ on $\Sigma\times\mathbb{R}$ for an arbitrary flat connection.

A less trivial example: Let $(\pi^*\nabla_B)v\in[0]\in H^1(\pi^*\nabla_B)$ be another representative, a pure gauge still within the zero cohomology class. The corresponding dreibein arises by taking $\eta_{[0]}=0$ in (12) with a fixed vector field $v_t=v|_{\Sigma\times\{t\}}$. Since $E=V\oplus\underline{\mathbb{R}}$ with a vector bundle $V\cong T\Sigma$ (non-canonical isomorphism) we have a decomposition $\Omega^0(\Sigma,\ E)\cong\Omega^0(\Sigma,\ V)\oplus\Omega^0(\Sigma)$. We put simply $v_t:=ta$ with $t>0$ and a constant $a=0+1\in\Omega^0(\Sigma,\ V)\oplus\Omega^0(\Sigma)$. Write γ for the metric whose "zweibein" is $\nabla_B a$ then the resulting metric is the incomplete cone metric $-\mathrm{d}t^2+t^2\gamma$ over $\Sigma\times\mathbb{R}^+$ as in [5] (cf. also [1] and [14]) and in particular γ is of constant -1 curvature.

We provide a further description of $H^1(\pi^*\nabla_B)$ which points out its relationship with the Teichmüller space of Σ (also cf. [9]). Let F be a real vector bundle over a manifold M and assume F carries a Lie algebra structure. For a section $\eta\in\Omega^1(M,F)$ define

$$\mathrm{e}^\eta:=1+\eta+\frac{1}{2!}[\eta,\eta]+\dots\in\Omega^0(M)\oplus\Omega^1(M,F)\oplus\Omega^2(M,F)\oplus\dots$$

Turning back to our case consider a cohomology class $[\xi]\in H^1(\pi^*\nabla_B)$ and its unique "Coulomb" representative $\pi^*\eta_{[\xi]}\in[\xi]$ with $\eta_{[\xi]}\in\Omega^1(\Sigma,E)$. Notice that referring to (5), E inherits an Abelian Lie algebra structure (the infinitesimal generators of the commuting fiberwise translations on the affine bundle $\hat{E}$) hence $\mathrm{e}^{\eta_{[\xi]}}=1+\eta_{[\xi]}\in\Omega^0(\Sigma)\oplus\Omega^1(\Sigma,E)$ is defined as above. Let $p:E\cong T\Sigma\oplus\underline{\mathbb{R}}\to V\cong T\Sigma$ be the projection then to any vector field X on Σ one can assign a new vector field $X\mapsto p(\mathrm{e}^{\eta_{[\xi]}}(X))=X+p(\eta_{[\xi]}(X))$.

Provided γ is a metric on Σ let $\tau:T\Sigma\to T^*\Sigma$ be the identification given by $X\mapsto\gamma(X,\cdot)$. Its induced Hodge operator $*_{[\gamma]}:T^*\Sigma\to T^*\Sigma$, depending only on the conformal class, defines an integrable almost complex structure $J_{[0]} := \tau^{-1}\circ *_{[\gamma]}\circ\tau$ on Σ since $*^2_{[\gamma]} = -\mathrm{Id}_{T^*\Sigma}$. One constructs a deformation of this complex structure like $J_{[\xi]} := \mathrm{e}^{-\eta_{[\xi]}}\circ J_{[0]}\circ \mathrm{e}^{\eta_{[\xi]}}$. It readly follows that it is also an almost complex structure since $J^2_{[\xi]} = -\mathrm{Id}_{T\Sigma}$, of course is integrable and differs from $J_{[0]}$. Hence we obtain

Proposition 2.4. *Consider a compact, orientable Riemann surface Σ of $g > 1$ and fix an orientation as well as a metric γ and its conformal class $[\gamma]$ on it. Take ∇_B, a smooth, irreducible flat $SO(2,1)$ connection on the bundle $E \cong T\Sigma\oplus\underline{\mathbb{R}}$ and its cohomology group $H^1(\pi^*\nabla_B)$. If $(\mathcal{T},[\gamma])$ denotes the punctured Teichmüller space of Σ then there is a natural homeomorphism*

$$H^1(\pi^*\nabla_B) \cong (\mathcal{T},[\gamma])$$

given by $[\xi]\mapsto \eta_{[\xi]}\mapsto J_{[\xi]}$ where $\pi^*\eta_{[\xi]}\in[\xi]$ is the unique representative $\eta_{[\xi]}\in \mathrm{Ker}(\nabla^*_B\oplus\nabla'_B)$ and $J_{[\xi]} := \mathrm{e}^{-\eta_{[\xi]}}\circ J_{[0]}\circ\mathrm{e}^{\eta_{[\xi]}}$ with $J_{[0]}$ being the complex structure induced by $[\gamma]$.

Notice that $[0]\in H^1(\pi^*\nabla_B)$ corresponds to the distinguished complex structure induced by $[\gamma]$.

Proof. If $p(\mathrm{e}^{\eta_{[\xi_1]}}(X)) = p(\mathrm{e}^{\eta_{[\xi_2]}}(X))$ for all vector fields then $\eta_{[\xi_1]} = \eta_{[\xi_2]}$ since they satisfy (11) hence $[\xi_1] = [\xi_2]$. Moreover by Proposition 2.2 the space $H^1(\pi^*\nabla_B)$ is homeomorphic to $\mathbb{R}^{6g-6}$ and the same is true for the Teichmüller space (cf. e.g. Corollary 11.10 in [6]). $\diamond$

Remark. The cohomology class $[\xi]\in H^1(\pi^*\nabla_B)$ consists of geometrically very different non-isometric metrics of the form (8), both singular and regular, as we have seen. Nevertheless we are able to assign a unique boundary conformal class to each cohomology class in a natural way via Propositions 2.3 and 2.4. Indeed, $[\xi]$ gives rise to a complex structure $J_{[\xi]}$ which is equivalent to a conformal class $[\gamma]_{[\xi]}$ on Σ.

However this assignment is less geometric in its nature: The conformal class to $[\xi]$ does *not* arise by simply restricting a particular metric within $[\xi]$ to some boundary at infinity.

Summing up we have the following characterization of singular Lorentzian metrics; the straightforward verification is left to the reader.

Proposition 2.5. *Let $\pi^*\nabla_B$ be a smooth, irreducible, flat $SO(2,1)$ connection on the bundle $\pi^*E\cong T(\Sigma\times\mathbb{R})$ over a compact, oriented Riemann surface of genus $g>1$. Consider a cohomology class $[\xi]\in H^1(\pi^*\nabla_B)$ representing an equivalence class of singular metrics g_ξ as in (8). Also fix a conformal class $[\gamma]$ on Σ. Then*

(i) *A generic element g_ξ is an inextensible, smooth, symmetric tensor field on $T(\Sigma\times\mathbb{R})$;*

(ii) *If $\xi\in[\xi]$ is moreover invertible everywhere then g_ξ is an inextensible, smooth, flat, globally hyperbolic Lorentzian metric on $\Sigma\times\mathbb{R}$ with $\Sigma\times\{t\}$ representing Cauchy surfaces. The metric g_ξ may fail to be complete;*

(iii) *In addition $[\xi]$ gives rise to a unique conformal class $[\gamma]_{[\xi]}$ on Σ providing a parameterization of the punctured Teichmüller space $(\mathcal{T},[\gamma])$ of the oriented Riemann surface.* $\diamond$

Before closing this section let us summarize what is known at this point. Starting with an irreducible, real solution $[(\nabla_A,\Phi)]$ of the SO(3) Hitchin equations over Σ which belongs to the connected component $\mathcal{M}_{2g-2}$ of Theorem 2.1, we have found an associated $6g-6$ dimensional moduli $H^1(\pi^*\nabla_B)$ of inequivalent singular solutions of the vacuum Einstein equation on $\Sigma\times\mathbb{R}$

via Proposition 2.2. However if we take into account not only the Hitchin pair itself but the conformal class $[\gamma]$ on Σ as well—which is implicitly present—then we can assign to $[(\nabla_A, \Phi)]$ the trivial cohomology class $[0] \in H^1(\pi^*\nabla_B)$ of singular metrics using Propositions 2.3 and 2.4. This distinguished class of solutions simply arises by picking up *that* cohomology class $[\xi]$ on the bulk whose corresponding boundary conformal class $[\gamma]_{[\xi]}$ in the sense of Propositions 2.3 and 2.4 yields precisely the originally given conformal class $[\gamma]$ on Σ regarded as the spacelike past boundary of $\Sigma \times \mathbb{R}$.

3 The inverse construction

Next we focus our attention to the reverse construction. This will turn out to be simple by referring to a powerful theorem of Donaldson. We continue to consider compact, orientable Riemann surfaces of genus greater than one.

Assume a smooth, flat, probably incomplete singular Lorentzian metric g is given on $\Sigma \times \mathbb{R}$ stemming from an irreducible, smooth, flat $\mathrm{ISO}(2,1)$ connection on an affine bundle $\hat{\tilde{E}}$ whose underlying vector bundle is $\tilde{E} \cong T(\Sigma \times \mathbb{R})$. We claim that any flat $\mathrm{ISO}(2,1)$ connection is of the form (6) hence this singular metric and its Levi–Civita connection look like (8) with a translation ξ and a flat connection ∇_B over the $\mathrm{SO}(2,1)$ bundle E on Σ whose principal bundle is Q_{2g-2} and $\tilde{E} = \pi^*E$.

Indeed, let Γ be a discretely embedded subgroup of $\mathrm{SL}(2,\mathbb{R})$, isomorphic to $\pi_1(\Sigma)$. Since $\mathrm{SL}(2,\mathbb{R})$ is the isometry group of the hyperbolic plane $\mathbb{H}^2$ we can construct a model for Σ as the quotient $\mathbb{H}^2/\Gamma \cong \Sigma$ together with the projection $p : \mathbb{H}^2 \to \Sigma$. We extend this to a map $p : \mathbb{H}^2 \times \mathbb{R} \to \Sigma \times \mathbb{R}$ acting as the identity on $\mathbb{R}$. Then given a flat $\mathrm{SO}(2,1)$ connection $\tilde{\nabla}_B$ on $\tilde{E}$ its pullback can be written as $p^*\tilde{\nabla}_B = \mathrm{d} + f^{-1}\mathrm{d}f$ with a Γ-periodic function $f : \mathbb{H}^2 \times \mathbb{R} \to \mathrm{SL}(2,\mathbb{R})$. We can always gauge away the $\mathbb{R}$-component of the pullback connection i.e., we can assume that $f^{-1}\partial_t f = 0$ yielding f is independent of t. Consequently in this "temporal gauge" the connection $p^*\tilde{\nabla}_B$ hence $\tilde{\nabla}_B$ looks like $\pi^*\nabla_B$ on $\pi^*E \cong \tilde{E}$ over $\Sigma \times \mathbb{R}$ i.e., gives rise to a flat $\mathrm{SO}(2,1)$ connection ∇_B on $E \cong T\Sigma \oplus \underline{\mathbb{R}}$. We conclude that a flat $\mathrm{ISO}(2,1)$ connection is of the form (6) with ∇_B a flat connection and ξ a translation. Consequently the associated singular metric g possesses the properties summarized in Proposition 2.5 hence we shall denote it as g_ξ.

In particular if ξ is a representative of a cohomology class $[\xi] \in H^1(\pi^*\nabla_B)$ and $[\gamma]$ is a conformal class on Σ then via Propositions 2.3 and 2.4 $[\xi]$ gives rise to a unique boundary conformal class $[\gamma]_{[\xi]}$ on Σ, regarded as the spacelike past boundary of $\Sigma \times \mathbb{R}$ with induced orientation. Moreover it also represents Lorentzian metrics and the restriction of their Levi–Civita connections to Σ yields a unique, irreducible flat $\mathrm{SO}(2,1)$ connection on E. Given these data: $[\gamma]_{[\xi]}$ and ∇_B on Σ one may wonder whether or not they correspond to a (real) solution of the $\mathrm{SO}(3)$ Hitchin equations. If yes, then we have $w_2(P) = 0$ for the corresponding $\mathrm{SO}(3)$ principal bundle over Σ since the Euler class of the underlying $\mathrm{SO}(2,1)$ principal bundle of $E = E_{2g-2}$ is even.

The question is answered in the affirmative by the following theorem [4]:

Theorem 3.1. (Donaldson, 1987) *Let P be an $SO(3)$ principal bundle over a compact, oriented Riemann surface $(\Sigma, [\gamma]_{[\xi]})$. Assume ∇_B is an irreducible flat $PSL(2,\mathbb{C})$ connection on $P^{\mathbb{C}}$. Then there exists an $PSL(2,\mathbb{C})$ gauge transformation on the complexified bundle $P^{\mathbb{C}}$ taking the flat connection into the form $\nabla_A + \Phi + \Phi^*$ where the pair (∇_A, Φ) satisfies the $SO(3)$ Hitchin equations (2) with respect to the conformal class $[\gamma]_{\xi]}$ and the orientation on Σ.* $\diamond$

Remark. If the flat, irreducible $\mathrm{PSL}(2,\mathbb{C})$ connection is real then the resulting Hitchin pair is also real in the sense of Theorem 2.1 and in particular our real solutions are mapped into the $\mathcal{M}_{2g-2}$ component.

4 An AdS/CFT-type correspondence

The time has come to bring all of our findings together. These lead us to an AdS/CFT-type correspondence between classical $2+1$ dimensional vacuum general relativity on the bulk space-time $\Sigma\times\mathbb{R}$ and 2 dimensional $\mathrm{SO}(3)$ Hitchin theory—regarded as a classical conformal field theory—on the spacelike past boundary Σ.

We find the most expressive way to present the duality equivalence by formulating it in terms of the corresponding field equations. Then we can rephrase it by referring to the solutions themselves. For notational simplicity we continue to denote a real Hitchin pair on the principal $\mathrm{SO}(3)$ bundle P with $w_2(P)=0$ as (∇_A,Φ) while $\nabla_A+\Phi+\Phi^*$ is the associated flat connection on the $\mathrm{SO}(2,1)$ vector bundle E of the $\mathrm{SO}(2,1)$ principal bundle Q_{2g-2} of Theorem 2.1.

Theorem 4.1. *Let $(\Sigma,[\gamma])$ be an oriented, compact Riemann surface Σ of genus $g>1$ with a fixed conformal class. Consider $[(\nabla_A,\Phi)]\in\mathcal{M}_{2g-2}$, an irreducible, real solution of the Hitchin equations on the SO(3) principal bundle P over Σ. Then this pair, consisting of the gauge equivalence class of an SO(3) connection ∇_A and a complex Higgs field Φ, satisfies the Hitchin equations over $(\Sigma,[\gamma])$:*

$$\begin{cases} F_A+\frac{1}{2}[\Phi,\Phi^*] &= 0\\ \overline{\partial}_A\Phi &= 0.\end{cases}$$

Then there is a unique associated pair $[(\pi^*\nabla_B,\xi)]$ consisting of the gauge equivalence class of a flat SO(2,1) connection $\nabla_B:=\nabla_A+\Phi+\Phi^*$ on $E\cong T\Sigma\oplus\mathbb{R}$ and the trivial cohomology class $[0]\in H^1(\pi^*\nabla_B)$ of a dreibein $\xi\in\Omega^1(\Sigma\times\mathbb{R},\,\pi^*E)$ with induced boundary conformal class being precisely $[\gamma]$, such that this pair satisfies the real vacuum Einstein equation over $\Sigma\times\mathbb{R}$ with induced natural orientation:

$$\begin{cases} \pi^*F_B &= 0\\ (\pi^*\nabla'_B)\xi &= 0.\end{cases}$$

Conversely, given a real solution $[(\pi^*\nabla_B,\xi)]$ of the vacuum Einstein equation over the oriented $\Sigma\times\mathbb{R}$ such that the induced boundary conformal class of $[\xi]\in H^1(\pi^*\nabla_B)$ is $[\gamma]_{[\xi]}$ with respect to some conformal class $[\gamma]$ and induced orientation on Σ, then there exists a unique irreducible, real solution $[(\nabla_A,\Phi)]$ of the Hitchin equations on the SO(3) principal bundle P over $(\Sigma,[\gamma]_{[\xi]})$, such that $\nabla_A+\Phi+\Phi^*$ is PSL$(2,\mathbb{C})$ gauge equivalent to ∇_B on E. $\diamond$

This implies that there is a kind of correspondence between certain smooth, real, irreducible solutions of the 2 dimensional $\mathrm{SO}(3)$ Hitchin equations and solutions of the $2+1$ dimensional vacuum Einstein equation expressed in the more usual form of a metric as follows.

Associated to $[(\nabla_A,\Phi)]\in\mathcal{M}_{2g-2}$ over the oriented $(\Sigma,[\gamma])$ there is a set of singular solutions g_ξ of the Lorentzian vacuum Einstein equation on $\Sigma\times\mathbb{R}$ with natural induced orientation such that

(i) The metric and its Levi–Civita connection are of the form (8) with $\nabla_B = \nabla_A + \Phi + \Phi^*$ and some ξ. The isometry classes of these singular metrics are parameterized by ξ's which belong to the trivial cohomology class $[0] \in H^1(\pi^*\nabla_B)$;

(ii) The boundary conformal class, induced by $[0]$ in the sense of Propositions 2.3 and 2.4 on Σ, regarded as the spacelike past boundary of $(\Sigma \times \mathbb{R}, g_\xi)$, is equal precisely to $[\gamma]$.

Conversely, assume Σ is equipped with some conformal structure $[\gamma]$. Given a set of singular solutions g_ξ of the Lorentzian vacuum Einstein equation on the oriented $\Sigma \times \mathbb{R}$, there is a unique real solution $[(\nabla_A, \Phi)] \in \mathcal{M}_{2g-2}$ of the SO(3) Hitchin equations over the spacelike past boundary $(\Sigma, [\gamma]_{[\xi]})$ with induced orientation such that

(i) The connection $\nabla_B := \nabla_A + \Phi + \Phi^*$ is $\mathrm{PSL}(2, \mathbb{C})$ gauge equivalent to the Levi–Civita connections of the g_ξ's restricted to Σ in temporal gauge and the set of these metrics are parameterized by a cohomology class $[\xi] \in H^1(\pi^*\nabla_B)$;

(ii) The boundary conformal class induced by this cohomology class $[\xi]$, in the sense of Propositions 2.3 and 2.4, is precisely $[\gamma]_{[\xi]}$ on Σ.

We can see at this point that this correspondence can be interpreted as a sort of *generalized* AdS/CFT correspondence between these theories. By "generalized" we mean the way of assigning a boundary conformal class to a bulk metric: It does *not* arise geometrically by taking the conformal class of the bulk metric and then restricting one of its representatives to the past or future boundary of the bulk. Rather we associate the same conformal geometry to metrics of probably very different asymptotics, parameterized by a cohomology class and the conformal geometry arises in an abstract way exploiting the conformal properties of a massless Dirac-like equation as explained in Propositions 2.3 and 2.4.

We decided to present the main result in terms of the field equations not only because of their impressive form but in this way we can also point out that the $2 + 1$ dimensional vacuum Einstein equation, if formulated in terms of a connection and a dreibein, can be viewed as a sort of "decoupled" version of the SO(3) Hitchin equations: It is challenging to view the flat $\mathrm{SO}(2, 1)$ connection ∇_B as a "dual" connection to the non-flat SO(3) connection ∇_A and the dreibein ξ as a "dual" Higgs field to Φ and vice versa. The straightforward advantage of the Einstein equation over the Hitchin equations is that the former is decoupled. Observe that at least formally we have no reason to restrict this description to real solutions hence this duality can in principle continue to hold for a generic complex solution of the Hitchin equations (in the sense that the associated flat connection may belong to $\mathrm{PSL}(2, \mathbb{C})$) and for the complex dual Higgs field one has $\xi^{\mathbb{C}} \in \Omega^1_{\mathbb{C}}(\Sigma \times \mathbb{R},\ \pi^*E^{\mathbb{C}})$.

5 Conclusions

In this paper we presented a natural classical AdS/CFT-type duality between three dimensional Lorentzian vacuum general relativity and two dimensional Hitchin conformal field theory. This correspondence might be considered as a physical interpretation of at least the real solutions of the SO(3) Hitchin equations (cf. the Introduction of [6]).

One may try to probe this correspondence beyond the classical level by calculating (1) in this context. Fix a cohomology class $[\xi] \in H^1(\pi^*\nabla_B)$ with corresponding $[\gamma]_{[\xi]}$ on Σ. Then on the conformal side we have the unique data $([\gamma]_{[\xi]}, [(\nabla_A, \Phi)])$ on Σ with a real Hitchin pair while on the gravitational side we find Lorentzian metrics g_ξ on $\Sigma \times \mathbb{R}$ parametrized by $[\xi]$. Then the partition function of the Hitchin conformal field theory is formally equal to

$$Z_{CFT}\left([\gamma]_{[\xi]}, [(\nabla_A, \Phi)]\right) = \frac{1}{\mathrm{Vol}([\xi])}\int\limits_{[\xi]} \mathrm{e}^{\mathbf{i}I(g_\xi)}\mathrm{D}\xi$$

where the integral is taken over the cohomology class $[\xi]$. A generic element is given as in (12) consequently $[\xi] \cong \Omega^0(\Sigma \times \mathbb{R}, \pi^*E)$, an infinite dimensional space. This integral shares some similarities with those considered in [10]. Bearing in mind that probably both sides of the above integral expression make no sense mathematically, we can calculate it formally as follows. The Einstein–Hilbert action on a Lorentzian manifold with vanishing cosmological constant and spacelike boundary looks like

$$I(g) = -\frac{1}{16\pi G}\int\limits_M s(g)\mathrm{d}g - \frac{1}{8\pi G}\int\limits_{\partial M} \mathrm{tr}k(g)\, \mathrm{d}(g|_{\partial M})$$

with $s(g)$ being the scalar curvature and $k(g)$ the second fundamental form of the boundary. In our case $s(g_\xi) = 0$ where ξ is invertible hence the first term vanishes for regular representatives however the second term may not exist for certain asymptotics of v_t in (12). We can overcome this difficulty if replace the action by its holographically renormalizied form $I^{ren}(g_\xi)$ as in [12] (also cf. [1]); this gives simply $I^{ren}(g_\xi) = 0$ in our case for all invertible representatives. Consequently, by arguing that non-invertible elements of the cohomology class form a "set of measure zero" we formally find for the particular Hitchin pairs in $\mathcal{M}_{2g-2}$ that

$$Z_{CFT}\left([\gamma]_{[\xi]}, [(\nabla_A, \Phi)]\right) = 1.$$

Interesting questions can be raised for future work. For instance, what is the physical interpretation of generic complex solutions of the SO(3) Hitchin equations? At first sight one can declare without problem that they correspond to complex flat metrics on $T^{\mathbb{C}}(\Sigma \times \mathbb{R})$ but this sounds rather unphysical. Taking into account that $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}(3,1)$, the identity component of the *four* dimensional Lorentz group, one may try to regard the complex solutions as real flat metrics on $\Sigma \times \mathbb{R}^2$; however in four dimensions flat metrics do not exhaust solutions of the vacuum Einstein equation therefore this interpretation would not be "tight" enough. Perhaps it is possible that a complex solution can be projected somehow to a *non-flat* real three dimensional connection therefore representing a non-vacuum solution or rather a solution with non-zero cosmological constant in $2+1$ dimensions. The presence of $\mathrm{SO}(3,1)$, the de Sitter isometry group, suggests this later possibility.

Finally, notice that in fact the whole construction proceeds through a complexification phase which cancels out the information of the original real group we began with; this was SO(3) in our case just because of convenience: Both the Hitchin and the Donaldson theorems are formulated with this group. However recently new smooth solutions of the $\mathrm{SO}(2,1)$ Hitchin equations have been discovered [7] pointing toward the possibility that even $\mathrm{SO}(2,1)$ Hitchin theory is interesting and can be used to formulate a duality if a Donaldson-type theorem could be worked out.

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